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Probability Theory and Related Fields

On quasi-linear stochastic partial differential equations

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Summary. We prove existence and uniqueness of the solution of a parabolic SPDE in one space dimension driven by space-time white noise, in the case of a measurable drift and a constant diffusion coefficient, as well as a comparison theorem.

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1 Introduction

In this paper, we consider the parabolic quasilinear stochastic partial differential equation

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + f(u)(t, x) + \frac{\partial^2 W}{\partial t \partial x}(t, x); t \geqq 0, \quad x \in (0, 1) .$$

with the initial condition

$$u(0, x) = u_0(x); u_0 \in C_0([0, 1]) ;$$

and Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, 1) = 0, \quad t \geqq 0 .$$

Here $\frac{\partial^2 W}{\partial t \partial x}$ denotes the space-time white noise, $f(u)(t, x) \triangleq f(t, x; u(t, x))$, where

$$(t, x, r) \to f(t, x, r)$$

satisfies some measurability and growth condition which will be specified below. Note that we won't make any continuity assumption on f.

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We shall also consider the same equation with Dirichlet boundary conditions.

A rigorous formulation of the above equation will be given in the next section.

The aim of this paper is to prove existence and uniqueness of a strong solution to our equation. Similar results are known to hold for a more general equation with a variable diffusion coefficient under Lipschitz continuity assumptions, see e.g. Walsh [11], Manthey [8], and under monotonicity type conditions, see Buckdahn and Pardoux [1].

Note that, as in the finite dimensional case, weak existence and uniqueness is easy to establish using Girsanov's theorem, and that strong existence and uniqueness depends essentially, via Yamada–Watanabe's argument (see e.g. Ikeda and Watanabe [4]), on pathwise uniqueness. Our argument for pathwise uniqueness will use in an essential way a comparison theorem, which depends heavily on the fact that the solution u takes values in $\mathbb{R}$ (i.e. we are dealing with an equation, not a system of equations). On the other hand, we shall not exploit the Yamada–Watanabe theorem, since our uniqueness proof will use the construction of a strong solution by an approximation procedure which is similar to that in Krylov [6]. The proof of the convergence of the approximating sequence uses an a priori estimate based on the fact that the law of $u(t, x)$ has a density whose $L^p(\mathbb{R})$ norm depends only on the sup of f, and plays the same role as deep estimates of Krylov [7] for finite dimensional Itô processes.

Our results generalize in a sense part of the results of Zvonkin [12] and Veretennikov [10] to SPDEs. Note that our results are restricted to scalar valued solutions (for which much more is known in the finite dimensional case – see e.g. Rogers and Williams [9]).

One open problem in our framework is the strong existence and uniqueness with measurable drift and nondegenerate Lipschitz diffusion coefficient, i.e. the equivalent of Veretennikov's full finite dimensional result.

The paper is organised as follows. In Sect. 2, we shall state rigorously the equation and our assumptions, and prove some preliminary results in Sect. 3. The main results will be proved in Sects. 4 and 5. We shall consider the case of Dirichlet boundary conditions in Sect. 6.

2 Statement of the problem

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqq 0}, P)$ be a stochastic basis and $\mathscr{P}$ denote the σ-algebra of progressively measurable subsets of $\Omega \times \mathbb{R}_+$. We are given a space-time white noise on $\mathbb{R}_+ \times [0, 1]$ defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$, i.e. an application $W: \mathscr{B}(\mathbb{R}_+ \times [0, 1]) \to \mathbb{H}$, where $\mathbb{H}$ is a Gaussian space, $\mathbb{H} \subset L^2(\Omega, \mathscr{F}, P)$, s.t.

(i) $\forall A, B \in \mathscr{B}(\mathbb{R}_+ \times [0, 1])$ with $A \cap B = \emptyset$, $W(A)$ and $W(B)$ are independent,

(ii) $\forall C \in \mathscr{B}([0, 1])$, $\{W([0, t] \times C); t \geqq 0\}$ is an $\mathscr{F}_t$-Brownian motion with covariance $t\lambda(C)$, where λ denotes Lebesgue measure.

