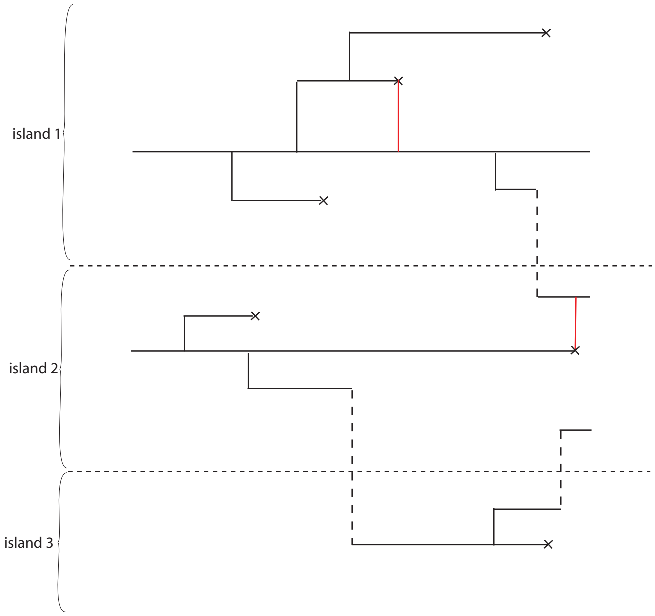
When mobility is not enough to make up for competition

Martin Hutzenthaler

Goethe-University Frankfurt

Marseille Luminy, May 29 2009


island 1
island 2
island 3

Interacting Feller diffusions with logistic drift

$$dX_t(i) = \Big(\sum_{j\in\mathbb{Z}^2} X_t(j)m(j,i) - X_t(i)\Big)\,dt$$
$$+\big(cX_t(i) - \gamma X_t^2(i)\big)dt + \sqrt{\beta X_t(i)}\,dB_t(i) \quad i \in \mathbb{Z}^2$$

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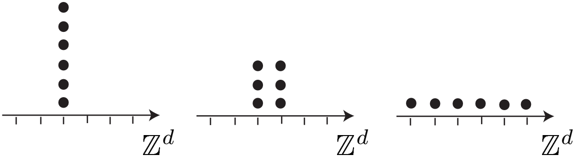
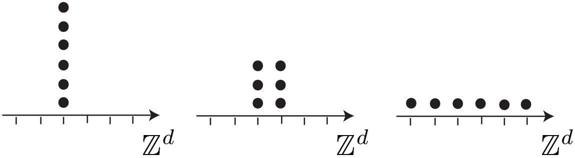
More generally: Interacting locally regulated diffusions

$$dX_t(i) = \Big(\sum_{j\in\mathbb{Z}^d} X_t(j)m(j,i) - X_t(i)\Big)\,dt$$
$$+\mu(X_t(i))dt + \sqrt{\sigma^2(X_t(i))}\,dB_t(i) \quad i \in \mathbb{Z}^d$$



Idea for a comparison result:

- If mass is evenly distributed, then competition is low.



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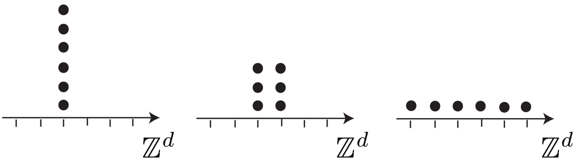

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Uniform migration on $\mathbb{Z}^d$???

Idea: Uniform migration on $\Lambda_N \subset \mathbb{Z}^d$, $\#\Lambda_N < \infty$.

Then $\Lambda_N \uparrow \mathbb{Z}^d$.

$$dX_t^N(i) = \left(\frac{1}{N}\sum_{j=1}^{N} X_t^N(j) - X_t^N(i)\right) dt$$

$$+ \mu\big(X_t^N(i)\big)\, dt + \sqrt{\sigma^2\big(X_t^N(i)\big)}\, dB_t(i) \quad i \leq N.$$

Idea: $\left(|X_t| := \sum_{i\in\mathbb{Z}^d} X_t(i)\right)$

$$\Big(|X_t|\Big)_{t\geq 0} \leq_? \lim_{N\to\infty} \Big(|X_t^N|\Big)_{t\geq 0}$$

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What is the limit of $|X_t^N|$ if $X_0^N(i) = x\mathbb{1}_{i=1}$?

