Λ-lookdown model with selection

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Abstract

The goal of this paper is to study the lookdown model with selection in the case of a population containing two types of individuals, with a reproduction model which is dual to the Λ-coalescent. In particular we formulate the infinite population "Λ-lookdown model with selection". When the measure Λ gives no mass to 0, we show that the proportion of one of the two types converges, as the population size N tends to infinity, towards the solution of a stochastic differential equation driven by a Poisson point process. We show that one of the two types fixates in finite time if and only if the Λ-coalescent comes down from infinity. We also consider the general case of a combination of the Kingman and the Λ-lookdown model.

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Keywords Look-down with selection, Lambda coalescent, Fixation and non fixation.

1 Introduction

In this paper we consider the lookdown (which is in fact usually called the "modified lookdown") model with selection when we replace the usual reproduction model by a population model dual to the Λ-coalescent. We first recall the models from [15] and [9], and then we will describe the variant which will be the subject of the present paper.

Pitman [15] and Sagitov [16] have pointed at an important class of exchangeable coalescents whose laws can be characterized by an arbitrary finite measure Λ on [0, 1]. Specifically, a Λ-coalescent is a Markov process $(\Pi_t, t \geq 0)$ on $\mathcal{P}_\infty$ (the set of partition of $\mathbf{N}$) started from the partition $0_\infty := \{1, 2, \dots\}$ and such that, for each integer $n \geq 2$, its restriction $(\Pi_t^{[n]}, t \geq 0)$ to $\mathcal{P}_n$ (the set of the partition of $\{1, 2, \dots, n\}$) is a continuous time Markov chain that evolves by coalescence events, and whose evolution can be described as follows.
Consider the rates

$$\lambda_{k,\ell} = \int_0^1 p^{\ell-2}(1-p)^{k-\ell}\Lambda(dp), \quad 2 \leq \ell \leq k. \tag{1.1}$$

Starting from a partition in $\mathcal{P}_n$ with k non-empty blocks, for each $\ell = 2, \dots, k$, every possible merging of ℓ blocks (the other $k-\ell$ blocks remaining unchanged) occurs at rate $\lambda_{k,\ell}$, and no other transition is possible. This description of the restricted processes Π^n determines the law of the Λ-coalescent Π.
Note that if $\Lambda(\{0\}) = \Lambda([0,1]) > 0$, then only pairwise merging occur, and the corresponding Λ-coalescent is just a time rescaling (by $\Lambda(0)$) of the Kingman coalescent. When $\Lambda(\{0\}) = 0$ which we will assume except in the very last section of this paper, a realization of the Λ-coalescent can be constructed (as in [15]) using a Poisson point process

$$m = \sum_{i=1}^{\infty} \delta_{t_i,p_i} \tag{1.2}$$

on $\mathbb{R}_+ \times (0,1]$ with intensity measure $dt \otimes \nu(dp)$ where $\nu(dp) = p^{-2}\Lambda(dp)$. The measure $\nu(dp)$ may have infinite total mass. Each atom (t,p) of p influences the evolution as follows :

- for each block of $\Pi(t^-)$ run an independent Bernoulli (p) random variable;
- all the blocks for which the Bernoulli outcome equals to 1 merge immediately into one single block, while all the other blocks remain unchanged.

In order to obtain a construction for a general measure Λ, one can superimpose onto the Λ-coalescent independent pairwise merges at rate $\Lambda(\{0\})$.

The lookdown construction was first introduced by Donnely and Kurtz in 1996 [9]. Their goal was to give a construction of the Fleming-Viot superprocess that provides an explicit description of the genealogy of the individuals in a population. Donnelly and Kurtz subsequently modified their construction in [10] to include more general measure-valued processes. Those authors extended their construction to the selective and recombination case [11].

We are going to present our model which we call Λ-lookdown model with selection. An important feature of our model is that we will describe it for a population of infinite size, thus retaining the great power of the lookdown cosntruction. As far as we know, this has not yet been done in the case of models with selection except in our previous publication [3], where we considered a model dual to Kingman's coalescent.

We consider the case of two alleles b and B, where B has a selective advantage over b. This selective advantage is modelled by a death rate α for the type b individuals. We will consider the proportion of b individuals. The type b individuals are coded by 1, and the type B individuals by 0. We assume that the individuals are placed at time 0 on levels $1, 2, \dots$, each one being, independently from the others, 1 with probability x, 0 with probability $1-x$, for some $0 < x < 1$. For each $i \geq 1$ and $t \geq 0$, let $\eta_t(i) \in \{0,1\}$ denote the type of the individual sitting on level i at time t. The evolution of $(\eta_t(i))_{i \geq 1}$ is governed by the two following mechanisms.

1. *Births* Each atom (t,p) of the Poisson point process m corresponds to a birth event. To each $(t,p) \in m$, we associate a sequence of i.i.d Bernoulli random variables $(Z_i, i \geq 1)$ with parameter p. Let

$$I_{t,p} = \{i \geq 1 : Z_i = 1\}.$$

and

$$\ell_{t,p} = \inf\{i \in I_{t,p} : i > \min I_{t,p}\}$$

At time t, those levels with Z_i=1 and $i \geq \ell_{t,p}$ modify their label to $\eta_{t^-}(\min I_{t,p})$. In other words, each level in $I_{t,p}$ immediately adopts the type of the smallest level participating in this birth event. For the remaining levels reassign the types so that their relative order immediately prior to this birth event is preserved. More precisely

$$\eta_t(i) = \begin{cases} \eta_{t^-}(i), & \text{if } i < \ell_{t,p} \\ \eta_{t^-}(\min I_{t,p}), & \text{if } i \in I_{t,p} \setminus \{\min I_{t,p}\} \\ \eta_{t^-}(i - (\#\{I_{t,p} \cap [1,\ldots,i]\} - 1)), & \text{otherwise} \end{cases}$$

We refer to the set $I_{t,p}$ as a multi-arrow at time t, originating from min $I_{t,p}$, and with tips at all other points of $I_{t,p}$. This procedure is usually referred to as the modified lookdown construction of Donnelly and Kurtz. In the original construction, the types of the levels in the complement of $I_{t,p}$ remained unchanged at time t, hence the types $\eta_{t^-}(i)$, for $i \in I_{t,p} \setminus \{\min I_{t,p}\}$ got erased from the population at time t.

2. *Deaths* Any type 1 individual dies at rate α, his vacant level being occupied by his right neighbor, who himself is replaced by his right neighbor, etc. In other words, independently of the above arrows, crosses are placed on all levels according to mutually independent rate α Poisson processes. Suppose there is a cross at level i at time t. If $\eta_{t^-}(i) = 0$, nothing happens. If $\eta_{t^-}(i) = 1$, then

$$\eta_t(k) = \begin{cases} \eta_{t^-}(k), & \text{if } k < i; \\ \eta_{t^-}(k+1), & \text{if } k \geq i. \end{cases}$$

We refer the reader to Figure 1 for a pictural representation of our model. Note that the type of the newborn individuals are found by "looking down", while the type of the individual who replaces a dead is found by looking up. So maybe our model could be called "look-down, look-up".

Since we have modelled selection by death events, the evolution of the N first individuals $\eta_t(1), \ldots, \eta_t(N)$ depends upon the next ones, and $X_t^N = N^{-1}(\eta_t(1) + \ldots \eta_t(N))$ is not a Markov process. We will show however that for each $t > 0$ the collection of r.v.'s $\{\eta_t(k),\ k \geq 1\}$ is well defined (which is not obvious in our setup) and constitutes an exchangeable sequence of $\{0,1\}$–valued random variables. We can then apply de Finetti's theorem, and prove that $X_t^N \to X_t$ a.s for any fixed $t \geq 0$, and in probability locally uniformly in t, where X_t is a $[0,1]$–valued Markov process, solution of the stochastic differential equation

$$X_t = x - \alpha \int_0^t X_s(1 - X_s)ds + \int_{[0,t]\times]0,1[^2} p(\mathbf{1}_{u \leq X_{s^-}} - X_{s^-})\bar{M}(ds,du,dp), \qquad (1.3)$$

where $\bar{M}(ds,du,dp) = M(ds,du,dp) - p^{-2}dsdu\Lambda(dp)$, and M is a Poisson point measure on $\mathbb{R}_+\times]0,1[\times]0,1]$ with intensity $dsdup^{-2}\Lambda(dp)$. X_t represents the proportion of

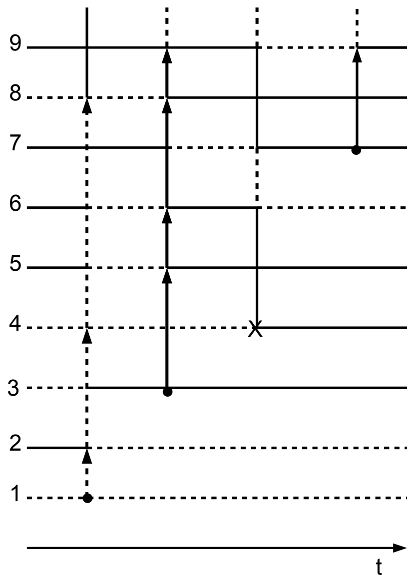

Figure 1: The graphical representation of the Λ-lookdown model with selection of size $N = 9$. Solid lines represent type B individuals, while dotted lines represent type b individuals.

type b individuals at time t in the infinite size population. Note that uniqueness of a solution to (1.3) is proved in [8].

The paper is organized as follows. We both construct our process, and establish the crucial exchangeability property satisfied by the Λ-lookdown model with selection in section 2. In section 3 we establish the convergence of X^N to the solution of (1.3). In section 4 we show that one of the two types fixates in finite time if and only if the Λ-coalescent comes down from infinity. Moreover, in the case of no fixation, we show that $X_t \to X_\infty \in \{0, 1\}$ as $t \to \infty$, and give both a condition ensuring that $X_\infty = 0$ $a.s$ and a condition ensuring that $\mathbf{P}(X_\infty = 1) > 0$. Finally, we extend our results to the case $\Lambda(\{0\}) > 0$ in the last section 5.

In this paper, we use $\mathbf{N}$ to denote the set of positive integers $\{1, 2, \ldots\}$, and $[n]$ to denote the set $\{1, \ldots, n\}$. We suppose that the measure Λ fulfills the condition

$$0 < \Lambda((0, 1)) < \infty, \quad \Lambda(\{1\}) = 0, \tag{1.4}$$

and in all the paper except section 5, we assume that $\Lambda(\{0\}) = 0$.

2 The lookdown process, exchangeability

2.1 Some results for general Λ

Throughout the paper, the notation

$$\mu_r := \int_{[0,1]} p^r \Lambda(dp)$$

is used for the rth moment of the finite measure Λ on [0, 1] for arbitrary real r. Note that μ_r is a decreasing function of r with $\infty > \mu_0 \geq \mu_r > 0$ for $r \geq 0$, while μ_r may be either finite or infinite for $r < 0$. For $r = 0, 1, \cdots$ observe from (1.1) that $\mu_r = \lambda_{r+2,r+2}$ is the rate at which Π_n jumps to its absorbing state $\{[n]\}$ from any state with $r + 2$ blocks. Let X denote a random variable with distribution $\mu_0^{-1}\Lambda$, defined on some background probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with expectation operator $\mathbf{E}$, so $\mathbf{E}(X^r) = \mu_r/\mu_0$. From (1.1), the transition rates of the Λ-coalescent are

$$\lambda_{k,\ell} = \mu_0 \mathbf{E}(X^{\ell-2}(1-X)^{k-\ell}) \text{ for all } 2 \leq \ell \leq k.$$

For any partition with a finite number $n \geq 2$ of blocks, the total rate of transitions of all kinds in a Λ-coalescent equals

$$\begin{aligned} \lambda_n := \sum_{\ell=2}^{n} \binom{n}{\ell} \lambda_{n,\ell} &= \int_0^1 \frac{1-(1-p)^n - np(1-p)^{n-1}}{p^2} \Lambda(dp) \\ &= \mu_0 \mathbf{E}\left[\frac{1-(1-X)^n - nX(1-X)^{n-1}}{X^2}\right]. \end{aligned}$$

By monotone convergence,

$$\lambda_n \uparrow \mu_{-2} = \int_{[0,1]} p^{-2} \Lambda(dp) \text{ as } n \uparrow \infty.$$

2.2 Construction of our process

In this section, we will construct the process $\{\eta_t(i), i \geq 1, t \geq 0\}$ corresponding to a given initial condition $(\eta_0(i), i \geq 1)$ defined in the Introduction.