Moreover,

$$f: \mathbb{R}_+ \times [0, 1] \times \mathbb{R} \to \mathbb{R}$$

is supposed to be $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}([0, 1]) \otimes \mathscr{B}_1$ measurable, where $\mathscr{B}(\mathbb{R}_+)$(resp. $\mathscr{B}([0, 1])$, $\mathscr{B}_1$) denotes the Borel σ-algebra over $\mathbb{R}_+$(resp. $[0, 1]$, $\mathbb{R}$), bounded on

$[0, T] \times [0, 1] \times [-R, R]$ for each $T, R > 0$, and has one-sided linear growth in the sense that there exists c s.t.

$$rf(t, x, r) \leqq c(1 + r^2); \quad t \geqq 0, 0 \leqq x \leqq 1, r \in \mathbb{R} .$$

We are finally given

$$u_0 \in C([0, 1]) .$$

We shall refer to our SPDE with the drift coefficient f as $Eq(f)$. More precisely, we say that a $\mathscr{P} \otimes \mathscr{B}([0, 1])$ measurable and continuous random field $\{u(t, x); t \geqq 0, 0 \leqq x \leqq 1\}$ solves equation $Eq(f)$ if for any $\varphi \in C^2([0, 1])$ s.t. $\varphi'(0) = \varphi'(1) = 0$,

$$\int_0^1 u(t, x)\varphi(x)dx = \int_0^1 u_0(x)\varphi(x)dx + \int_0^t \int_0^1 \left[u(s, x)\frac{\partial^2 \varphi}{\partial x^2}(x) + f(u)(s, x)\varphi(x) \right] dxds$$
$$+ \int_0^t \int_0^1 \varphi(x) W(ds, dx), \quad t \geqq 0, \text{ p.s.}$$

where the last integral is a Wiener integral and

$$f(u)(s, x) \triangleq f(s, x; u(s, x)) .$$

It is shown in Walsh [11] that the continuous and adapted process u solves $Eq(f)$ iff u satisfies:

$$u(t, x) = \int_0^1 G_t(x, y)u_0(y)dy$$
$$+ \int_0^t \int_0^1 G_{t-s}(x, y)f(u)(s, y)dyds + \int_0^t \int_0^1 G_{t-s}(x, y) W(ds, dy);$$
$$t \geqq 0, \quad 0 \leqq x \leqq 1, \quad \text{a.s.}$$

where

$$G_t(x, y) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left(-\frac{(y - x - 2n)^2}{4t} \right) + \exp\left(-\frac{(y + x - 2n)^2}{4t} \right) \right\} \tag{1}$$

is the fundamental solution of the heat equation on $\mathbb{R}_+ \times (0, 1)$ with Neumann boundary conditions.

3 Preliminary results

In this section, we assume that *f is bounded.*

In this case, it is easily seen, using Girsanov's theorem exactly as for finite dimensional Itô equations, that $Eq(f)$ has a unique weak solution. The details can be found in the companion paper [3].

We now establish an estimate which will play an essential role below.

Proposition 3.1 *Let u denote the solution of $Eq(f)$. For any $T > 0$ and $\rho > 5/4$, there exists a constant $K(T, \rho)$ (which depends also on the uniform bound for f) such that for any $x \in [0, 1]$ and any Borel measurable function $g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,*

$$E \int_0^T |g(t, u(t, x))| dt \leqq K(T, \rho) \left(\int_0^T \int_{-\infty}^{\infty} |g(t, r)|^\rho dr dt \right)^{1/\rho} .$$

Proof. Define $\tilde{P}$ by

$$d\tilde{P} := Z dP$$

$$Z := \exp\left(- \int_0^T \int_0^1 f(u)(s, x) W(ds, dx) - \frac{1}{2} \int_0^T \int_0^1 f^2(u)(s, x) dx ds \right) .$$

Then $\tilde{P}$ is a probability measure and under $\tilde{P}$

$$\tilde{W}(dt, dx) := f(u)(t, x) dt dx + W(dt, dx)$$

is a space-time white noise. Thus u solves $Eq(0)$ driven by $\tilde{W}$ in place of W, and

$$u(t, x) = \int_0^1 G_t(x, y) u_0(y) dy + \int_0^t \int_0^1 G_{t-s}(x, y) \tilde{W}(ds, dy) .$$