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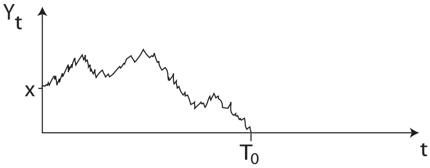
What is the limit of $|X_t^N|$ if $X_0^N(i) = x\mathbb{1}_{i=1}$?

Heuristic: The first emigrant moves to some island. The probability that a later emigrant moves to the same island is $\frac{1}{N}$. In the limit $N \to \infty$ no two emigrants move to the same island.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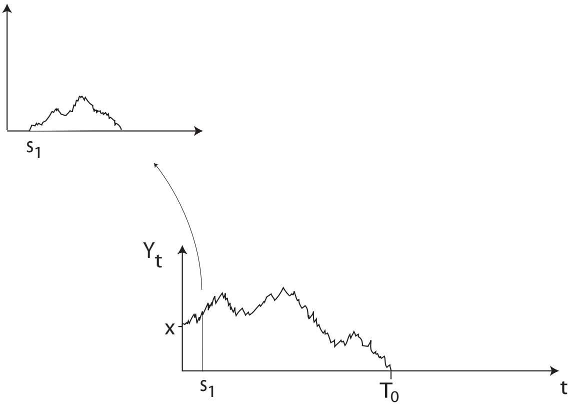

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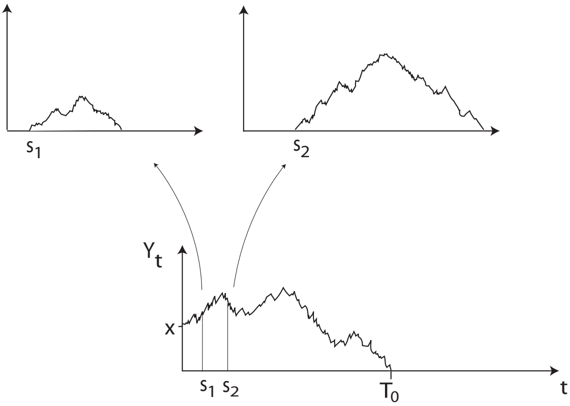
s1
T0
t



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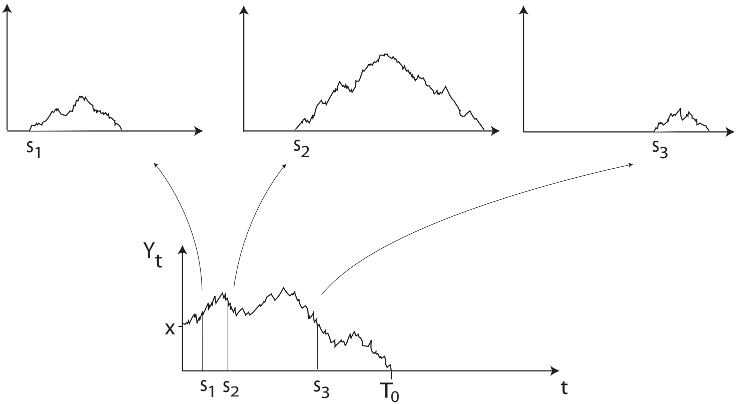
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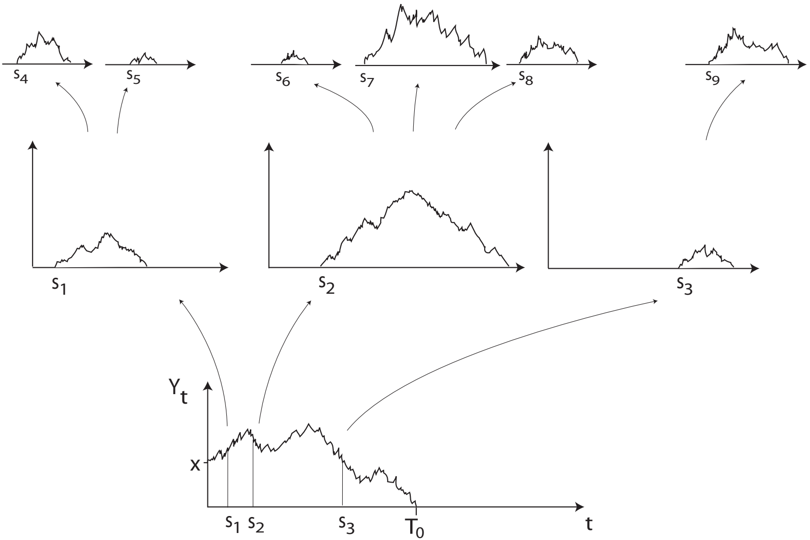