Recall the Poisson point process m defined in (1.2). For each $n \geq 1$ and $t \geq 0$, let

$$I(n,t) = \{k \geq 1 : t_k \in [0,t] \text{ and } \#\{I_{t_k,p_k} \cap [n]\} \geq 2\}.$$

We have

Lemma 2.1. *For each $n \geq 1$ and $t \geq 0$,*

$$\#I(n,t) < \infty \quad a.s$$

Proof : Each atom (t,p) of m affects the n first individuals with probability

$$1-(1-p)^n - np(1-p)^{n-1} \leq \binom{n}{2} p^2.$$

Consequently

$$\mathbf{E}(\#I(n,t)) \leq \binom{n}{2} t \int_0^1 \Lambda(dp) < \infty.$$

The result follows. □

2.2.1 Λ-lookdown model without selection

For each $N \geq 1$, one can define the vector $\xi_t^N = (\xi_t^N(1), \ldots, \xi_t^N(N)), t \geq 0$ with values in $\{0,1\}^N$, by

1. $\xi_0^N(i) := \eta_0(i)$ for all $i \geq 1$.
2. At any birth event $(t,p) \in m$ and such that $\{I_{t,p} \cap [N]\} \geq 2$, for each $i \in [N]$, $\xi_t^N(i)$ evolves as follows

$$\xi_t^N(i) = \begin{cases} \xi_{t-}^N(i), & \text{if } i < \ell_{t,p} \\ \xi_{t-}^N(\min I_{t,p}), & \text{if } i \in I_{t,p} \setminus \{\min I_{t,p}\} \\ \xi_{t-}^N(i - (\#\{I_{t,p} \cap [1,\ldots,i]\} - 1)), & \text{otherwise.} \end{cases}$$

Using the above lemma, we see that the process ξ_t^N has finitely many jump on $[0,t]$ for all $t > 0$, hence its evolution is well defined. From this definition, one can easily deduce that, the evolution of the type at level i depends only upon the types at levels up to i. Consequently, if $1 \leq N < M$, the restriction of ξ^M to the first N levels yields ξ^N, in other words :

$$\{\xi_t^M(1), \ldots, \xi_t^M(N), t \geq 0\} \equiv \{\xi_t^N(1), \ldots, \xi_t^N(N), t \geq 0\}.$$

Hence, the process $\eta = \xi^\infty$ is easily defined by a projective limit argument as a $\{0,1\}^\infty$-valued process.

2.2.2 Λ-lookdown model with selection

This section is devoted to the construction of the infinite population lookdown model with selection.

For each $M \geq 1$, we consider the process $(\eta_t^M(i), i \geq 1, t \geq 0)$ obtained by applying all the arrows between $1 \leq i < j < \infty$, and only the crosses on levels 1 to M. Using the fact that we have a finite number of crosses on any finite time interval, it is not hard to see that the process $(\eta_t^M, t \geq 0)$ is well defined by applying the model without selection between two consecutive crosses, and applying the recipe described in the Introduction at a death time. More generally, our model is well defined if we suppress all the crosses above a curve which is bounded on any time interval $[0,T]$. Note also that, if we remove or modify the arrows and or the crosses above the evolution curve of a type B individual, this does not affect her evolution as well as that of those sitting below her.

At any time $t \geq 0$, let K_t denote the lowest level occupied by a B individual. Of course, if $K_0 = 1, K_t = 1$, for all $t \geq 0$. If for any T, $\sup_{0 \leq t \leq T} K_t < \infty$ a.s, then the process $\{K_t, t \geq 0\}$ is well defined by taking into account only those crosses below the curve K_t, and evolves as follows. When in state $n > 1$, K_t jumps to

1. $n + k$ at rate $\binom{n+k-1}{k+1}\lambda_{n+k,k+1}, \quad k \geq 1;$
2. $n - 1$ at rate $\alpha(n-1)$, $\alpha > 0$.

In other words, the infinitesimal generator of the Markov process $\{K_t, t \geq 0\}$ is given by:

$$\mathcal{L}g(n) = \sum_{k=1}^{\infty} \binom{n+k-1}{k+1} \lambda_{n+k-1,k+1}[g(n+k) - g(n)] + \alpha(n-1)[g(n-1) - g(n)]. \quad (2.1)$$

Now, we are going to show that the process $\{\eta_t(i), i \geq 1, t \geq 0\}$ is well defined. For this, we study two cases.

Case 1: $K_t \to \infty, t \to \infty$.

For each $N \geq 1, t \geq 0$, we define

$$K_t^N = \text{the level of the } N-\text{ th individual of type } B \text{ at time } t.$$

and

$$T_\infty^N = \inf\{t \geq 0 : K_t^N = \infty\}.$$

We have $T_\infty^1 \geq T_\infty^2 \geq \cdots > 0$. For each $N \geq 1$, we define

$$\mathcal{H}_N = \{(s,k); k \leq K_s^N\}.$$

Consider first the event

$$A = \{T_\infty^N = \infty \;, \forall\; N \geq 1\}^1.$$

Recall the Poisson point measure m defined in (1.2). Now, for each $N \geq 1$, we define the process$(\eta_t^N(i), i \geq 1, t \geq 0)$, with values in $\{0,1\}^\infty$, by

1. $\eta_0^N(i) := \eta_0(i)$ for all $i \geq 1$.
2. At any birth event $(t,p) \in m$, η_t^N evolves as follows

$$\eta_t^N(i) = \begin{cases} \eta_{t^-}^N(i), & \text{if } \; i < \ell_{t,p} \\ \eta_{t^-}^N(\min I_{t,p}), & \text{if } \; i \in I_{t,p} \setminus \{\min I_{t,p}\} \\ \eta_{t^-}^N(i - (\#\{I_{t,p} \cap [1,\ldots,i]\} - 1)), & \text{otherwise,} \end{cases}$$

3. Suppose there is a cross on level j at time s. If $(s,j) \notin \mathcal{H}_N$ or $(s,j) \in \mathcal{H}_N$ and $\eta_{s^-}(j) = 0$, nothing happens. If $(s,j) \in \mathcal{H}_N$ and $\eta_{s^-}(j) = 1$, then

$$\eta_s^N(i) = \begin{cases} \eta_{s^-}^N(i), & \text{if } \; i < j; \\ \eta_{s^-}^N(i+1), & \text{if } \; i \geq j. \end{cases}$$

In other words, the process $\{\eta_t^N(i), i \geq 1, t \geq 0\}$ is obtained by applying all the arrows between $1 \leq i < j < \infty$, and only the crosses on levels 1 to K_t^N. On the event A, we have a finite number of crosses on any finite time interval, and $(\eta_t^N(i), i \geq 1, t \geq 0)$ is constructed as explained above. Now, let

$$\mathcal{H} = \cup_N \mathcal{H}_N.$$

[1] We shall see below that $\mathbf{P}(A) = 1$ if the Λ-coalescent does not come down from infinity and $\mathbf{P}(A) = 0$ otherwise.

By a projective limit argument, we can easily deduce that the process $\{\eta_t(i), i \geq 1, t \geq 0\}$ is well defined on the set $\mathcal{H}$. Our model is defined on the event A.

Now we consider the event A^c. We first work on the event $\{T^1_\infty < \infty\}$. This means that the allele b fixates in finite time. It implies that for each $N \geq 2, T^N_\infty$ is finite as well. Consider first the process $\{\eta^1_t(i), i \geq 1, t \geq 0\}$ defined on $\mathcal{H}_1$, i.e we take into account all the arrows between $1 \leq i < j \leq K^1_t$, and only the crosses on levels 1 to K^1_t. This process is well defined on the time interval $[0, T^1_\infty)$. However, on the interval $[T^1_\infty, \infty), \eta^1_t(i) = 1, \forall i \geq 1$, hence the process is well defined in $\mathcal{H}_1$. We next consider the process $\{\eta^2_t(i), i \geq 1, t \geq 0\}$ defined on $\mathcal{H}_2$. This process is well defined on the time interval $[0, T^2_\infty)$. But on the interval $[T^2_\infty, \infty)$, there is at most one B, whose position is completely specified from the previous step. Iterating that procedure, and using again a projective limit argument, we define the full Λ-lookdown model with selection.

If $T^1_\infty = +\infty$, but $T^N_\infty < +\infty$ for some N, the construction is easily adapted to that case. In fact some arguments in section 4 below show that this cannot happen with positive probability.

Case 2 : $K_t \nrightarrow \infty, t \to \infty$.

Let

$$T_1 = \inf\{t \geq 0 : K_t = 1\}.$$

It is not hard to deduce from the strong Markov property of the process K_t that $\{T_1 < \infty\}$ a.s on the set $\{K_t \nrightarrow \infty, t \to \infty\}$. In that case the idea is to show that there exists an increasing mapping $\psi : \mathbf{N} \to \mathbf{N}$ such that a.s. for N large enough, any individual sitting on level $\psi(N)$ at any time never visits a level below N, with the convention that if that individual dies, we replace him by his neighbor below. Once this is true, the evolution of the individuals sitting on levels $1, 2, \ldots, N$ is not affected by deleting the crosses above level $\psi(N)$. Hence it is well defined. If this holds for all N large enough, the whole model is well defined.

Let

$$M = \sup_{0 \leq t < T_1} K_t.$$

For each $N \geq M$, let $\varphi(N) = Ne^{\alpha S_N}(Ne^{\alpha S_N} + 1) + K_0$, where S_N is defined below. We want to show that an individual sitting on level $\varphi(N) + N$ at any time $t \geq 0$ never visits a level below N. In order to prove this, we couple our model with the following one.

On the interval $[0, T_1]$, we erase all the arrows pointing to levels above K_t, and pretend that all individuals above level K_t, $0 \leq t \leq T_1$, are of type b, i.e coded by 1, and we apply all the crosses above level K_t. This model is clearly well defined since until T_1 there is only one 0, all other sites being occupied by 1's. We next extend this model for $t > T_1$ as follows :

For each $t \geq T_1$, let $\bar{K}_t$ denote the lowest level occupied by a b individual. At time T_1, $\eta_{T_1}(1) = 0, \eta_{T_1}(i) = 1$, for all $i \geq 2$. At any time $t > T_1$, we shall have $\eta_t(i) = 0$ for $i < \bar{K}_t$, and $\eta_t(i) = 1$ for $i \geq \bar{K}_t$. Again all crosses are kept, and we keep only those arrows whose tip hits a level $j \leq \bar{K}_t$.

This model is well defined. For each $N \geq 1$, we define by S_N the first time where all the N first individuals of this model are of type B. We have

Lemma 2.2. *If $T_1 < \infty$, then for each $N \geq 1$,*

$$S_N < \infty \ a.s$$

PROOF : The result follows from $T_1 < \infty$ and the fact that the process of arrows from 1 to 2 is a Poisson process with rate $\Lambda((0,1))$.

□

Now, let $\{\xi_t^{\varphi(N)},\ t \geq 0\}$ denote the process which describes the position at time t of the individual sitting on level $\varphi(N)$ at time 0 in the present model.

The individual who sits on level $\varphi(N)$ at time 0 will remain below the level $\varphi(N)+N$ on the time interval $[0, S_N]$. If she does not visit any level below N before time S_N, she will never visit any level below N at any time, and moreover any individual who visits level $\varphi(N)+N$ before time S_N will remain above the individual who was sitting at level $\varphi(N)$ at time 0 until S_N, hence will never visit any level below N.

Since the "true" model has more arrows and less "active crosses" than the present model, if we show that in the present model a.s. there exists N such that the individual who starts from level $\varphi(N)$ at time 0 never visits a level below N, we will have that in the true model a.s. for N large enough the evolution within the box $(t,i) \in [0,\infty) \times \{1, 2, \ldots, N\}$ is not altered by removing all the crosses above $\varphi(N)+N$. A projective limiting argument allows us then to conclude that the full model is well defined.

The result will follow from the Borel-Cantelli and the next lemma.

Lemma 2.3. *If $T_1 < \infty$, then for each $N \geq M$,*

$$\widehat{\mathbf{P}}_N(\exists 0 < t \leq S_N \ \textit{such that} \ \xi_t^{\varphi(N)} \leq N) \leq \frac{2}{N^2},$$

where $\widehat{\mathbf{P}}_N[.] = \mathbf{P}(. \mid S_N)$

PROOF : It is clear from the definition of $\xi_t^{\varphi(N)}$ that there exists a death process $(D_t, t \geq 0)$, which is independent of $(K_t, t \geq 0)$ conditionally upon $D_0 = \varphi(N) - K_0$, and such that

$$\xi_t^{\varphi(N)} = \tilde{K}_t + D_t, \ \forall t \geq 0,$$

where

$$\tilde{K}_t = \begin{cases} K_t, & 0 \leq t \leq T_1; \\ \bar{K}_t - 1, & t > T_1. \end{cases}$$

On the other hand, we have

$$\{\inf_{0 \leq t \leq S_N} \xi_t^{\varphi(N)} > N\} \supset \{\inf_{0 \leq t \leq S_N} D_t > N\} \supset \{D_{S_N} > N\}.$$

All we need to prove is that

$$\widehat{\mathbf{P}}_N(D_{S_N} \leq N) \leq \frac{2}{N^2}.$$

The process $(D_t, t \geq 0)$ is a jump Markov death process which takes values in the space $\{0, 1, \ldots, \varphi(N) - K_0\}$. When in state n, D_t jumps to $n-1$ at rate αn. In other words the infinitesimal generator of $\{D_t, t \geq 0\}$ is given by

$$Qf(n) = \alpha n[f(n-1) - f(n)].$$

Let $f : \mathbf{N} \to \mathbb{R}$. the process $(M_t^f)_{t\geq 0}$ given by

$$M_t^f = f(D_t) - f(D_0) - \alpha \int_0^t D_s[f(D_s - 1) - f(D_s)]ds \tag{2.2}$$

is a martingale. Applying (2.2) with the particular choice $f(n) = n$, there exists a martingale $(M_t^1)_{t\geq 0}$ such that $M_0^1 = 0$ and

$$D_t = D_0 - \alpha \int_0^t D_s ds + M_t^1, \qquad t \geq 0. \tag{2.3}$$

We note that $\{M_t^1, t \geq 0\}$ is a martingale under $\widehat{\mathbf{P}}_N[.]$. This is due to the fact that the Poisson process of crosses above K_t is independent of K_t. We first deduce from (2.3) that $\widehat{\mathbf{E}}_N(D_s) = D_0 e^{-\alpha s}$.