Hence we see that under $\tilde{P}$ the solution u is a Gaussian random field with expectation

$$m(t, x) = \int_0^1 G_t(x, y) u_0(y) dy ,$$

and variance

$$\sigma^2(t, x) = \int_0^t \int_0^1 G_{t-s}^2(x, y) dy ds .$$

By (1) we have

$$\begin{aligned} \sigma^2(t, x) &\geqq \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_0^t 1/s \int_0^1 \left\{ \exp\left(- \frac{(y - x - 2n)^2}{2s} \right) \right. \\ &\quad \left. + \exp\left(- \frac{(y + x - 2n)^2}{2s} \right) \right\} dy ds \\ &= \frac{1}{2\pi} \int_0^t 1/s \left(\sum_n \int_{-x-2n}^{-x-2n+1} \exp\left(- \frac{z^2}{2s} \right) dz \right. \\ &\quad \left. + \sum_n \int_{-x-2n}^{-x-2n+1} \exp\left(- \frac{z^2}{2s} \right) dz \right) ds \\ &\geqq \frac{1}{\pi} \int_0^t 1/s \int_{-\infty}^{\infty} \exp\left(- \frac{z^2}{2s} \right) dz ds = \frac{1}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{s}} ds = \sqrt{\frac{2}{\pi}} \sqrt{t} . \quad (2) \end{aligned}$$

By Hölder's inequality

$$E \int_0^T |g(t, u(t,x))| dt \leqq K_T (\tilde{E} Z^{-\alpha})^{1/\alpha} \left(\tilde{E} \int_0^T |g(t, u(t,x))|^\beta dt \right)^{1/\beta}$$

$$\leqq K_{\alpha, T} \left(\int_0^T \int_{-\infty}^{\infty} |g(t, r)|^{\beta\gamma} dt dr \right)^{1/\beta\gamma} \left(\int_0^T \int_{-\infty}^{\infty} |p_{t,x}(r)|^\delta dr dt \right)^{1/\delta\beta} \tag{3}$$

for every $\alpha, \beta, \gamma, \delta \in (1, \infty)$ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1,$$

where $p_{t,x}(r)$ is the density function of the distribution of $u(t,x)$, $\tilde{E}$ is the expectation with respect to $\tilde{P}$, and K_T, $K_{T,\alpha}$ are constants.

By (2) we get

$$\int_0^T \int_{-\infty}^{\infty} |p_{t,x}(r)|^\delta dr dt = \frac{1}{(2\pi)^{\delta/2}} \int_0^T \frac{1}{|\sigma(t,x)|^\delta} \int \exp\left(-\frac{\delta(m(t,x) - r)^2}{2\sigma^2(t,x)} \right) dr dt$$

$$= \frac{1}{(2\pi)^{\delta/2}} \frac{1}{\sqrt{\delta}} \int_0^T \frac{1}{|\sigma(t,x)|^{\delta-1}} dt \leqq K \int_0^T \frac{1}{t^{\delta/4 - 1/4}} dt$$

which is finite if $\delta < 5$, i.e., when $\gamma > \frac{5}{4}$. Therefore, taking β close to 1 and $\gamma > \frac{5}{4}$ we get our estimate from (3). □

We now exploit the above estimate and establish a technical result which will be crucial for the convergence proof of our approximation procedure.

Let $\{f_n(t,x,r); t \geqq 0, 0 \leqq x \leqq 1, r \in \mathbb{R}\}$ and $\{h_n(t,r); t \geqq 0, r \in \mathbb{R}\}$, $n = 1, 2 \ldots$ be $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}([0,1] \otimes \mathscr{B}_1$ (resp. $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}([0,1])$) measurable functions satisfying

(A) f_n is bounded uniformly in n.

(B) Suppose there exists an $\mathscr{F}_t$-adapted solution u_n to $Eq(f_n)$ such that for every every (t,x)

$$\lim_{n \to \infty} u_n(t,x) = u(t,x)$$

for almost every $\omega \in \Omega$, where u is some random field with values in $\mathbb{R}$.

(C) h_n is bounded, uniformly in n, and the set $\{h_n; n \in \mathbb{N}\}$ is relatively compact in $L^2([0,T] \times [-R, R])$ for any $T, R > 0$.