s_4
s_5
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s_7
s_8
s_9
s_1
s_2
s_3
Y_t
x
s_1
s_2
s_3
T_0
t

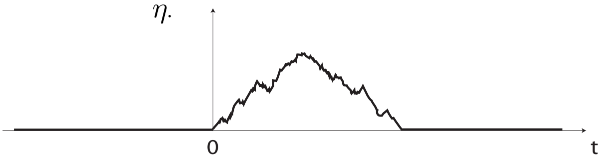
The virgin island model

- The population $(Y_t)_{t\geq 0}$ on island 1 evolves as

$$dY_t = -Y_t\,dt + \mu(Y_t)\,dt + \sqrt{\sigma^2(Y_t)}dB_t \qquad Y_0 = x \geq 0,$$

- Every emigrant migrates to an unpopulated island
- The evolution on a newly populated island is modeled by excursions from zero of $(Y_t)_{t\geq 0}$. The excursion measure Q is defined by

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbf{E}^{\varepsilon}F\big((Y_t)_{t\geq 0}\big) =: \int F\big((\eta_t)_{t\geq 0}\big)Q(d\eta)$$



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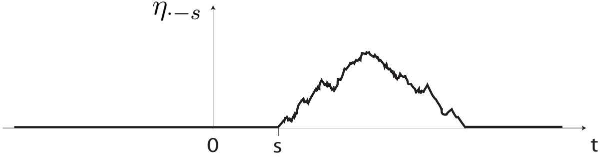

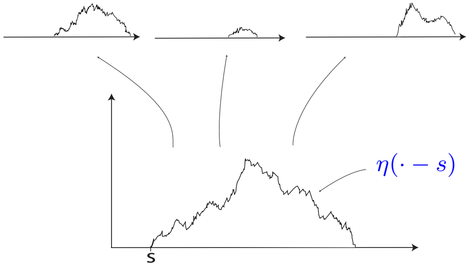

If the mother island is populated at time s and has population $\eta(t-s)_{t\geq 0}$, then offspring islands are a Poisson point process Π with intensity measure

$$\mathbf{E}\big[\Pi(dt \otimes d\psi)\big] = \eta(t-s)\,dt \otimes Q(d\psi)$$

Theorem: (Comparison)
Let μ be sublinear $\big(\mu(x+y) \leq \mu(x) + \mu(y)\big)$ and σ^2 be linear.
If $X_0(i) = x\mathbb{1}_{i=0}$, then

$$|X_t| \leq_{\text{st}} |V_t| \qquad |V_0| = x, \;\; \forall\, t \geq 0$$

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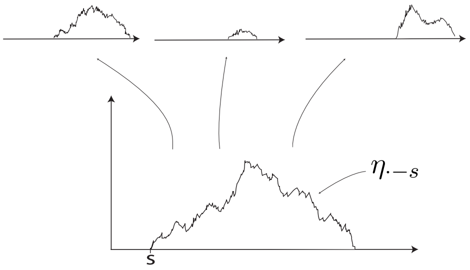
Corollary: If $|V_t| \xrightarrow{w} 0$ as $t \to \infty$, then $|X_t| \xrightarrow{w} 0$.