Using the fact that D_t is a pure death process, we obtain the identity

$$[M^1]_t = D_0 - D_t,$$

which, together with (2.3), implies

$$< M^1 >_t = \alpha \int_0^t D_s ds.$$

From (2.3), it is easy to deduce that

$$D_t = e^{-\alpha t}(\varphi(N) - K_0) + \int_0^t e^{-\alpha(t-s)} dM_s^1,$$

which implies that

$$\begin{aligned}
\widehat{\mathbf{P}}_N(D_{S_N} \leq N) &\leq \widehat{\mathbf{P}}_N\Big(|\int_0^{S_N} e^{-\alpha(S_N - s)} dM_s^1| \geq N^2 e^{\alpha S_N}\Big) \\
&= \widehat{\mathbf{P}}_N\Big(|\int_0^{S_N} e^{\alpha s} dM_s^1| \geq N^2 e^{2\alpha S_N}\Big) \\
&\leq \frac{1}{N^4 e^{4\alpha S_N}} \int_0^{S_N} \alpha e^{2\alpha s} \widehat{\mathbf{E}}_N(D_s) ds \\
&\leq \frac{2}{N^2}.
\end{aligned}$$

The result is proved .

□

From now on, we equip the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the filtration defined by $\mathcal{F}_t = \sigma\{\eta_s(i),\ i \geq 1,\ 0 \leq s \leq t\}$. Any stopping time will be defined with respect to that filtration.

2.3 Exchangeability

In this subsection, we will show that the Λ-lookdown model with selection preserves the exchangeability property, by an argument similar to that which we developed in [3].

Let S_n denote the group of permutations of the set $\{1, 2, \dots, n\}$. For all $\pi \in S_n$ and $a^{[n]} = (a_i)_{1\le i\le n} \in \{0,1\}^n$, we define the vectors

$$\pi^{-1}(a^{[n]}) = (a_{\pi^{-1}(1)}, \dots, a_{\pi^{-1}(n)}) = (a_i^\pi)_{1\le i\le n}$$
$$\pi(\xi_t^{[n]}) = (\xi_t(\pi(1)), \dots, \xi_t(\pi(n)))$$

We should point out that $\pi(\xi_t^{[n]})$ is a permutation of $(\xi_t(1), \dots, \xi_t(n))$ and it is clear from the definitions that

$$\{\pi(\xi_t^{[n]}) = a^{[n]}\} = \{\xi_t^{[n]} = \pi^{-1}(a^{[n]})\}, \quad \text{for any} \quad \pi \in S_n. \tag{2.4}$$

The main result of this subsection is

Theorem 2.4. *If $(\eta_0(i))_{i\ge1}$ are exchangeable random variables, then for all $t > 0$, $(\eta_t(i))_{i\ge1}$ are exchangeable.*

We first establish two lemmas

Lemma 2.5. *For any finite stopping time τ, any $\mathbf{N}$–valued $\mathcal{F}_\tau$–measurable random variable n^*, if the random vector $\eta_\tau^{[n^*]} = (\eta_\tau(1), \dots, \eta_\tau(n^*))$ is exchangeable, and T is the first time after τ of an arrow pointing to a level $\le n^*$ or a death at a level $\le n^*$, then conditionally upon the fact that $T = t_{i_0}$, for some $i_0 \ge 1$ and $I_{t_{i_0},p_{i_0}} \cap [n^*] = k$, where $k \ge 2$, the random vector $\eta_{t_{i_0}}^{[n^*-1+k]} = \Big(\eta_{t_{i_0}}(1), \dots, \eta_{t_{i_0}}(n^*-2+k), \eta_{t_{i_0}}(n^*-1+k)\Big)$ is exchangeable.*

Proof : For the sake of simplifying the notations, we condition upon $n^* = n, t_{i_0} = t$, $p_{t_{i_0}} = p$ and $I_{t_{i_0},p_{i_0}} \cap [n^*] = k$. We start with some notation.

$$A_t^{j_0,\dots,j_{k-1}} := \{\text{the } k \text{ levels selected by the point } (t,p) \text{ between levels 1 and } n \text{ are } j_0, j_1, \dots, j_{k-1}\}.$$

We define

$$\widehat{\mathbf{P}}_{t,n}[.] = \mathbf{P}(.|t_{i_0} = t, n^* = n, I_{t,p} \cap [n] = k).$$

Thanks to (2.4), we deduce that, for $\pi \in S_{n-1+k}$, $a^{[n-1+k]} \in \{0,1\}^{n-1+k}$,

$$\begin{aligned}&\widehat{\mathbf{P}}_{t,n}(\pi(\eta_t^{[n-1+k]}) = a^{[n-1+k]})\\ &= \sum_{1\le j_0<j_1<\dots<j_{k-1}\le n} \widehat{\mathbf{P}}_{t,n}\Big(\{\eta_t^{[n-1+k]} = (a_1^\pi, \dots, a_{n-1+k}^\pi)\}, A_t^{j_0,\dots,j_{k-1}}\Big)\end{aligned} \tag{2.5}$$

On the event $A_t^{j_0,\dots,j_{k-1}}$, we have :

$$\eta_t(i) = \begin{cases} \eta_{t^-}(i), & \text{if } 1 \le i < j_1\\ \eta_{t^-}(j_0), & \text{if } i \in \{j_1, j_2, \dots, j_{k-1}\}\\ \eta_{t^-}(i - (\#\{\{j_1, j_2, \dots, j_{k-1}\} \cap [i]\})), & \text{if } j_1 < i \le n-1+k,\ i \notin \{j_2, \dots, j_{k-1}\}. \end{cases}$$

This implies that

$$A_t^{j_0,\dots,j_{k-1}} \cap \{\eta_t^{[n-1+k]} = (a_1^\pi, \dots, a_{n-1+k}^\pi)\} \subset \{a_{j_0}^\pi = a_{j_1}^\pi = a_{j_2}^\pi = \cdots = a_{j_{k-1}}^\pi\}.$$

For $1 < j_1 < j_2 < \cdots < j_{k-1} \le n$, define the mapping $\rho_{j_1,j_2,\dots,j_{k-1}} : \{0,1\}^{n+k-1} \longrightarrow \{0,1\}^n$ by :

$$\rho_{j_1,j_2,\dots,j_{k-1}}(b_1, \dots, b_{n-1+k}) = (B_{j_1}, \dots, B_{j_{k-1}}),$$

where

$$\begin{aligned} B_{j_1} &= (b_1, \dots, b_{j_1-1}), \\ B_{j_m} &= (b_{j_{m-1}+1}, b_{j_{m-1}+2}, \dots, b_{j_m-1}), \quad 2 \le m \le k-1 \\ B_{j_{k-1}} &= (b_{j_{k-1}+1}, b_{j_{k-1}+2}, \dots, b_{n-1+k}). \end{aligned}$$

In other words, $\rho_{j_1,j_2,\dots,j_{k-1}}(z)$ is the vector z from which the coordinates with indices $j_1, \dots, j_{k-1}$ have been suppressed. The right hand side of (2.5) is equal to

$$\sum_{1 \le j_0 < j_1 < \cdots < j_{k-1} \le n} \mathbf{1}_{\{a_{j_0}^\pi = a_{j_1}^\pi \cdots = a_{j_{k-1}}^\pi\}} \widehat{\mathbf{P}}_{t,n} \left(\{\eta_{t^-}^{[n]} = \rho_{j_1,j_2,\dots,j_{k-1}}(\pi^{-1}(a^{[n-1+k]}))\}, A_t^{j_0,\dots,j_{k-1}} \right),$$

It is easy to see that the events $(\eta_{t^-}^{[n]} = \rho_{j_1,j_2,\dots,j_{k-1}}(\pi^{-1}(a^{[n-1+k]})))$ and $A_t^{j_0,\dots,j_{k-1}}$ are independent. Thus

$$\begin{aligned} \widehat{\mathbf{P}}_{t,n}(\pi(\eta_t^{[n-1+k]}) = a^{[n-1+k]}) &= \sum_{1 \le j_0 < j_1 < \cdots < j_{k-1} \le n} \mathbf{1}_{\{a_{j_0}^\pi = a_{j_1}^\pi \cdots = a_{j_{k-1}}^\pi\}} \\ &\quad \times \widehat{\mathbf{P}}_{t,n} \left(\eta_{t^-}^{[n-1+k]} = \rho_{j_1,j_2,\dots,j_{k-1}}(\pi^{-1}(a^{[n-1+k]})) \right) \widehat{\mathbf{P}}_{t,n}(A_t^{j_0,\dots,j_{k-1}}) \\ &= \binom{n}{k}^{-1} \sum_{1 \le j_0 < j_1 < \cdots < j_{k-1} \le n} \mathbf{1}_{\{a_{j_0}^\pi = a_{j_1}^\pi \cdots = a_{j_{k-1}}^\pi\}} \\ &\quad \times \widehat{\mathbf{P}}_{t,n} \left(\eta_{t^-}^{[n-1+k]} = \rho_{j_1,j_2,\dots,j_{k-1}}(\pi^{-1}(a^{[n-1+k]})) \right). \end{aligned}$$

On the other hand, we have

$$\#\{1 \le j_0 < \cdots < j_{k-1} \le n : a_{j_0} = \cdots = a_{j_{k-1}}\} = \#\{1 \le j_0 < \cdots < j_{k-1} \le n : a_{j_0}^\pi \cdots = a_{j_{k-1}}^\pi\}$$

Let $\ell_0 < \ell_1 < \cdots < \ell_{k-1}$ be the increasing reordering of the set $\{\pi(j_0), \pi(j_1), \cdots, \pi(j_{k-1})\}$. If $a_{j_0} = a_{j_1} = \cdots = a_{j_{k-1}}$, then we have $a_{\ell_0}^\pi = a_{\ell_1}^\pi \cdots = a_{\ell_{k-1}}^\pi = a_{j_0} = a_{j_1} = \cdots = a_{j_{k-1}}$, and consequently $\rho_{j_1,j_2,\dots,j_{k-1}}(a^{[n-1+k]})$ and $\rho_{\ell_1,\ell_2,\dots,\ell_{k-1}}(\pi^{-1}(a^{[n-1+k]}))$ contain the same number of 0's and 1's. Since $\eta_{t^-}^{[n]}$ is exchangeable,

$$\begin{aligned}\widehat{\mathbf{P}}_{t,n}(\pi(\eta_t^{[n-1+k]}) = a^{[n]}) &= \binom{n}{k}^{-1} \sum_{\gamma\in\{0,1\}} \sum_{1\le \ell_0<\ell_1<\cdots<\ell_{k-1}\le n} \mathbf{1}_{\{a^\pi_{\ell_0}=a^\pi_{\ell_1}\cdots=a^\pi_{\ell_{k-1}}=\gamma\}} \\ &\qquad \times \widehat{\mathbf{P}}_{t,n}\left(\eta_{t^-}^{[n]} = \rho_{\ell_1,\ell_2,\dots,\ell_{k-1}}(\pi^{-1}(a^{[n-1+k]}))\right) \\ &= \binom{n}{k}^{-1} \sum_{\gamma\in\{0,1\}} \sum_{1\le j_0<j_1<\cdots<j_{k-1}\le n} \mathbf{1}_{\{a_{j_0}=a_{j_1}\cdots=a_{j_{k-1}}=\gamma\}} \\ &\qquad \times \widehat{\mathbf{P}}_{t,n}\left(\eta_{t^-}^{[n]} = \rho_{j_1,j_2,\dots,j_{k-1}}(a^{[n-1+k]})\right) \\ &= \widehat{\mathbf{P}}_{t,n}(\eta_t^{[n-1+k]} = a^{[n-1+k]}).\end{aligned}$$

The result follows. □

Lemma 2.6. *For any finite stopping time τ, any $\mathbf{N}$–valued $\mathcal{F}_\tau$–measurable random variable n^*, if the random vector $\eta_\tau^{[n^*]} = (\eta_\tau(1), \dots, \eta_\tau(n^*))$ is exchangeable, and T is the first time after τ of an arrow pointing to a level $\le n^*$ or a death at a level $\le n^*$, then conditionally upon the fact that T is the time of a death, the random vector $\eta_T^{[n^*-1]} = (\eta_T(1), \dots, \eta_T(n^*-1))$ is exchangeable.*

Proof : To ease the notation we will condition upon $n^* = n$ and $T = t$. Let $\pi \in S_{n-1}$ be arbitrary. We consider the events :

$$B_t^i := \{\text{the level of the dying individual at time } t \text{ is } i\}.$$