We first prove:

Proposition 3.2 *Assume* (A) *and* (B). *Then the result of Proposition* 3.1 *applies to* $u := \lim_n u_n$.

Proof. From Proposition 3.1 and (A), there exists a constant $K(T,\rho)$, which is independent of n, such that:

$$E \int_0^T |g(t, u_n(t,x))| dt \leqq K(T,\rho) \left(\int_0^T \int_{-\infty}^{+\infty} |g(t,r)|^\rho dr dt \right)^{1/\rho}.$$

Let g be continuous, bounded and non negative. We can then take the limit in the above, yielding:

$$E\int_0^T g(t,u(t,x))dt \leqq K(T,\rho)\left(\int_0^T\int_{-\infty}^{\infty} |g(t,r)|^\rho dr\,dt\right)^{1/\rho}.$$

By the monotone class theorem the last inequality holds for any non negative and Borel measurable g. The result follows. □

Proposition 3.3 *Assume* (A)–(C). *Then*

$$\lim_{n\to\infty}\sup_k E\int_0^T |h_k(t,u_n(t,x)) - h_k(t,u(t,x))|\,dt = 0$$

for every $x \in [0,1]$ *and* $T \geqq 0$.

Proof. Let $\kappa:\mathbb{R}\to\mathbb{R}$ be a smooth function such that $0 \leqq \kappa(z) \leqq 1$ for every z, $\kappa(z) = 0$ for $|z| \geqq 1$ and $\kappa(0) = 1$. Let us fix $x \in \mathbb{R}$ and $T > 0$. For a given $\varepsilon > 0$, let $R > 0$ be such that

$$E\int_0^T |1 - \kappa(u(t,x)/R)|dt < \varepsilon .$$

We can find finitely many bounded smooth functions $H_1, \ldots, H_N$, such that for every k

$$\left(\int_0^T\int_{-R}^{R} |h_k(t,r) - H_i(t,r)|^2 dr\,dt\right)^{1/2} < \varepsilon$$

for some H_i. Obviously

$$I(n,k) := E\int_0^T |h_k(u_n) - h_k(u)|dt \leqq I_1(n,k) + I_2(n) + I_3(k)\,,$$

where

$$I_1(n,k) := E\int_0^T |h_k(u_n) - H_i(u_n)|dt\,,$$

$$I_2(n) := \sum_{j=1}^N E\int_0^T |H_j(u_n) - H_j(u)|dt\,,$$

$$I_3(k) := E\int_0^T |h_k(u) - H_i(u)|dt\,.$$

(For simplicity of writing we omit the variables t, x of the integrands). It is clear that $\lim_{n+\infty} I_2(n) = 0$. By Proposition 3.1

$$\begin{aligned} I_1(n,k) &= E\int_0^T \kappa(u_n/R)|h_k(u_n) - H_i(u_n)|dt \\ &\quad + E\int_0^T |1-\kappa(u_n/R)|\,|h_k(u_n) - H_i(u_n)|dt \\ &\leqq K\left(\int_0^T\int_{-R}^{R} |h_k(t,r) - H_i(t,r)|^2 dr\,dt\right)^{1/2} + KE\int_0^T |1-\kappa(u_n/R)|dt\,, \end{aligned}$$

where K is a constant. Hence

$$\lim_{n\to\infty}\sup_k I_1(n,k) \leqq K\varepsilon + KE\int_0^T |1-\kappa(u/R)|dt \leqq 2K\varepsilon\,.$$

Similarly, using Proposition 3.2 instead of Proposition 3.1

$$\sup_k I_3(k) \leqq 2L\varepsilon\,,$$

where L is a constant. Consequently,

$$\lim_{n\to\infty}\sup_k I(n,k) \leqq 2(K+L)\varepsilon\,,$$

and the proof is complete, since $\varepsilon > 0$ can be chosen arbitrarily small.

Corollary 3.4 *Assume* (A) *and* (B) *and suppose that for* $n \to \infty$

$$h_n \to h \text{ in } L_2([0,T]\times[-R,R])$$

for every $T \geqq 0$, $R \geqq 0$, *and* h_n *is bounded uniformly in* n. *Then*

$$\lim_{n\to\infty} E\int_0^T |h_n(t,u_n(t,x)) - h(t,u(t,x))|dt = 0$$

for every $x \in [0,1]$ *and* $T \geqq 0$.