$\eta_{\cdot - s}$
s

Theorem: (Global Extinction of VIM)

The virgin island process dies out globally for every initial mass $x > 0$ if and only if

$$\int \Big(\int_0^\infty \eta_t \, dt \Big) Q(d\eta) \leq 1$$

Theorem: (Global Extinction of VIM)

The virgin island process dies out globally for every initial mass $x > 0$ if and only if

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Explicit formula for this:

$$\begin{aligned} \int \Big(\int_{-\infty}^\infty \eta_t \, dt \Big) Q(d\eta) &= \int y \int_{-\infty}^\infty Q(\eta_t \in dy) \, dt \\ &= \int y \, m(dy) \\ &= \int_0^\infty \frac{y}{\sigma^2(y)/2} \exp \Big(\int_0^y \frac{-x + \mu(x)}{\sigma^2(x)/2} \, dx \Big) \, dy \end{aligned}$$

$$dX_t(i) = \Big(\sum_{j\in\mathbb{Z}^d} X_t(j)m(j,i) - X_t(i)\Big)\,dt$$

$$+\mu\big(X_t(i)\big)dt + \sqrt{\sigma^2\big(X_t(i)\big)}\,dB_t(i) \quad i \in \mathbb{Z}^d$$

Corollary: (Global Extinction of X)

If μ is sublinear, σ^2 is linear and if

$$\int_0^\infty \frac{y}{\sigma^2(y)/2}\exp\Big(\int_0^y \frac{-x+\mu(x)}{\sigma^2(x)/2}\,dx\Big)\,dy \le 1,$$

then every system of interacting locally regulated diffusions dies out globally ($|X_t| \xrightarrow{w} 0$) for every migration kernel m.

There exist comparison results if σ^2 is either superlinear or sublinear. Then the stochastic order is more complicated.

Example: Stepping stone model with selection and mutation

$$dX_t(i) = \left(\sum_{j \in \mathbb{Z}^d} X_t(j) m(j,i) - X_t(i)\right) dt$$
$$+ \Big(sX_t(i)\big(1 - X_t(i)\big) - uX_t(i)\Big) dt + \sqrt{2X_t(i)\big(1 - X_t(i)\big)}\, dB_t(i)$$

for $i \in \mathbb{Z}^d$.

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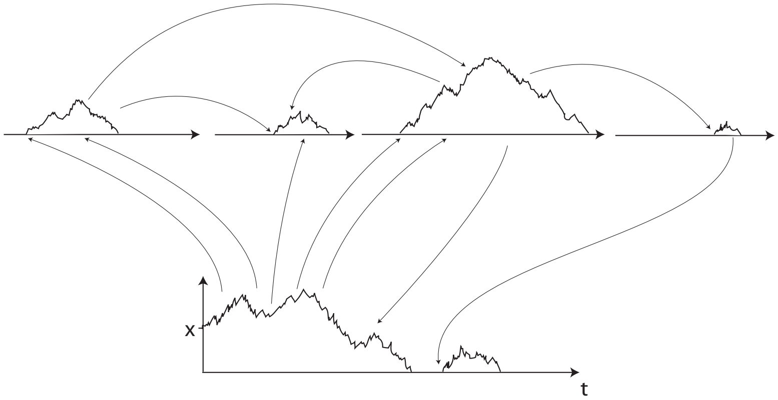
for $i \in \mathbb{Z}^d$. If

$$\int_0^1 (1-y)^u e^{sy}\, dy < 1$$

then $|X_t| \to 0$ as $t \to \infty$ in L^1.