Let $\widehat{\mathbf{P}}_{t,n}[.] = \mathbf{P}(.|T = t, n^* = n)$. We have

$$\begin{aligned}\widehat{\mathbf{P}}_{t,n}(\pi(\eta_t^{[n-1]}) = a^{[n-1]}) &= \sum_{1\le i\le n} \widehat{\mathbf{P}}_{t,n}\left(\eta_t^{[n-1]} = \pi^{-1}(a^{[n-1]}), B_t^i\right) \\ &= \sum_{1\le i\le n} \widehat{\mathbf{P}}\left(\eta_t(1) = a_1^\pi, \dots, \eta_t(n-1) = a^\pi_{n-1}, B_t^i\right).\end{aligned}$$

Define

$$c_i^{\pi,n} = (a_1^\pi, \dots, a_{i-1}^\pi, 1, a_i^\pi, \dots, a^\pi_{n-1}), \qquad c_i^n = (a_1, \dots, a_{i-1}, 1, a_i, \dots, a_{n-1}).$$

The last term in the previous relation is equal to

$$\begin{aligned}\sum_{1\le i\le n} \widehat{\mathbf{P}}_{t,n}\left(\eta_{t^-}^{[n]} = c_i^{\pi,n}, B_t^i\right) &= \sum_{1\le i\le n} \mathbf{P}\left(\eta_{t^-}^{[n]} = c_i^{\pi,n}\right) \widehat{\mathbf{P}}_{t,n}\left(B_t^i \mid \eta_{t^-}^{[n]} = c_i^{\pi,n}\right) \\ &= \frac{1}{1+\sum_{j=1}^{n-1} a_j^\pi} \sum_{1\le i\le n} \mathbf{P}\left(\eta_{t^-}^{[n]} = c_i^{\pi,n}\right).\end{aligned}$$

Thanks to the exchangeability of $(\eta_{t^-}(1), \dots, \eta_{t^-}(n))$, we have

$$\widehat{\mathbf{P}}_{t,n}(\pi(\eta_t^{[n-1]}) = a^{[n-1]}) = \frac{1}{1+\sum_{j=1}^{n-1} a_j} \sum_{1\le i\le n} \mathbf{P}\left(\eta_{t^-}^{[n]} = c_i^n\right),$$

since $\sum_{j=1}^{n-1} a_j^\pi = \sum_{j=1}^{[n-1]} a_j$ and $c_i^{\pi,n}$ is a permutation of c_i^π. The result follows. $\square$

We can now proceed with the
PROOF OF THEOREM 2.4 For each $N \geq 1$, let $\{V_t^N,\ t \geq 0\}$ denote the $\mathbf{N}$–valued process which describes the position at time t of the individual sitting on level N at time 0, with the convention that, if that individual dies, we replace him by his neighbor below. The construction of our process $\{\eta_t(i), i \geq 1, t \geq 0\}$ in section 2.2 shows that $\inf_{t\geq 0} V_t^N \to \infty$, as $N \to \infty$.

It follows from Lemma 2.5 and 2.6 that for each $t > 0$, $N \geq 1$, $(\eta_t(1), \dots, \eta_t(V_t^N))$ is an exchangeable random vector.

Consequently, for any $t > 0$, $n \geq 1$, $\pi \in S_n$, $a^{[n]} \in \{0, 1\}$,

$$|\mathbf{P}(\eta_t^{[n]} = a^{[n]}) - \mathbf{P}(\eta_t^{[n]} = \pi^{-1}(a^{[n]}))| \leq \mathbf{P}(V_t^N < n),$$

which goes to zero, as $N \to \infty$. The result follows.

$\square$

For each $N \geq 1$ and $t \geq 0$, denote by X_t^N the proportion of type b individuals at time t among the first N individuals, i.e.

$$X_t^N = \frac{1}{N} \sum_{i=1}^{N} \eta_t(i). \tag{2.6}$$

We are interested in the limit of $(X_t^N)_{t\geq 0}$ as N tends to infinity. For this, let us recall the following useful result due to de Finetti (see e. g. [2]). In this statement, $\mathcal{G}$ denotes the tail $\sigma-$field of the sequence $\{X_n, n \geq 1\}$.

Theorem 2.7. *An exchangeable (countably infinite) sequence $\{X_n, n \geq 1\}$ of random variables is a mixture of i.i.d. r.v.'s, in the sense that conditionally upon $\mathcal{G}$, the X_n's are i.i.d.*

As a consequence, we have the following asymptotic property for fixed t of the sequence $(X_t^N)_{N\geq 1}$ defined by (2.6)

Corollary 2.8. *For each $t \geq 0$,*

$$X_t = \lim_{N\to\infty} X_t^N \quad \textit{exist a.s.} \tag{2.7}$$

3 Tightness and Convergence to the Λ-W-F SDE with selection

3.1 Tightness of $\{X^N, N \geq 1, t \geq 0\}$

In this part, we will prove tightness of $(X^N)_{N\geq 1}$ in $D([0, \infty[)$, where for each $N \geq 1$ and $t \geq 0$, X_t^N is defined by (2.6). We start with some notations.

For any N, n, r, p such that $N \geq 1, Nr \in \mathbf{N}, r \in]0,1], p \in [0,1]$, we define $Y(\cdot, N, p)$ to be the binomial distribution function with parameter N and p; $H(\cdot, N, n, r)$ the hypergeometric distribution function with parameter $(N-1, n-1, \frac{Nr-1}{N-1})$; $\bar{H}(\cdot, N, n, r)$ the hypergeometric distribution function with parameter $(N-1, n-1, \frac{Nr}{N-1})$. For every $v, w \in [0,1]$, let

$$\begin{aligned} F_p^N(v) &= \inf\{s; Y(s, N, p) \geq v\}, \\ G_{N,n,r}(w) &= \inf\{s; H(s, N, n, r) \geq w\}, \\ \bar{G}_{N,n,r}(w) &= \inf\{s; \bar{H}(s, N, n, r) \geq w\}. \end{aligned}$$

It follows that if V, W are $\mathcal{U}([0,1])$ r.v.'s, then the law of $F_p^N(V)$ is binomial with parameter N, p. $G_{N,n,r}(W)$(resp $\bar{G}_{N,n,r}(W)$) is hypergeometric with parameters $N-1, n-1, \frac{Nr-1}{N-1}$ (resp $N-1, n-1, \frac{Nr}{N-1}$). Note that $F_p^N(\cdot) = Y^{-1}(\cdot, N, p), G_{N,n,r}(\cdot) = H^{-1}(\cdot, N, n, r)$ and $\bar{G}_{N,n,r}(\cdot) = \bar{H}^{-1}(\cdot, N, n, r)$. We recall that if X is hypergeometric with parameters (N, n, p) such that $Np \in \mathbf{N}$ and $p \in [0,1]$, then

$$\mathbf{E}(X) = np \quad \text{and} \quad Var(X) = \frac{N-n}{N-1} np(1-p).$$

Now, for every $r, u, p, v, w \in [0,1]$, let

$$\psi^N(r, u, p, v, w) = \frac{1}{N} \mathbf{1}_{F_p^N(v) \geq 2} \Big[\mathbf{1}_{u \leq r} \Big(F_p^N(v) - 1 - G_{N, F_p^N(v), r}(w) \Big) - \mathbf{1}_{u > r} \bar{G}_{N, F_p^N(v), r}(w) \Big].$$

From the identity $r(n-1-\mathbf{E}[G_{N,n,r}(w)]) = (1-r)\mathbf{E}[\bar{G}_{N,n,r}(w)]$, we deduce the

Lemma 3.1. *For each $N \geq 1, r, p, v \in [0,1]$ and $t \geq 0$,*

$$\int_{]0,1]^2} \psi^N(r, u, p, v, w) du dw = 0.$$

Using the definition of the model, it is easy to see that

$$X_t^N = X_0^N + \int_{[0,t] \times]0,1]^4} \psi^N(X_{s^-}^N, u, p, v, w) M_0(ds, du, dp, dv, dw)$$

$$- \frac{1}{N} \int_{[0,t] \times [0,1]^2} \mathbf{1}_{u \leq X_{s^-}^N} \mathbf{1}_{v \geq X_{s^-}^{N+1}} M_1^N(ds, du, dv)$$

where M_0 and M_1^N are two mutually independent Poisson point processes. M_0 is a Poisson point process on $\mathbb{R}_+ \times [0,1] \times [0,1] \times [0,1] \times [0,1]$ with intensity measure $\mu(ds, du, dp, dv, dw) = ds du p^{-2} \Lambda(dp) dv dw$, M_1^N is a Poisson point process on $\mathbb{R}_+ \times [0,1] \times [0,1]$ with intensity measure $\alpha N ds du dv$. Now, let

$$\bar{M_0} = M_0 - \mu, \quad \bar{M_1^N} = M_1^N - \alpha N \lambda. \tag{3.1}$$

Using Lemma 3.1, we have

$$X_t^N = X_0^N + \int_{[0,t]\times]0,1]^4} \psi^N(X_{s^-}^N, u, p, v, w)\bar{M}_0(ds, du, dp, dv, dw)$$
$$- \frac{1}{N}\int_{[0,t]\times[0,1]^2} \mathbf{1}_{u\leq X_{s^-}^N}\mathbf{1}_{v\geq X_{s^-}^{N+1}}\bar{M}_1^N(ds, du, dv) - \alpha\int_0^t X_s^N(1-X_s^{N+1})ds. \tag{3.2}$$

For each $N \geq 1, t \geq 0$, we define

$$\begin{aligned}
\mathcal{M}_t^N &= \int_{[0,t]\times]0,1]^4} \psi^N(X_{s^-}^N, u, p, v, w)\bar{M}_0(ds, du, dp, dv, dw)\\
\mathcal{N}_t^N &= \frac{1}{N}\int_{[0,t]\times]0,1]^2} \mathbf{1}_{u\leq X_{s^-}^N}\mathbf{1}_{v\geq X_{s^-}^{N+1}}\bar{M}_1^N(ds, du, dv)\\
V_t^N &= -\alpha\int_0^t X_s^N(1-X_s^{N+1})ds.
\end{aligned}$$

$\mathcal{M}_t^N$ and $\mathcal{N}_t^N$ are two orthogonal martingales. It is easy to see that

$$X_t^N = X_0^N + V_t^N + \mathcal{M}_t^N - \mathcal{N}_t^N.$$

$\forall N \geq 1, X_0^N \in [0,1]$, which implies that it is tight. Moreover, we have

Proposition 3.2. *The sequence $(X^N, N \geq 1)$ is tight in $D([0,\infty])$.*

We first establish the lemma :

Lemma 3.3. *For each $N \geq 1$ and $t \geq 0$,*

$$\begin{aligned}
\langle \mathcal{M}^N\rangle_t &= \Lambda((0,1))\int_0^t X_s^N(1-X_s^N)ds\\
\langle \mathcal{N}^N\rangle_t &= \frac{\alpha}{N}\int_0^t X_s^N(1-X_s^{N+1})ds
\end{aligned}$$

Proof : Using the fact that $\mathcal{M}^N$ and $\mathcal{N}^N$ are pure-jump martingales, we deduce that

$$\langle \mathcal{M}^N\rangle_t = \int_{[0,t]\times]0,1]^4} (\psi^N(X_s^N, u, p, v, w))^2 ds du p^{-2}\Lambda(dp) dv dw.$$

Let

$$\begin{aligned}
\mathcal{A}^N(X_s^N, p) &= \int_{]0,1]^3} (\psi^N(X_s^N, u, p, v, w))^2 du dv dw\\
&= \frac{1}{N^2}\int_{]0,1]^2} \mathbf{1}_{F_p^N(v)\geq 2}\Big[X_s^N\Big(F_p^N(v) - 1 - G_{N,F_p^N(v),X_{s^-}^N}(w)\Big)^2\\
&\qquad + (1-X_s^N)(\bar{G}_{N,F_p^N(v),X_s^N}(w))^2\Big] dv dw.
\end{aligned}$$

Tedious but standard calculations yield

$$\int_{[0,1]} \Big[r\Big(F_p^N(v) - 1 - G_{N,F_p^N(v),r}(w)\Big)^2 dw + (1-r)\Big(\bar{G}_{N,F_p^N(v),r}(w)\Big)^2 \Big] dw$$

$$= \frac{N}{N-1} r(1-r) F_p^N(v)(F_p^N(v) - 1),$$

for every $v, r \in [0,1]^2$. Consequently

$$\begin{aligned}\mathcal{A}^N(X_s^N, p) &= \frac{X_s^N(1-X_s^N)}{N(N-1)} \int_{[0,1]} \mathbf{1}_{F_p^N(v)\geq 2} F_p^N(v)(F_p^N(v)-1)dv \\ &= p^2 X_s^N(1-X_s^N).\end{aligned}$$

We deduce that

$$\begin{aligned}\langle \mathcal{M}^N \rangle_t &= \int_{[0,t]\times[0,1]} \mathcal{A}^N(X_s^N, p) ds p^{-2} \Lambda(dp) \\ &= \Lambda((0,1)) \int_{[0,t]} X_s^N(1-X_s^N) ds.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\langle \mathcal{N}^N \rangle_t &= \frac{\alpha}{N} \int_{[0,t]\times[0,1]\times[0,1]} \mathbf{1}_{u \leq X_{s-}^N} \mathbf{1}_{v \geq X_{s-}^{N+1}} ds du dv \\ &= \frac{\alpha}{N} \int_{[0,t]} X_s^N(1-X_s^{N+1}) ds.\end{aligned}$$