Proof. Obviously

$$J(n) := E\int_0^T |h_n(u_n) - h(u)|dt \leqq J_1(n) + J_2(n)\,,$$

where

$$J_1(n) := \sup_k E\int_0^T |h_k(u_n) - h_k(u)|dt$$

$$J_2(n) := E\int_0^T |h_n(u) - h(u)|dt.$$

By Proposition 3.3 we have $\lim_{n\to\infty} J_1(n) = 0$. By Proposition 3.2

$$J_2(n) = E\int_0^T \kappa(u/R)|h_n(u) - h(u)|dt + E\int_0^T |1-\kappa(u/R)|\,|h_n(u)-h(u)|dt$$

$$\leqq K\left(\int_0^T\int_{-R}^R |h_n(t,r) - h(t,r)|^2 dr\,dt\right)^{1/2} + KE\int_0^T |1-\kappa(u/R)|dt\,,$$

where K is a constant. Letting here first $n \to \infty$ then $R \to \infty$ we get $\lim_{n\to\infty} J_2(n) = 0$, and the proof is complete. □

Corollary 3.5 *Let* $f_n = f_n(t,x,r)$ *be* $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{B}([0,1])\otimes\mathscr{B}_1$ *measurable functions which are bounded uniformly in* n *and converge to a measurable function* f *for almost all* t, x, r. *Assume that* $Eq(f_n)$ *admits a* $\mathscr{P}\otimes\mathscr{B}([0,1])$ *measurable solution* u_n *such that for every* t, x

$$u_n(t,x) \to u(t,x)$$

for almost every $\omega \in \Omega$, where u is some random field. Then u is a solution of $Eq(f)$.

Proof. If suffices to show that for $\varphi \in C^\infty[0, 1]$

$$\int_0^t \int_0^1 f_n(u_n)(s, x)\varphi(x)dx\,ds \to \int_0^t \int_0^1 f(u)(s, x)\varphi(x)dx\,ds$$

in probability as $n \to \infty$ for every $t \geqq 0$. Clearly

$$E\left|\int_0^t \int_0^1 (f_n(u_n) - f(u)(s, x))\varphi(x)dx\,ds\right|$$

$$\leqq \int_0^1 |\varphi(x)| E \int_0^t |f_n(u_n)(s, x) - f(u)(s, x)|\,ds\,dx \to 0$$

by Corollary 3.4. □

We will construct converging sequences of solutions by using the following known result on comparison of solutions (see Buckdahn and Pardoux [1]). For more general comparison results for SPDEs with non constant diffusion coefficients, see e.g. Kotelenez [5] and Donati-Martin and Pardoux [2].

Proposition 3.6 *Let $f = (f(t, x, r))$ and $F = (F(t, x, r))$ be $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}([0, 1]) \otimes \mathscr{B}_1$ measurable and bounded functions. Assume that one of them is Lipschitz in r, uniformly in (t, x). Let u and v be solutions to $Eq(f)$ and $Eq(F)$ respectively. Assume that for every r*

$$f(t, x, r) \leqq F(t, x, r)$$

for $dt \times dx$ almost every t, x. Then almost surely

$$u(t, x) \leqq v(t, x)$$

for all t, x.

4 Existence and uniqueness with bounded drift

In this section, we assume that

$$f = \mathbb{R}_+ \times [0, 1] \times \mathbb{R} \to \mathbb{R}$$

is $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}[0, 1] \otimes \mathscr{B}_1$ measurable and *bounded.*

Theorem 4.1 *There exists a continuous and $\mathscr{P} \otimes \mathscr{B}([0, 1])$ measurable solution u of $Eq(f)$.*

Proof. Let ρ be a smooth non negative function with compact support in $\mathbb{R}$ s.t. $\int_{\mathbb{R}} \rho(z)dz = 1$. For $j \in \mathbb{N}$, we define:

$$f_j(t, x, r) = j \int_{\mathbb{R}} f(t, x, z)\rho(j(r - z))dz\ .$$

Moreover, let

$$\tilde{f}_{n,k} \triangleq \bigwedge_{j=n}^{k} f_j,\ n \leqq k$$

$$F_n \triangleq \bigwedge_{j=n}^{\infty} f_j\ .$$

Clearly, $\tilde{f}_{n,k}$ is Lipschitz in r, uniformly with respect to (t, x), and:

$$\tilde{f}_{n,k} \downarrow F_n, \quad \text{as } k \to \infty$$

$$F_n \uparrow f, \quad \text{as } n \to \infty$$

dr a.e., for any (t, x).