Theorem: (Convergence to the VIM)
If $X_0(i) = x\mathbb{1}_{i=0}$, then

$$\sum_{i=1}^{N} X_t^N(i) \xrightarrow[N\to\infty]{\text{w}} |V_t| \qquad \forall\, t \geq 0,\ |V_0| = x$$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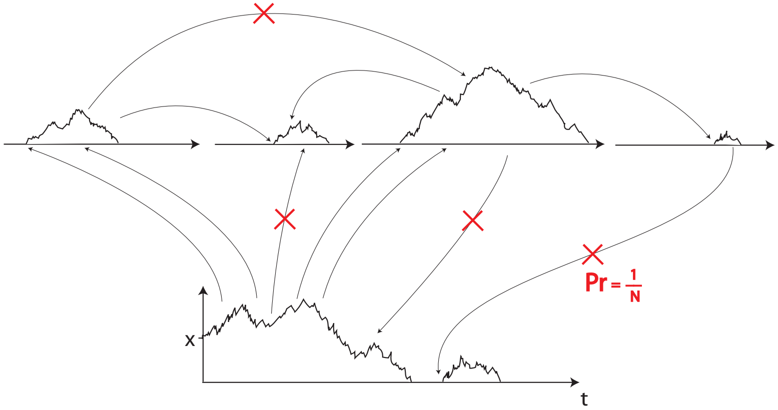

Pr = 1/N
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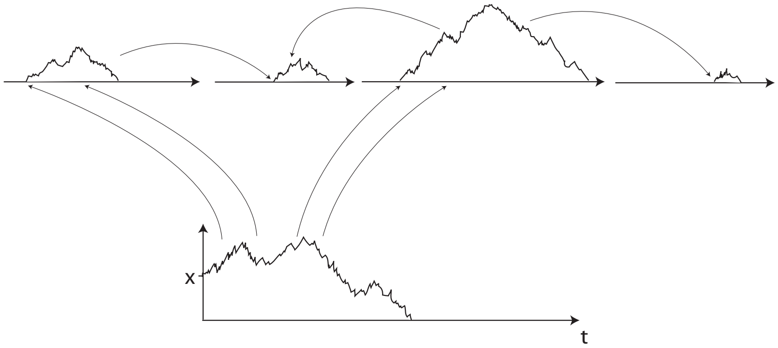
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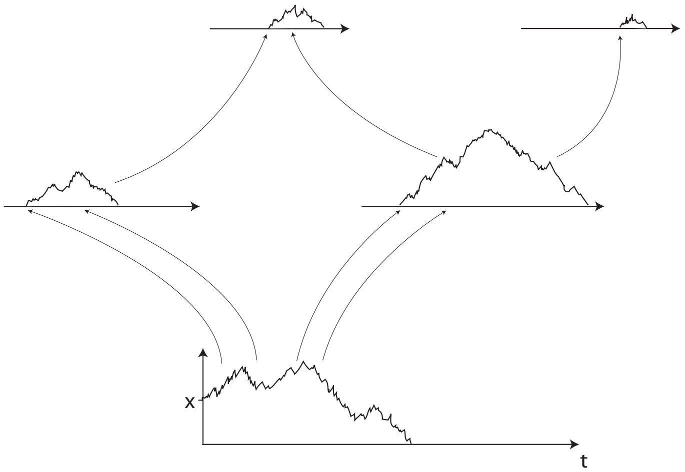
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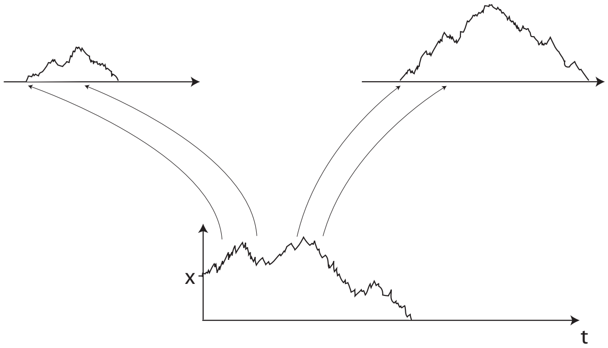
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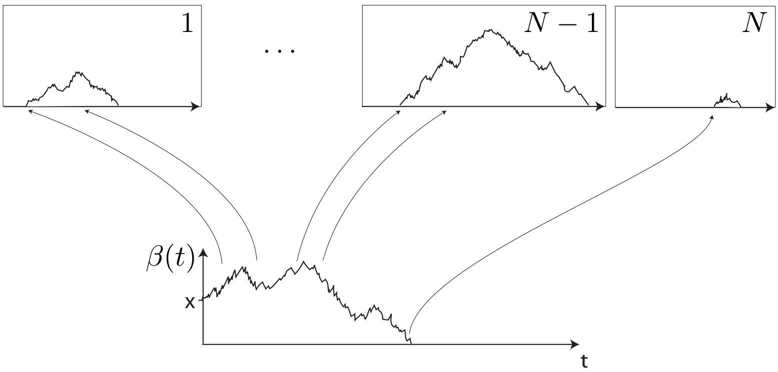
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. . .
N − 1
N
β(t)
x
t