The lemma has been established. $\square$

We can now proceed with the
Proof of Proposition 3.2 Let

$$\begin{aligned}\varphi(x,y) &= -\alpha x(1-y) \\ \Psi_N(x,y) &= \Big[\Lambda((0,1))(1-x) + \frac{\alpha}{N}(1-y)\Big] x.\end{aligned}$$

It follows from lemma 3.3 that for each $t \geq 0$,

$$X_t^N = X_0 + \int_0^t \varphi(X_s^N, X_s^{N+1}) ds + \mathcal{M}_t^N - \mathcal{N}_t^N$$

and

$$\langle \mathcal{M}^N - \mathcal{N}^N \rangle_t = \langle \mathcal{M}^N \rangle_t + \langle \mathcal{N}^N \rangle_t = \int_0^t \Psi_N(X_s^N, X_s^{N+1}) ds.$$

Moreover, $\forall T \geq 0$

$$\sup_{0\leq t\leq T} \sup_{N\geq 1} \Big(\mid \varphi(X_t^N, X_t^{N+1}) \mid + \Psi_N(X_t^N, X_t^{N+1}) \Big) \leq C \quad a.s. \tag{3.3}$$

Aldous' tightness criterion (see Aldous [1]) is an easy consequence of (3.3). $\square$

Now, from Proposition 3.2 and (2.7), it is not hard to show there exists a process $X \in D([0,\infty))$, such that for all $t \geq 0$,

$$X_t^N \to X_t \quad a.s, \tag{3.4}$$

and

$$X^N \Rightarrow X \text{ weakly in } D([0,\infty)).$$

We deduce for the above facts

Theorem 3.4. *For all $T > 0$,*

$$\sup_{0 \leq t \leq T} |X_t^N - X_t| \to 0 \textit{ in probability, as } N \to \infty.$$

PROOF : To each $\delta > 0$, we associate $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n = T$, such that $\sup_{1 \leq i \leq n}(t_i - t_{i-1}) \leq \delta$. We have, with the notation $y \wedge z = \inf(y, z)$,

$$\begin{aligned}
\sup_{0 \leq t \leq T} |X_t^N - X_t| &\leq \sup_i \sup_{t_{i-1} \leq t \leq t_i} |X_t^N - X_{t_{i-1}}^N| \wedge |X_t^N - X_{t_i}^N| + \sup_i |X_{t_i}^N - X_{t_i}| \\
&\quad + \sup_i \sup_{t_{i-1} \leq t \leq t_i} |X_t - X_{t_i}| \wedge |X_t - X_{t_{i-1}}| \\
&\leq w_T''(X^N, \delta) + \sup_i |X_{t_i}^N - X_{t_i}| + w_T''(X, \delta),
\end{aligned}$$

where

$$w_T''(x, \delta) = \sup_{0 \leq t_1 < t < t_2 \leq T, t_2 - t_1 \leq \delta} |x(t) - x(t_1)| \wedge |x(t) - x(t_2)|.$$

Since from (3.4),

$$\sup_{1 \leq i \leq n} |X_{t_i}^N - X_{t_i}| \to 0 \quad \text{a. s., as } N \to \infty,$$

$$\limsup_{N \to \infty} \mathbf{P}\Big(\sup_{0 \leq t \leq T} |X_t^N - X_t| > \varepsilon\Big) \leq \limsup_{N \to \infty} \mathbf{P}\Big(w_T''(X^N, \delta) > \varepsilon/2\Big) + \mathbf{P}\Big(w_T''(X, \delta) > \varepsilon/2\Big).$$

From the proof of Theorem 13.5 in [7], we know that tightness of X^N implies that

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbf{P}(w_T''(X^N, \delta) > \varepsilon/2) = 0.$$

Since X is cadlag a.s., for each $\varepsilon > 0$,

$$\mathbf{P}(w_T''(X, \delta) > \varepsilon/2) \to 0, \quad \text{as } \delta \to 0.$$

The result follows. $\square$

3.2 Convergence to the Λ-Wright-Fisher SDE with selection

Our goal is to get a representation of the process X_t defined in (3.4) as the unique weak solution of the stochastic differential equation (1.3).

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a fixed probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Recall the Poisson point measure $M = \sum_{i=1}^{\infty} \delta_{t_i,u_i,p_i}$ defined in the Introduction, and for every $u \in]0,1[$ and $r \in [0,1]$, we introduce the elementary function

$$\Psi(u,r) = \mathbf{1}_{u\leq r} - r.$$

We rewrite equation (1.3) as

$$X_t = x - \alpha \int_0^t X_s(1-X_s)ds + \int_{[0,t]\times]0,1[^2} p\Psi(u, X_{s^-})\bar{M}(ds,du,dp), \ t > 0, 0 < x < 1, \tag{3.5}$$

which we call the Λ-Wright-Fisher SDE with selection. Without loss of generality, we shall assume that $\alpha > 0$, which means that X_t represents the proportion of non-advantageous alleles.

The proof of the following identity is standard and left to the reader.

Lemma 3.5. *For each $r \in [0,1]$,*

$$\int_{[0,1]^4} \left(\psi^N(r,u,p,v,w) - p\Psi(u,r)\right)^2 dup^{-2}\Lambda(dp)dvdw$$
$$= 2r(1-r)\Big[\frac{N}{N-1}\int_{[0,1]^2}(1-up)^{N-1}du\Lambda(dp) - \frac{\Lambda([0,1])}{N-1}\Big].$$

We are now in position to prove our main result.

Theorem 3.6. *Suppose that $X_0^N \to x$ a.s., as $N \to \infty$. Then the $[0,1]-$valued process $\{X_t, t \geq 0\}$ defined by (3.4) is the (unique in law) solution of the Λ-W-F SDE with selection (3.5).*

Proof : Uniqueness of the solution of (3.5) follows from Theorem 4.1 in [8]. We now prove that $(X_t)_{t\geq 0}$ defined by (3.4) is a solution of the Λ-Wright-Fisher (3.5).

Since $X_t^N \to X_t$ *a.s.*, from (3.1), it suffices to show that for each $t \geq 0$, as $N \to \infty$

$$V_t^N + \mathcal{M}_t^N - \mathcal{N}_t^N \to -\alpha \int_0^t X_s(1-X_s)ds + \int_{[0,t]\times[0,1]^2} p\Psi(X_{s^-}, u)\bar{M}(ds,du,dp)$$

in probability. Now using (3.4) and Lemma 3.3, it is easy to see that, for each $t \geq 0$

$$V_t^N \to -\alpha \int_0^t X_s(1-X_s)ds \text{ a.s, and } \mathcal{N}_t^N \to 0 \quad \text{in probability,}$$

as $N \to \infty$. The main work is to show that

$$\mathcal{M}_t^N \to \int_{[0,t]\times]0,1[\times]0,1[} p\Psi(X_{t^-}, u)\bar{M}(ds,du,dp) \text{ in probability,} \quad \textit{as } N \to \infty. \tag{3.6}$$

For each $N \geq 1$ and $t \geq 0$, let

$$h^N(t) = \int_{[0,t]\times[0,1]^4} \left(\psi^N(X^N_{s^-}, u, p, v, w) - p\Psi(X_{s^-}, u)\right) \bar{M}_0(ds, du, dp, dv, dw),$$

where $\bar{M}_0$ is defined by (3.1). $h^N(t)$ is a martingale, and

$$\langle h^N\rangle_t = \int_{[0,t]\times[0,1]^4} \left(\psi^N(X^N_s, u, p, v, w) - p\Psi(X_s, u)\right)^2 dsdup^{-2}\Lambda(dp)dvdw.$$

We have

$$\begin{aligned}\langle h^N\rangle_t &\leq 2\int_{[0,t]\times[0,1]^2} \left(p\Psi(u, X^N_s) - p\Psi(u, X_s)\right)^2 dsdup^{-2}\Lambda(dp)\\ &\quad + 2t\sup_{0\leq s\leq t}\int_{[0,1]^4} \left(\psi^N(X^N_s, u, p, v, w) - p\Psi(u, X^N_s)\right)^2 dup^{-2}\Lambda(dp)dvdw\\ &\leq 2\int_{[0,t]\times[0,1]^2} \left(p\Psi(u, X^N_s) - p\Psi(u, X_s)\right)^2 dsdup^{-2}\Lambda(dp)\\ &\quad + 2t\sup_{0\leq r\leq 1}\int_{[0,1]^4} \left(\psi^N(r, u, p, v, w) - p\Psi(u, r)\right)^2 dup^{-2}\Lambda(dp)dvdw.\end{aligned}$$

Using the fact that $X^N_s \to X_s$ a.s., it is not hard to show by the dominated convergence theorem that as $N \to \infty$,

$$\int_{[0,t]\times[0,1]^2} p^2\left(\Psi(u, X^N_s) - \Psi(u, X_s)\right)^2 dsdu\Lambda(dp) \to 0 \quad \text{a.s.} \tag{3.7}$$

Now from lemma 3.5 , it is easy to show that as $N \to \infty$,

$$\sup_{0\leq r\leq 1}\int_{[0,1]^4} \left(\psi^N(r, u, p, v, w) - p\Psi(u, r)\right)^2 dup^{-2}\Lambda(dp)dvdw \to 0 \tag{3.8}$$

Combining (3.7) and (3.8), we deduce that

$$\forall t \geq 0, \quad \langle h^N\rangle_t \to 0 \quad a.s, \quad as\ N \to \infty.$$

On the other hand, we have

$$\langle h^N\rangle_t \leq Ct\Lambda([0,1]),\ \forall N \geq 2.$$

We deduce from the dominated convergence theorem that

$$\lim_{N\to\infty} \mathbf{E}[h^N(t)^2] = 0 \quad \forall t \geq 0$$

i.e

$$\mathcal{M}^N_t = \int_{[0,t]\times[0,1]^4} \Psi^N(X^N_{s^-}, u, p, v, w)\bar{M}_0(ds, du, dp, dv, dw) \xrightarrow{L^2} \int_{[0,t]\times[0,1]^4} p\Psi(X_{s^-}, u)\bar{M}(ds, du, dp)$$

as $N \to \infty$, in particularly

$$\mathcal{M}^N_t \to \int_{[0,t]\times[0,1]^2} p\Psi(X_{s^-}, u)\bar{M}(ds, du, dp) \text{ in probability },as\ N \to \infty\ .$$

(3.6) is established.

$\square$

Remark 3.7. *Uniqueness in law could also by proved as in [4], (where the case $\alpha = 0$ is treated) by a duality argument, which we now sketch .*

Recall the notation $\Psi(u, y) = \mathbf{1}_{u \le y} - y$. For every $y \in [0, 1]$ and every function $g : [0, 1] \to \mathbb{R}$ of class $\mathcal{C}^2$, we set

$$\mathcal{L}g(y) = \int_{[0,1]\times[0,1]} \Big[g(y + p\Psi(u, y)) - g(y) - p\Psi(u, y)g'(y) \Big] p^{-2}\Lambda(dp)du - \alpha g'(y)(1-y)y.$$

A solution $(Y_t)_{t\ge 0}$ of (3.5) is a Markov process with generator $\mathcal{L}$. Hence for every $g : [0, 1] \to \mathbb{R}$ of class $\mathcal{C}^2$, the process

$$g(Y_t) - \int_0^t ds \mathcal{L}g(Y_s), \qquad t \ge 0$$

is a martingale.

It is plain that for $g(z) = z^n$

$$\mathcal{L}g(z) = \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k}(z^{n-k+1} - z^n) + \alpha n(z^{n+1} - z^n). \tag{3.9}$$

Let $\{R_t, t \ge 0\}$ be a $\mathbf{N}$-valued jump Markov process which, when in state k, jumps to

1. *$k - \ell + 1$ at rate $\binom{k}{\ell}\lambda_{k,\ell}$, $\quad 2 \le \ell \le k$;*
2. *$k + 1$ at rate αk, $\alpha > 0$.*

In other words, the infinitesimal generator of $\{R_t, t \ge 0\}$ is given by:

$$\mathcal{L}^* f(k) = \sum_{\ell=2}^{k} \binom{k}{\ell} \lambda_{k,\ell}[f(k - \ell + 1) - f(k)] + \alpha k[f(k + 1) - f(k)].$$

For every $z \in [0, 1]$ and every $r \in \mathbf{N}$, we set

$$P(z, r) = z^r. \tag{3.10}$$

Viewing $P(z, r)$ as a function of r, we have

$$\mathcal{L}^* P(z, r) = \sum_{k=2}^{r} \binom{r}{k} \lambda_{r,k}[z^{r-k+1} - z^r] + \alpha r[z^{r+1} - z^r]$$

On the other hand, viewing $P(z, r)$ as a function of z we can easily evaluate $\mathcal{L}P(z, r)$ from formula (3.9), and we deduce that

$$\mathcal{L}P(z, r) = \mathcal{L}^* P(z, r). \tag{3.11}$$

Now suppose that $(Y_t)_{t\ge 0}$ is a solution of (3.5), and let $R_0 = n$. By standard argument (see Section 4.4 in [12]) we deduce from (3.11) that

$$\mathbf{E}[P(Y_t, R_0)] = \mathbf{E}[P(Y_0, R_t)],$$

i.e

$$\mathbf{E}[Y_t^n|Y_0 = x] = \mathbf{E}[x^{R_t}|R_0 = n].$$

Since this is true for each $n \geq 1$ and Y_t take values in the compact set [0, 1], this is enough to identify the conditional law of Y_t, given that $Y_0 = x$, for all $0 \leq x \leq 1$. Since $(Y_t)_{t\geq 0}$ is a homogeneous Markov process, this implies that the law of $(Y_t)_{t\geq 0}$ is uniquely determined.