$Eq(\tilde{f}_{nk})$ has a unique solution (see e.g. Walsh [11]) $\tilde{u}_{nk}$. From Proposition 3.6, the sequence $\{\tilde{u}_{nk}\}$ decreases with k, hence it has a limit

$$u_n \triangleq \lim_{k \to \infty} \tilde{u}_{nk} .$$

Note that again by Proposition 3.6 $\tilde{u}_{nk}$ – and hence u_n – is bounded from above (resp. from below) by the solution of the equation with f replaced by $\|f\|_\infty$ (resp. $-\|f\|_\infty$). From Corollary 3.4, u_n solves $Eq(F_n)$. Moreover, u_n increases as n increases, since

$$\tilde{u}_{nk} \geqq \tilde{u}_{mk}, \quad n \leqq m \leqq k .$$

So again u_n converges, and

$$u \triangleq \lim_{n \uparrow \infty} u_n$$

solves $Eq(f)$ by Corollary 3.5. □

Theorem 4.2 *$Eq(f)$ has at most one continuous and $\mathscr{P} \otimes \mathscr{B}([0, 1])$ measurable solution.*

Proof. Let u denote the solution constructed in Theorem 4.1, and v denote another solution.

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(v) + \frac{\partial^2 W}{\partial t \partial x} .$$

Let $\varphi \triangleq f(v)$. Then φ is a bounded $\mathscr{P} \otimes \mathscr{B}$ ($[0, 1]$)-measurable random field. F_n being defined as in the proof of Theorem 4.1, $F_n(v) \leqq f(v)$, hence v solves also (n is fixed below):

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + F_n(v) \vee \varphi + \frac{\partial^2 W}{\partial t \partial x} .$$

Also $F_n(\cdot) \vee \varphi$ is random, from $|a \vee \varphi - b \vee \varphi| \leqq |a - b|$ it is easily seen that the arguments used in the proof of Theorem 4.1 and Proposition 3.6 permit to construct a solution $\tilde{u}$ to this last equation, s.t. $v \leqq \tilde{u}$. From Girsanov's theorem, v and $\tilde{u}$ have the same law. Hence $v = \tilde{u}$ a.s. Hence $v_k \to v$, where v_k is the unique solution of:

$$\frac{\partial v_k}{\partial t} = \frac{\partial^2 v_k}{\partial x^2} + \tilde{f}_{nk}(v_k) \vee \varphi + \frac{\partial^2 W}{\partial t \partial x} .$$

Again by Proposition 3.6,

$$v_k(t, x) \geqq \tilde{u}_{n,k}(t, x) \text{ a.s.}$$

Hence

$$v(t, x) \geqq u_n(t, x) \quad \text{a.s. } \forall n \in \mathbb{N}$$

and

$$v(t,x) \geqq u(t,x) \quad \text{a.s.}$$

Finally from Girsanov's theorem the laws u and v coincide, and consequently

$$v(t,x) = u(t,x);\ t \geqq 0,\ x \in [0,1], \text{ a.s.} \quad \square$$

From the above results and the procedure used in the existence theorem, we deduce the following extension of Proposition 3.6:

Theorem 4.3 *Let f and g be two $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}([0,1]) \otimes \mathscr{B}_1$-measurable coefficients which are bounded on $[0,T] \times [0,1] \times [-R,R]$, for any $T, R > 0$, and satisfy:*

$$f(t,x,r) \leqq g(t,x,r)\,,$$

$dt \times dx \times dr$ a.e.

Let u(resp. v) denote a continuous and $\mathscr{P} \otimes \mathscr{B}([0,1])$-measurable solution of $Eq(f)$ [resp. $Eq(g)$]. Then:

$$u(t,x) \leqq v(t,x);\ t \geqq 0, x \in [0,1];\ \text{a.s.}$$

Proof. In the case f and g are bounded, the result follows from Proposition 3.6 and the way the solutions of $Eq(f)$ and $Eq(g)$ are constructed in Theorem 4.1.