Consider the one-dimensional diffusion

$$dY_t^N = \left(\frac{\beta(t)}{N} - Y_t^N + \mu(Y_t^N)\right) dt + \sqrt{\sigma^2\left(Y_t^N\right)}\, dB_t$$

where $Y_0^N = 0$.

Lemma:

$$\lim_{N\to\infty} N\,\mathbf{E}^0 f\left(Y_t^N\right) = \int \int_0^t f(\eta_{t-s})\beta(s)\, ds Q(d\eta)$$

if f vanishes in a neighbourhood of zero.

Fancy duality: If $\mu(z) = \gamma x(K - x)$ and $\sigma^2(x) = 2\gamma x$, then

$$\mathbf{E}^x \exp(-yM_t) = \mathbf{E}^y \exp(-x|V_t|) \qquad \forall\, x, y \in [0, \infty)$$

where

$$dM_t = (\mathbf{E}M_t - M_t)\,dt + \mu(M_t)\,dt + \sqrt{\sigma^2(M_t)}\,dB_t \;\; M_0 = x$$

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Thank you

$$
\begin{aligned}
dX_t^{(N,k)}(i) =& \alpha \left(\frac{1}{N} \sum_{j=1}^{N} X_t^{(N,k-1)}(j) - X_t^{(N,k)}(i) \right) dt \\
&+ \frac{X_t^{(N,k)}(i)}{\sum_{m\geq 0} X_t^{(N,m)}(i)} \mu \Big(\sum_{m\geq 0} X_t^{(N,m)}(i) \Big) dt \\
&+ \sqrt{\frac{X_t^{(N,k)}(i)}{\sum_{m\geq 0} X_t^{(N,m)}(i)} \sigma^2 \Big(\sum_{m\geq 0} X_t^{(N,m)}(i) \Big)}\, dB_t^k(i),
\end{aligned}
$$

where $i = 1, ..., N, k \geq 0$ and where $X_0^{(N,k)}(i) = X_0^N(i) \mathbb{1}_{k=0}$.

$$
\begin{aligned}
dZ_t^{(N,k)}(i) =& \alpha \Big(\frac{1}{N} \sum_{j=1}^{N} Z_t^{(N,k-1)}(j) - Z_t^{(N,k)}(i) + \mu\big(Z_t^{(N,k)}(i)\big) \Big) dt \\
&+ \sqrt{\sigma^2\big(Z_t^{(N,k)}(i)\big)}\, dB_t^k(i) \qquad i = 1, ..., N.
\end{aligned}
$$

Define the scale function of $(Y_t)_{t\geq 0}$ by

$$S(y) := \int_0^y s(z)\,dz \qquad s(z) := \exp\Big(-\int_0^z \frac{-x+\mu(x)}{\sigma^2(x)}\,dx\Big).$$

$\big(S(Y_t)\big)_{t\geq 0}$ is a local martingale and

$$\mathbf{P}^y(T_b < T_0) = \frac{S(y)}{S(b)} \qquad 0 < y < b$$

where $T_b := \inf\{t > 0 \colon Y_t = b\}$.

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where $T_b := \inf\{t > 0 \colon Y_t = b\}$.
The excursion measure Q satisfies

$$\lim_{y\to 0} \frac{1}{S(y)} \mathbf{E}^y F(Y) = \int F(\eta) Q(d\eta)$$

for all bounded continuous $F\colon \mathbf{C}\big([0,\infty),[0,\infty)\big) \to \mathbb{R}$ for which there exists an $\varepsilon > 0$ such that $F(\eta) = 0$ whenever $\sup_{t\geq 0} \eta_t < \varepsilon$.