If we prove a priori that $(X_t)_{t\geq 0}$ defined by (3.4) is a Markov process, we can use the following Remark to prove that X_t is a weak sense solution of the Λ-Wright-Fisher SDE (3.5).

Remark 3.8. *Suppose we know that X_t defined by* (3.4) *is a Markov process. Let us look backwards from time t to time 0. For each $0 \leq s \leq t$, we denote by $Z_s^{n,t}$ the highest level occupied by the ancestors at time s of the n first individuals at time t. We know that conditionally upon X_t, the $\{\eta_t(i), i \geq 1\}$ are i.i.d Bernoulli with parameter X_t. Consequently, for any $n \geq 1$,*

$$X_t^n = \mathbf{P}(\eta_t(1) = \cdots = \eta_t(n) = 1 \mid X_t),$$

this implies that

$$\begin{aligned}
\mathbf{E}_x[X_t^n] &= \mathbf{E}_x[\mathbf{P}(\eta_t(1) = \cdots = \eta_t(n) = 1 \mid X_t)] \\
&= \mathbf{P}_x(\eta_t(1) = \cdots = \eta_t(n) = 1) \\
&= \mathbf{P}_x(\textit{the } 1 \ldots Z_0^{n,t} \textit{ individuals at time } 0 \textit{ are all } b) \\
&= \mathbf{E}_n[x^{Z_0^{n,t}}].
\end{aligned}$$

It is plain that the conditional law of $Z_0^{n,t}$, given that $(\eta_t(1) = \cdots = \eta_t(n) = 1)$ equal the conditional law of R_t, given that $R_0 = n$. Consequently, for each $n \geq 1$

$$\mathbf{E}[X_t^n \mid X_0 = x] = \mathbf{E}[Y_t^n \mid Y_0 = x],$$

where $(Y_t)_{t\geq 0}$ is a solution of (3.5)*. But for all $t > 0, r \in [0, 1]$, the conditional law of X_t, given that $X_0 = x$ is determined by its moments, since X_t is a bounded r. v. So X_t and Y_t have the same transition densities, that is $\{X_t, t \geq 0\}$ is the unique weak solution of* (3.5)*.*

4 Fixation and non-fixation in the Λ-W-F SDE

4.1 The CDI property of the Λ-coalescent

In this subsection, we recall a remarkable property of the Λ-coalescent $(\Pi_t)_{t\geq 0}$ defined in the introduction. For each $n \geq 1$, let $\#\Pi_t^{[n]}$ denote the number of blocks in the partition $\Pi_t^{[n]}$ ($\Pi_t^{[n]}$ is the restriction of Π_t to $[n]$). Then let $T_n = \inf\{t \geq 0 : \#\Pi_t^{[n]} = 1\}$. As stated in (31) of [15], we have

$$0 = T_1 < T_2 \leq T_3 \leq \ldots \uparrow T_\infty \leq \infty.$$

We say the Λ-coalescent comes down from infinity ($\Lambda \in \mathbf{CDI}$) if $\mathbf{P}(\#\Pi_t < \infty) = 1$ for all $t > 0$, and we say it stays infinite if $\mathbf{P}(\#\Pi_t = \infty) = 1$ for all $t > 0$. In terms of the population model, this means that for any $t > 0$, we can find a finite number of individuals in the initial population which generate the entire population at time t. The coalescent comes down from infinity if and only if $T_\infty < \infty$ a.s. We will show that this is equivalent to fixation. Kingman showed that the δ_0-coalescent comes down from infinity.

A necessary and sufficient condition for a Λ-coalescent to come down from infinity was given by Schweinsberg [17]: define

$$\phi(n) = \sum_{k=2}^{n} (k-1) \binom{n}{k} \lambda_{n,k},$$

and

$$\nu(dp) = p^{-2} \Lambda(dp).$$

It is not hard to deduce from the binomial formula

$$\phi(n) = \int_0^1 [np - 1 + (1-p)^n] \nu(dp).$$

Schweinsberg's result [17] says that the Λ-coalescent comes down from infinity if and only if

$$\sum_{n=2}^{\infty} \frac{1}{\phi(n)} < \infty. \tag{4.1}$$

The condition of convergence of this series is also necessary and sufficient for fixation in finite time. Using the fact that the function $f_n(p) = (1-p)^n - 1$ is decreasing for any fixed n, we have

$$\int_0^1 (np-1)\nu(dp) \leq \phi(n) \leq n \int_0^1 p\nu(dp), \qquad \forall n \geq 1.$$

The last assertion together with (4.1), implies that if $\int_0^1 p\nu(dp) < \infty$ then the Λ-coalescent stays infinite. The result has been proved by Pitman (see lemma 25 in [15]).

Theorem 3.6 shows that X_t is a bounded supermartingale. Indeed, if $(X_t)_{t\geq 0}$ is a solution of 3.5 , then for all $0 \leq t \leq s$,

$$\begin{aligned}\mathbf{E}(X_t \mid \mathcal{F}_s) &\leq x - \alpha \int_0^s X_r(1-X_r)dr + \mathbf{E}\Big[\int_{[0,t]\times]0,1[\times]0,1[} p\Psi(u, X_{s-})\bar{M}(ds,du,dp) \mid \mathcal{F}_s \Big] \\ &= X_s.\end{aligned}$$

Consequently the following limit exists a.s

$$X_\infty = \lim_{t\to\infty} X_t \in \{0,1\}. \tag{4.2}$$

Indeed, 0 and 1 are the only possible limit values.

4.2 Fixation and non-fixation in the Λ-W-F SDE

We assume that the initial proportion x of type B individuals satisfies $0 < x < 1$. In this section, we prove that if the condition (4.1) is satisfied, then we have fixation in our model in finite time. Before establishing the main result of this section, we collect some results which will be required for its proof

Lemma 4.1.

$$\frac{\phi(n)}{n} \uparrow \int_0^1 p\nu(dp) \ \ as \ \ n \uparrow \infty,$$

where $\nu(dp) = p^{-2}\Lambda(dp)$.

PROOF :

$$\begin{aligned}
\phi(n) &= \int_0^1 [np - 1 + (1-p)^n]\, \nu(dp) \\
&= \int_0^1 \left[\frac{n}{p}\left(1 - \int_0^1 (1-up)^{n-1}du\right)\right] \Lambda(dp).
\end{aligned}$$

On the last line, we have made use of the identity

$$(1-p)^n - 1 = \int_0^1 -np(1-up)^{n-1}du.$$

For each $p \in]0,1]$, let

$$f^n(p) = \frac{1}{p}\left(1 - \int_0^1 (1-up)^{n-1}du\right).$$

We have,

$$n^{-1}\phi(n) = \int_0^1 f^n(p)\Lambda(dp).$$

The result follows from the monotone convergence theorem.

□

We now deduce that

Lemma 4.2. *The function ϕ increases, and*

$$\sum_{n=2}^{\infty} \frac{1}{\phi(n)} < \infty \quad \Rightarrow \quad \sum_{n=2}^{\infty} \frac{1}{\phi(n) - \alpha n} < \infty.$$

PROOF : We have

$$\begin{aligned}
\phi(n+1) - \phi(n) &= \int_0^1 [p + (1-p)^{n+1} - (1-p)^n]\nu(dp) \\
&= \int_0^1 p(1-(1-p)^n)\nu(dp) \\
&\geq 0.
\end{aligned}$$

Which implies the first claim. Now, we already know that if $\sum_{n=2}^{\infty}\frac{1}{\phi(n)}<\infty$, then $\int_0^1 p\nu(dp)=\infty$. Thus, the second assertion is a consequence of the last lemma and the following relation

$$\sum_{n=2}^{\infty}\frac{1}{\phi(n)-\alpha n}=\sum_{n=2}^{\infty}\frac{\alpha}{\phi(n)(n^{-1}\phi(n)-\alpha)}+\sum_{n=2}^{\infty}\frac{1}{\phi(n)}.$$

The lemma is proved.

□

For each $t\geq 0$, we define again

$$K_t=\inf\{i\geq 1:\eta_t(i)=0\}.$$

and

$$T_1=\inf\{t\geq 0:K_t=1\}.$$

We have the following

Theorem 4.3. *If $\Lambda\in$ **CDI**, then one of the two types (b or B) fixates in finite time, i.e.*

$$\exists\,\zeta<\infty\ a.s: X_\zeta=X_\infty\in\{0,1\}$$

*If $\Lambda\notin$ **CDI**, then*

$$\forall t\geq 0,\ 0<X_t<1\ a.s.$$

Proof : The proof has been inspired by [6] (see Section 4).

Step 1 : Suppose that $\Lambda\in$ **CDI**. We consider two cases.

Case 1 : $K_0=1$.

In this case, the allele B fixates in the population. Indeed, the individual at level 1 never dies and, he cannot be pushed to an upper level. Let

$$\zeta=\inf\{t>0:\eta_t(i)=0,\forall i\geq 1\}.$$

ζ is the time of fixation of allele B. We are going to show that $\zeta<\infty$ $a.s$.

Let $N>1$ denote a fixed integer, and we define S_N the first time when the N first individuals are all of type B. As $S_N<\infty$ $a.s$, for all $N\geq 1$, we wait until all the N first individuals are of B type and we look backward in time, and let $\bar{Z}_t^N$ denote the number of ancestors at time t of the N first individuals at time S_N (i.e of the individuals sitting on levels $1,\dots,N$ at time S_N). The process $(\bar{Z}_t^N)_{t\geq 0}$ is a jump Markov process with state-space $\{1,2,\dots,N\}$. When $\bar{Z}_t^N=n$, $\bar{Z}_t^N$ is shifted to $n-\ell+1$, where $2\leq\ell\leq n$, at rate $\binom{n}{\ell}\lambda_{n,\ell}$. In other words, the infinitesimal generator Q^N of $\bar{Z}^N$ is given by

$$Q^N f(n)=\sum_{\ell=2}^{n}\binom{n}{\ell}\lambda_{n,\ell}[f(n-\ell+1)-f(n)].$$

Let

$$\zeta_N=\inf\{t\geq 0:Z_t^N=1\}.$$

ζ_N is also an upper bound for the time taken in the forward time direction, for the progeny of a type B individual sitting on level 1, to invade all the levels $\{1, 2, \ldots, N\}$. The sequence $(\zeta_N)_{N\geq 1}$ increases and it is easy to see that $\lim_{N\to\infty} \zeta_N = \zeta$, where ζ is the time of fixation of the type B. We already know that since $\Lambda \in \mathbf{CDI}$,

$$\sum_{n=2}^{\infty} \frac{1}{\phi(n)} < \infty. \tag{4.3}$$

Now, for each $n \geq 1$, we define

$$f(n) = \sum_{k=n+1}^{\infty} \frac{1}{\phi(k)}.$$

We have for $2 \leq \ell \leq n$

$$f(n-\ell+1) - f(n) = \sum_{k=n-\ell+2}^{n} \frac{1}{\phi(k)}.$$

Recall lemma 4.2. Since $1/\phi$ is decreasing, we have for $2 \leq \ell \leq n$,

$$f(n-\ell+1) - f(n) \geq (\ell-1)\frac{1}{\phi(n)},$$

and therefore

$$Q^N f(n) \geq \frac{1}{\phi(n)} \sum_{\ell=2}^{n} \binom{n}{\ell} (\ell-1)\lambda_{n,\ell} = 1.$$

Using the fact that the process

$$f(\bar{Z}_t^N) - f(N) - \int_0^t Q^N f(\bar{Z}_s^N) ds, \qquad t \geq 0$$

is a martingale, we obtain

$$\begin{aligned}
\mathbf{E}(\zeta_N) &\leq \mathbf{E}\left(\int_0^{\zeta_N} Q^N f(\zeta_s^N) ds\right) \\
&= f(1) - f(N) \\
&\leq f(1)
\end{aligned}$$

The last assertion, together with the monotone convergence theorem, implies that

$$\mathbf{E}(\zeta) \leq f(1) < \infty.$$

Case 2 : $K_0 > 1$.
If $T_1 < \infty$ then B fixates in finite time. Indeed, wait until T_1 which is a stopping time

at which the Markov process $\{\eta_t(i), i \geq 1\}_{t\geq 0}$ starts afresh, and then use the argument from Case 1.