In the general case, let $T, R > 0$, and $f_{T,R}$(resp. $g_{T,R}$) be $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}([0,1]) \otimes \mathscr{B}_1$ measurable and bounded, and coincide with f (resp. with g) on $[0,T] \times [0,1] \times [-R,R]$. The unique solution u_R (resp. v_R) of $Eq(f_{T,R})$ (resp. of $Eq(g_{T,R})$) coïncides with u (resp. v) on

$$[0, \tau_R \wedge T] \times [0,1]$$

where $\tau_R = \inf\{t; \sup_x |u(t,x)| \vee |v(t,x)| \geqq R\}$. Hence from the result in the case of bounded coefficients,

$$u \leqq v \text{ on } [0, \tau_R \wedge T] \times [0,1]\,.$$

But $\tau_R \wedge T \to \infty$ a.s., as R and $T \to \infty$. $\square$

5 Existence and uniqueness in the general case

In this section, we finally assume that f is $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}([0,1]) \otimes \mathscr{B}_1$ measurable, locally bounded and satisfies a one sided linear growth condition, i.e.

(i) For any $R > 0$, $\exists c_R$ s.t. $|f(t,x,r)| \leqq c_R;\ t \geqq 0, 0 \leqq x \leqq 1,\ -R \leqq r \leqq R$

(ii) $rf(t,x,r) \leqq c(1+r^2)$

for a constant c independent of (t,x,r).

We first prove uniqueness:

Theorem 5.1 *There is at most one continuous and $\mathscr{P} \otimes \mathscr{B}([0,1])$ measurable solution of $Eq(f)$.*

Proof. Let u and v be two such solutions,

$$R > 0,\ \tau_R = \inf\left\{t;\ \sup_{0 \leqq x \leqq 1} |u(t,x)| \vee |v(t,x)| \geqq R\right\},$$

$f_R(t,x,r) = f(t,x,(r \wedge R) \vee (-R))$. The restrictions of u and v to $[0,\tau_R]\times[0,1]$ are restrictions of the unique solution of $Eq(f_R)$, so they coincide. The result follows, since $\tau_R \to +\infty$ a.s. as $R \to +\infty$. □

We finally prove existence:

Theorem 5.2 *There exists a continuous and* $\mathscr{P}\otimes\mathscr{B}([0,1])$ *measurable solution u of* $Eq(f)$.

Proof. First step. We suppose that f has linear growth in r, i.e. $\exists c$ s.t. $|f((t,x,r)| \leqq c(1+|r|)$.

For any R, let f_R be defined as in the previous proof, and let u_R be the unique solution of $Eq(f_R)$. We define $\tau_R = \inf\{t; \sup_x |u_R(t,x)| \geqq R\}$. Then u_R solves $Eq(f)$ on the random interval $[0,\tau_R]$, and it is easily seen that by this procedure we construct a solution u of $Eq(f)$ on the random interval $[0,\tau[$, with $\tau = \lim_{R\uparrow\infty}\tau_R$. It remains to show that $\tau = +\infty$. This follows from the fact that $\bar{u} = u - v$ (v being the solution of $Eq(0)$) is continuous on $([0,\tau]\cap\mathbb{R})\times[0,1]$. Indeed,

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial x^2} + f(\bar{u}+v) \quad \text{on } [0,\tau[$$

and satisfies the Neumann boundary condition. For almost ω, $(t,x)\to v(t,x,\omega)$ is continuous, and hence standard estimates on the equation for $\bar{u}$ show that $\bar{u}(\cdot,\cdot,\omega)$ is continuous on $([0,\tau(\omega)]\cap[0,T])\times[0,1]$. Hence $\tau > T$ a.s. for any $T>0$, which establishes our result.