We suppose now that $T_1 = \infty$, which implies that $K_t \to \infty, t \to \infty$, as already noted in section 2.2.2. In other word, if $T_1 = \infty$, then the allele B does not fixate in the population. Let

$$n_0 = \inf\{n \geq 1 : \phi(n) - \alpha n \geq 1\}.$$

Such an n_0 exists because since $\Lambda \in$ **CDI**, $\int_0^1 p\nu(dp) = +\infty$, hence by Lemma 4.1, we have $\lim_{n\to\infty} n^{-1}\phi(n) = +\infty$. Let $N \geq n_0$ denote a fixed integer. We let

$$S_N = \inf\{t \geq 0 : \eta_t(1) = \cdots = \eta_t(N) = 1\}$$

Since $K_t \to \infty, t \to \infty$, we have that $S_N < \infty$ *a.s.* Now, we are going to use a similar argument as in **Case 1** . We wait until the N first individuals are all of b type and we look backward in time, and we denote by Z_t^N the highest level occupied by the ancestors at time t of the N first individuals at time S_N. The process $(Z_t^N)_{t\geq 0}$ is a jump Markov process with state-space $\{1, 2, \ldots, \infty\}$. When $Z_t^N = n$, Z_t^N jumps to $n - \ell + 1$, where $2 \leq \ell \leq n$, at rate $\binom{n}{\ell}\lambda_{n,\ell}$, and jumps to $n+1$ at rate αn. The infinitesimal generator Q^N of Z^N is given by

$$Q^N f(n) = \sum_{\ell=2}^{n} \binom{n}{\ell} \lambda_{n,\ell}[f(n-\ell+1) - f(n)] + \alpha n(f(n+1) - f(n)).$$

Let

$$\zeta_N^{n_0} = \inf\{t \geq 0 : Z_t^N \leq n_0\}.$$

Now, we are going to show that the allele b fixates in finite time. For this, we need only prove that $\zeta = \lim_{N\to\infty} \zeta_N^{n_0} < \infty$ *a.s.*.

Recall lemma 4.2. For each $n \geq 1$, we define

$$f(n) = \sum_{k=n+1}^{\infty} \frac{1}{(\phi(k) - \alpha k) \vee 1}.$$

By Lemma 4.2, for each $n \geq 2$, $f(n)$ is finite. We have for $2 \leq \ell \leq n$

$$f(n-\ell+1) - f(n) = \sum_{k=n-\ell+2}^{n} \frac{1}{(\phi(k) - \alpha k) \vee 1}.$$

Since $k \to 1/(\phi(k) - \alpha k) \vee 1$ is decreasing, we obtain

$$f(n-\ell+1) - f(n) \geq (\ell - 1)\frac{1}{(\phi(n) - \alpha n) \vee 1},$$

and therefore

$$\begin{aligned} Q^N f(n) &\geq \frac{1}{(\phi(n) - \alpha n) \vee 1} \sum_{\ell=2}^{n} \binom{n}{\ell} (\ell - 1)\lambda_{n,\ell} - \frac{\alpha n}{(\phi(n+1) - \alpha(n+1)) \vee 1} \\ &= \frac{\phi(n)}{(\phi(n) - \alpha n) \vee 1} - \frac{\alpha n}{(\phi(n+1) - \alpha(n+1)) \vee 1} \\ &\geq \frac{\phi(n)}{(\phi(n) - \alpha n) \vee 1} - \frac{\alpha n}{(\phi(n) - \alpha n) \vee 1} \end{aligned}$$

hence $Q^N f(n) \geq 1$, for each $n \geq n_0$. Using the fact that the process

$$f(Z_t^N) - f(N) - \int_0^t Q^N f(Z_s^N) ds, \qquad t \geq 0$$

is a Martingale, we obtain

$$\begin{aligned}\mathbf{E}(\zeta_N^{n_0}) &\leq \mathbf{E}\left(\int_0^{\zeta_N^{n_0}} Q^N f(Z_s^N) ds\right) \\ &= f(Z_{\zeta_N^{n_0}}^N) - f(N) \\ &\leq f(1)\end{aligned}$$

Since the sequences $\zeta_N^{n_0} \uparrow \zeta$ as $N \uparrow \infty$, the last assertions, together with the monotone convergence theorem, implies that

$$\mathbf{E}(\zeta) \leq f(1) < \infty,$$

and hence $\zeta < \infty$ $a.s.$

STEP 2 : Suppose $\Lambda \notin \mathbf{CDI}$, that is the Λ-coalescent does not come down from infinity. We have

$$\sum_{n=2}^{\infty} \frac{1}{\phi(n)} = +\infty. \tag{4.4}$$

We claim that $(K_t, t \geq 0)$ does not reach ∞ in finite time. The contrary would imply that $\exists T < \infty$ such that $K_T = \infty$ a.s., so the number of ancestors of the infinite population at time T of the Λ-lookdown model would be finite and bounded by to $K_0 - 1$, which contradicts the fact that $\Lambda \notin \mathbf{CDI}$. Hence $K_t < \infty$ a.s. This implies that $X_t < 1$, for all $t \geq 0$. Indeed if $X_t = 1$, for some $t > 0$, by applying de Finetti's Theorem, we deduce that $\eta_t(i) = 1, \forall i \geq 1$, which contradicts the fact that $K_t < \infty$. It remains to show that $X_t > 0$ for all $t \geq 0$.

For any $m \geq 1, t > 0$, we define the event

$$A_t^m = \{\text{The } m \text{ first individuals of type } b \text{ at time 0 are dead at time } t\}$$

We have

$$\mathbf{P}(A_t^m) = (1 - e^{-\alpha t})^m,$$

and then

$$\mathbf{P}(\cap_m A_t^m) = 0 \quad \forall t > 0.$$

From this, we deduce that $\exists i \geq 1$ such that $\eta_t(i) = 1$. The same argument for the proof of $X_t < 1$ shows that $X_t > 0$, for all $t \geq 0$. □

4.3 The law of X_∞

Let x be the proportion of type b individuals at time 0, where $0 < x < 1$. As the individual at level 1 cannot be pushed to an upper level, we have

$$\{\eta_0(1) = 0\} \subset \{X_\infty = 0\}, \text{ hence } \mathbf{P}(X_\infty = 0) \geq 1 - x.$$

If $\alpha = 0$, X_t is a bounded martingale, so

$$\mathbf{P}(X_\infty = 1) = \mathbf{E}(X_\infty) = \mathbf{E}(X_0) = x.$$

If $\alpha > 0$, by using (3.5) together with (4.2), we deduce that

$$\mathbf{P}(X_\infty = 1) = \mathbf{E}X_\infty < x.$$

In this subsection we want to describe those cases where can we decide whether $\mathbf{P}(X_\infty = 1) > 0$ or $\mathbf{P}(X_\infty = 1) = 0$. We first prove

Proposition 4.4. *If $\Lambda \in$ **CDI**, then*

$$\mathbf{P}(X_\infty = 1) > 0.$$

PROOF : Since $\Lambda \in \mathbf{CDI}$, if all individuals at time 0 would be of type b, there would be a level J such that the individual sitting on level J at time 0 reaches $+\infty$ in finite time. Now $\mathbf{P}(X_\infty = 1) > 0$ follows from the fact that $\mathbf{P}(K_0 > J) > 0$, where K_0 denotes the lowest level occupied by a type B individual at time 0. $\square$

In the rest of this subsection, we assume that $\Lambda \notin \mathbf{CDI}$, and want to decide whether $\mathbf{P}(X_\infty = 1) > 0$ or $\mathbf{P}(X_\infty = 1) = 0$. We shall see that these two situations are possible.

We first consider the case of a population model dual to Bolthausen-Sznitman's coalescent, i.e $\Lambda(dp) = dp$. It is know that in that case the Λ-coalescent does not come down from infinity.

For any partition with a finite number $n \geq 2$ of blocks, the total rate of transitions of all kinds in a Bolthausen-Sznitman coalescent equals

$$\begin{aligned}
\lambda_n &= \sum_{\ell=2}^{n} \binom{n}{\ell} \lambda_{n,\ell} = \sum_{\ell=2}^{n} \binom{n}{\ell} \int_0^1 p^{\ell-2}(1-p)^{n-\ell} dp \\
&= \sum_{\ell=2}^{n} \binom{n}{\ell} \frac{\Gamma(\ell-1)\Gamma(n-\ell+1)}{\Gamma(n)} \\
&= \sum_{\ell=2}^{n} \frac{n}{\ell(\ell-1)} \\
&= n-1.
\end{aligned} \tag{4.5}$$

Recall the definition of K_t. For each $n \geq 1$, we have

$$\mathbf{P}(K_0 = n) = (1-x)x^{n-1},$$

that is K_0 follows the geometric distribution with parameter $1-x$. Hence, for each $x \in]0,1[$, $\mathbf{E}K_0 < \infty$. We have the following theorem

Proposition 4.5. *If $\Lambda(dp) = dp$, then for any $\alpha \geq 0$,*

$$\mathbf{P}(X_\infty = 1) > 0.$$

PROOF : It suffices to show that

$$\mathbf{P}(\lim_{t\to\infty} K_t = \infty) > 0. \tag{4.6}$$

Denote by $T_1, T_2, \ldots$ the jumps times of the process K_t. Let

$$X_n = K_{T_n} \text{ for each } n \geq 1.$$

For each integer $k \geq 1$, we define by V_k the size of the jump to the right of the individual sitting on level k. Using (4.5), it is not hard to show that conditionally upon $X_n = k$;

$$X_{n+1} = \begin{cases} k + V_k, & \text{with probability } \frac{1}{\alpha+1} \\ k-1, & \text{with probability } \frac{\alpha}{\alpha+1}. \end{cases}$$

Now, we consider the model where we erase all the arrows pointing to levels above K_t for each birth event $(t,p) \in m$ (the Poisson point process m is defined by (1.2)). We also define V'_k the size of the jump to the right of the individual sitting on level k in this modified model. We can couple the process X_n with the Markov Chain Y_n, which jumps at the same birth and death time than X_n, and which evolves as follows

$$Y_0 = X_0 = K_0,$$

and conditionally upon $Y_n = k$;

$$Y_{n+1} = \begin{cases} k + V'_k \wedge C, & \text{with probability } \frac{1}{\alpha+1} \\ k-1, & \text{with probability } \frac{\alpha}{\alpha+1}, \end{cases}$$

where $C \geq 2$ is a constant to be chosen below.

Since the “true ” model has more arrows than the present model, we have

$$Y_n \leq X_n \quad \forall n \geq 1,$$

and

$$\mathbf{P}(\lim_{n\to\infty} Y_n = \infty) \leq \mathbf{P}(\lim_{n\to\infty} X_n = \infty) = \mathbf{P}(\lim_{t\to\infty} K_t = \infty).$$

Using the last statement, to prove (4.6), it suffices to show that

$$\mathbf{P}(\lim_{n\to\infty} Y_n = \infty) > 0. \tag{4.7}$$

For this, we will use Theorem 3.1 in [13]. Let $k > 1$ denote a fixed integer, and we define

$$\begin{aligned} \mu(k) &= \mathbf{E}(Y_{n+1} - Y_n \mid Y_n = k) \\ &= \frac{\mathbf{E}(V'_k \wedge C) - \alpha}{1+\alpha}. \end{aligned}$$

and

$$\begin{aligned}v(k) &= \mathbf{E}\left[(Y_{n+1}-Y_n)^2 \mid Y_n = k\right] \\ &= \frac{\mathbf{E}\left[(V'_k \wedge C)^2\right]+\alpha}{1+\alpha}.\end{aligned}$$

Let us compute $\mathbf{E}(V'_k \wedge C)$. For each $p \in]0,1]$ and $k > C$, we have

$$\begin{aligned}\mathbf{E}_p(V'_k \wedge C) &= \mathbf{E}_p(V'_k; V'_k \leq C) + C\mathbf{P}_p(V'_k > C) \\ &= \sum_{\ell=2}^{C}(\ell-1)\mathbf{P}_p(Z_1+\cdots+Z_k=\ell) + C\sum_{\ell=C+1}^{k}\mathbf{P}_p(Z_1+\cdots+Z_k=\ell) \\ &= \sum_{\ell=2}^{C}(\ell-1)\binom{k}{\ell}p^\ell(1-p)^{k-\ell} + C\sum_{\ell=C+1}^{k}\binom{k}{\ell}p^\ell(1-p)^{k-\ell}.\end{aligned}$$

Next,

$$\begin{aligned}\int_0^1 \mathbf{E}_p(V'_k \wedge C)\nu(dp) &= \sum_{\ell=2}^{C}(\ell-1)\binom{k}{\ell}\int_0^1 p^{\ell-2}(1-p)^{k-\ell}dp + C\sum_{\ell=C+1}^{k}\binom{k}{\ell}\int_0^1 p^{\ell-2}(1-p)^{k-\ell}dp \\ &= \sum_{\ell=2}^{C}(\ell-1)\binom{k}{\ell}\frac{\Gamma(\ell-1)\Gamma(k-\ell+1)}{\Gamma(k)} + C\sum_{\ell=C+1}^{k}\binom{k}{\ell}\frac{\Gamma(\ell-1)\Gamma(k-\ell+1)}{\Gamma(k)} \\ &= \sum_{\ell=2}^{C}\frac{k}{\ell} + C\sum_{\ell=C+1}^{k}\frac{k}{\ell(\ell-1)} \\ &= k\sum_{\ell=1}^{C}\frac{1}{\ell} - C.\end{aligned}$$

We deduce that

$$\begin{aligned}\mathbf{E}(V'_k \wedge C) &= \frac{1}{\lambda_k}\int_0^1 \mathbf{E}_p(V'_k \wedge C)\nu(dp) \\ &= \frac{1}{k-1}(k\sum_{\ell=1}^{C}\frac{1}{\ell} - C).\end{aligned}$$

Using the last identity, we obtain

$$\begin{aligned}\frac{k\mu(k)}{v(k)} &= \frac{k\mathbf{E}(V'_k \wedge C) - \alpha k}{\mathbf{E}(V'_k \wedge C)^2 + \alpha} \\ &\geq \frac{k}{C^2+\alpha}\Big[\mathbf{E}(V'_k \wedge C) - \alpha\Big] \\ &\geq \frac{k}{C^2+\alpha}\Big[\frac{1}{k-1}(k\log C - C) - \alpha\Big].\end{aligned}$$

we choose C such that $\log C \geq \alpha$, and deduce that

$$\lim_{k\to\infty} \frac{k\mu(k)}{v(k)} = +\infty.$$

From Theorem 3.1 in [13], the result follows. $\square$

In the rest of this subsection, we will give a condition under which $\mathbf{P}(X_\infty = 0)$. Recall the Poisson point measure m defined in the introduction. Let (t,p) be a point of the measure m. Let us first give one possible description of the size of the jump to the right of an individual sitting on level n at time t^-.