Second step. Suppose f is bounded from below by a function with linear growth, i.e. $\exists c$ s.t.

$$f(t,x,r) \geqq -c(1+|r|)\,.$$

It follows from the assumption of the theorem that

$$f(t,x,r) \leqq F(r)$$

where F is continuous has at most linear growth as $r\to+\infty$, and arbitrary growth as $r\to-\infty$. For any $N>0$, let $C_N = \sup_{-N\leqq r\leqq 0}(F^+(r))$,

$$F_N(r) = \begin{cases} F(r) \vee C_N, & r \geqq 0 \\ F(r) \wedge C_N, & r < 0 \end{cases}$$

and $f_N(t,x,r) = F_N(r) \wedge f(t,x,r)$. Clearly, f_N has at most linear growth, f_N increases with N, and

$$f_N \equiv f \text{ on } \Omega\times\mathbb{R}_+\times[0,1]\times[-N,+\infty[\,.$$

From the previous step, $Eq(f_N)$ has a unique solution u_N. From Proposition 3.6 u_N increases, hence u_N solves $Eq(f)$ on $\Omega_N^T\times[0,T]\times[0,1]$, where

$$\Omega_N^T = \{u_0(t,x) \geqq -N; 0\leqq t\leqq T, 0\leqq x\leqq 1\},$$

But $\bigcup_n \Omega_N^T = \Omega$ a.s. Hence we have constructed a solution up to time T, for any $T>0$.

Third step. The general case.

We construct an approximating decreasing sequence of functions which are bounded from below by linear functions of r, in a way which is very similar to what we have done in the second step, and apply the result of that step. □

6 The case of Dirichlet boundary conditions

We have proved a strong existence and uniqueness result of solutions of SPDEs driven by addive white noise in the case of a measurable drift satisfying some boundedness assumption. We have treated the case of Neumann boundary conditions only. However, the result extends to other types of boundary conditions. The only place in the paper where we have used explicitly the precise form of the kernel $G_t(x, y)$ associated with the Neumann boundary conditions is the derivation of the estimate (2). Clearly, the same estimate holds if we choose periodic boundary conditions (i.e. we identify 0 and 1, and replace $[0, 1]$ by the circle S^1). In the case of Dirichlet boundary conditions, the estimate (2) cannot possibly hold at the boundary ($x = 0$ or 1). However, the following is still true:

Lemma 6.1 *Let* $\{G_t(x, y); t \geqq 0; x, y \in [0, 1]\}$ *denote the fundamental solution of the heat equation with Dirichlet boundary conditions. For any* $x \in (0, 1)$, $T > 0$, *there exists* $c(T, x) > 0$ s.t.

$$\sigma^2(t, x) = \int_0^t \int_0^1 G_s^2(x, y)dyds \geqq c(T, x)\sqrt{t}, \quad 0 \leqq t \leqq T.$$

Proof. Let $p_t(x, y) = \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{(x-y)^2}{4t} \right]$ denote the transition probability of the one-dimensional Brownian motion (multiplied by $\sqrt{2}$) and τ denote the first exit time from (0, 1) of Brownian motion $\{\sqrt{2}B_t\}$. Then

$$G_s(x, y) = P_x(\tau > s/\sqrt{2}B_s = y)p_s(x, y).$$

Let $0 < a < x < b < 1$. Clearly,

$$\bar{c}(T, x) \triangleq \inf_{\substack{0 \leqq s \leqq T \\ a \leqq y \leqq b}} P_x(\tau > s/\sqrt{2}B_s = y) > 0 .$$

Hence, for $0 \leqq t \leqq T$,

$$\int_0^t \int_0^1 G_s^2(x, y)dsdy \geqq \bar{c}(T, x) \int_0^t \int_a^b p_s^2(x, y)dyds$$

$$\geqq \frac{\bar{c}(T, x)}{4\pi} \int_0^t \frac{1}{s} \int_0^h \exp\left(\frac{y^2}{2s}\right) dyds$$

$$\geqq \frac{\bar{c}(T, x)}{4\pi} \int_0^t \frac{1}{\sqrt{s}} \int_0^{h/\sqrt{2T}} \exp(-z^2)dzds$$

$$= c(T, x)\sqrt{t}. \quad \square$$

This allows us to prove Proposition 3.1 up to Corollary 3.4 at any point $x \in (0, 1)$, and that is enough in order to deduce Corollary 3.5. Hence the results of the paper are still true in the case of Dirichlet boundary conditions (in that case we of course use an initial condition $\{u_0(x)\}$ which vanishes at 0 and 1).

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