Let L^n be a random variable with the binomial law $B(n,p)$ and $W_0^n = (L^n-1)^+$. If $W_0^n = 0$, then there is no jump. If $W_0^n \geq 1$, we put W_0^n balls in a box B_0. Each of those W_0^n balls, independently of the others and of the value of W_0^n, is copied with probability p, and all copies are put in a box B_1. Let W_1^n denote the number of balls in B_1. Again, each of those is copied with probability p, all copies being put in a box B_2, etc.... The just described r.v.'s $(W_i^n, i \geq 1)$ have the same joint law as the following r.v.'s :

$$W_i^n = \begin{cases} \#(I_{t,p} \cap [1,\dots,n]) - 1, & \text{if } i = 0 \\ \#(I_{t,p} \cap [n+1,\dots,n+W_0^n]), & \text{if } i = 1 \\ \#\left(I_{t,p} \cap [n+\sum_{k=0}^{i-2} W_k^n + 1,\dots,n+\sum_{k=0}^{i-1} W_k^n]\right), & \text{otherwise} \end{cases}$$

The size of the jump equals $\sum_{i=0}^\infty W_i^n$. Let us compute

$$\mathbf{E}_p\left(\sum_{i=0}^\infty W_i^n\right).$$

$$\mathbf{E}_p W_0^n = \sum_{k=1}^{n-1} \binom{n}{k+1} k p^{k+1}(1-p)^{n-k-1} = np - 1 + (1-p)^n.$$

It is plain that

$$\mathbf{E}_p[W_i^n] = p\mathbf{E}_p[W_{i-1}^n], \quad i \geq 1.$$

Indeed, conditionally upon $W_{i-1}^n = j, W_j^n$ is $B(j,p)$ distributed. Consequently

$$\mathbf{E}_p\left(\sum_{i=0}^\infty W_i^n\right) = (1-p)^{-1}\mathbf{E}_p[W_0^n] = (1-p)^{-1}[np-1+(1-p)^n].$$

Now the mean speed of the movement to the right, starting from n, equals the above quantity integrated over ν, hence equals

$$\Phi(n) = \int_0^1 \mathbf{E}_p\left(\sum_{i=0}^\infty W_i^n\right)\nu(dp) = \int_0^1 (1-p)^{-1}[np-1+(1-p)^n]\nu(dp),$$

where $\nu(dp) = p^{-2}\Lambda(dp)$. Using the same argument as in the proof of Lemma 4.1, one can easily show that

$$\frac{\Phi(n)}{n} \uparrow \int_0^1 \frac{1}{p(1-p)}\Lambda(dp) \text{ as } n \uparrow \infty. \tag{4.8}$$

Recall that $\Lambda \notin \mathbf{CDI}$. We introduce the notation

$$\mu = \int_0^1 \frac{1}{p(1-p)}\Lambda(dp).$$

The second main result in this section is

Theorem 4.6. *If $\mu < \alpha$, then*

$$\mathbf{P}(X_\infty = 1) = \mathbf{E}(X_\infty) = 0.$$

PROOF : Suppose $\mu < \alpha$. It follows from lemma 25 of [15] that $\Lambda \notin \mathbf{CDI}$. Using Theorem 4.3, we deduce that $0 < X_t < 1$, for all $t > 0$. Let T_1 be the first time when the level 1 is occupied by a type B individual.

The Theorem is a consequence of the following lemma

Lemma 4.7. *If $\mu < \alpha$, then*

$$T_1 < \infty \ a.s$$

PROOF : Let

$$N_0 = \inf\{n \geq 1, \Phi(n) \leq \alpha(n-1)\}.$$

From (4.8) and $\mu < \alpha$, it is plain that $N_0 < \infty$. Recall the infinitesimal generator of $\{K_t, t \geq 0\}$ defined in section 2. Applying (2.1) with the particular choice $g(n) = n$, the process $(M_t)_{t\geq 0}$ given by

$$M_t = K_t - K_0 - \int_0^t [\Phi(K_s) - \alpha(K_s - 1)]ds$$

is a martingale. Let us show that K_t comes back below level N_0 after any time t. Suppose that $K_0 > N_0$ and define

$$S_0 = \inf\{t, K_t \leq N_0\}.$$

The process $(K_{t\wedge S_0})_{t\geq 0}$ is a positive supermartingale. Consequently,

$$K_{t\wedge S_0} \to K_\infty \in \{N_0, \infty\} \ a.s.$$

and

$$\mathbf{E}(K_{t\wedge S_0}) \leq \mathbf{E}(K_0) < \infty, \quad \forall t \geq 0,$$

hence by Fatou's lemma, $\mathbf{E}K_\infty < \infty$, consequently $K_\infty = N_0$ a.s. As the process $(K_{t\wedge S_0})_{t\geq 0}$ takes values in the set $\{N_0, \ldots, \infty\}$, it is easy to deduce that $S_0 < \infty$ *a.s.* We now define recursively S_k for $k \geq 1$ by the formula

$$S_k = \inf\{t > S_{k-1} + 1 : K_t \leq N_0\}$$

We have

$$\mathbf{P}_{N_0}(K_1 = 1) \geq e^{-\phi(N_0)}(1 - e^{-\alpha})^{N_0 - 1}. \tag{4.9}$$

Indeed, for $(K_t)_{t\geq 0}$ to reach the level 1 during the time interval $[0, 1]$, it suffices that during that time interval there are no arrows between the level 1 and N_0 and during the same time all the individuals between 1 and $N_0 - 1$ die. Using (4.9), we deduce by the Markov property

$$\begin{aligned}
\mathbf{P}(K_{S_k} > 1) &= \mathbf{E}\left(\prod_{i=1}^{k} K_{S_i} > 1\right) \\
&\leq (1 - e^{-\phi(N_0)}(1 - e^{-\alpha})^{N_0 - 1})\mathbf{P}(K_{S_{k-1}} > 1) \\
&\leq \left(1 - e^{-\phi(N_0)}(1 - e^{-\alpha})^{N_0 - 1}\right)^k .
\end{aligned}$$

The result follows by taking the limit as $k \to \infty$. □

5 Kingman and Λ-coalescent

In this last section we suppose that the measure Λ is general (i.e $\Lambda(\{0\}) > 0$) and we show that the proportion X_t of type b individuals at time t in the population of infinite size is a solution of the stochastic differential equation with selection

$$\begin{aligned}
X_t = x &- \alpha \int_0^t X_s(1 - X_s)ds \\
&+ \int_0^t \sqrt{cX_s(1 - X_s)}dB_s \\
&+ \int_{[0,t]\times]0,1[\times]0,1[} p(\mathbf{1}_{u\leq X_{s^-}} - X_{s^-})\bar{M}(ds, du, dp),
\end{aligned} \tag{5.1}$$

where $c = \Lambda(\{0\}), \bar{M}$ is the compensated measure M defined in section 3.2, and B is a standard Brownian motion. Let $\{W(ds, du)\}$ be a white noise on $(0, \infty) \times (0, 1]$ based on the Lebesgue measure $dsdu$. We remark that if X_t satisfies (5.1), then X_t is a solution in law of the following stochastic differential equation

$$\begin{aligned}
X_t = x &- \alpha \int_0^t X_s(1 - X_s)ds \\
&+ \sqrt{c} \int_{[0,t]\times]0,1[} (\mathbf{1}_{u\leq X_s} - X_s)W(ds, du) \\
&+ \int_{[0,t]\times]0,1[^2} p(\mathbf{1}_{u\leq X_{s^-}} - X_{s^-})\bar{M}(ds, du, dp).
\end{aligned}$$

We first define the model. Recall the process $\{\eta_t(i), i \geq 1, t \geq 0\}$ defined in the introduction. The evolution of the population is the same as that described in the case $\Lambda(\{0\}) = 0$ except that we superimpose single births, which are described as follows

For any $1 \le i < j$, arrows are placed from i to j according to a rate $\Lambda(\{0\})$ Poisson process, independently of the other pairs $i' < j'$. Suppose there is an arrow from i to j at time t. Then a descendent (of the same type) of the individual sitting on level i at time t^- occupies the level j at time t, while for any $k \ge j$, the individual occupying the level k at time t^- is shifted to level $k+1$ at time t. In other words, $\eta_t(k) = \eta_{t^-}(k)$ for $k < j$, $\eta_t(j) = \eta_{t^-}(i)$, $\eta_t(k) = \eta_{t^-}(k-1)$ for $k > j$.

By coupling our model with the simplest lookdown model with selection defined in [3], it is not hard to show that for N large enough, the individual sitting on level $2N$ at time 0 never visits a level bellow N, that is the evolution within the box $(t,i) \in [0,\infty) \times \{1,2,\dots,N\}$ is not altered by removing all crosses above $2N$. The process $\{\eta_t(i), i \ge 1, t \ge 0\}$ is well-defined.

For each $N \ge 1$ and $t \ge 0$, denote by X_t^N the proportion of type b individuals at time t among the first N individuals, i.e.

$$X_t^N = \frac{1}{N}\sum_{i=1}^{N} \eta_t(i). \tag{5.2}$$

Combining the arguments in [3] and section 2.3 (see above), it is easy to show if $(\eta_0(i))_{i\ge1}$ are exchangeable random variables, then for all $t > 0$, $(\eta_t(i))_{i\ge1}$ are exchangeable. An application of de Finetti's theorem, yields that

$$X_t = \lim_{N\to\infty} X_t^N \quad \text{exist a.s.} \tag{5.3}$$

Using the definition of the model, it is easy to see that

$$\begin{aligned} X_t^N = X_0^N + \mathcal{K}_t^N + \int_{[0,t]\times]0,1]^4} \psi^N(X_{s^-}^N, u, p, v, w)\bar{M}(ds, du, dp, dv, dw) \\ - \frac{1}{N}\int_{[0,t]\times[0,1]^2} \mathbf{1}_{u\le X_{s^-}^N}\mathbf{1}_{v\ge X_{s^-}^{N+1}} M_1^N(ds, du, dv), \end{aligned}$$

where $\mathcal{K}_t^N$ is a martingale of jump size $\pm\frac{1}{N}$. We have

Lemma 5.1.

$$\langle \mathcal{K}^N \rangle_t = \int_0^t \varphi^N(s)ds$$

where, $\varphi^N(s) = \Lambda(0)X_s^N(1 - X_s^N)$.

Proof : For each $1 \le i < N$, let P^i be a Poisson process with intensity $\Lambda(0)(N-i)$. At time $t \in P^i$, we have

$$\Delta X_t^N = \begin{cases} \frac{1}{N}, & \text{if } \eta_{t^-}(i) = 1 \text{ and } \eta_{t^-}(N) = 0 \\ -\frac{1}{N}, & \text{if } \eta_{t^-}(i) = 0 \text{ and } \eta_{t^-}(N) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now, let

$$A_i = \{\eta_t(i) = 1, \eta_t(N) = 0\},$$
$$B_i = \{\eta_t(i) = 0, \eta_t(N) = 1\}.$$

We have

$$\mathbf{P}(A_i \mid X_t^N) = \mathbf{P}(B_i \mid X_t^N) = \frac{N}{N-1} X_t^N (1 - X_t^N),$$

from which, we deduce that

$$\begin{aligned}\langle \mathcal{K}^N \rangle_t &= \frac{1}{N^2} \Lambda(0) \frac{N(N-1)}{2} \frac{2N}{N-1} X_t^N (1 - X_t^N) \\ &= \Lambda(0) X_t^N (1 - X_t^N)\end{aligned}$$

The result is proved.

□

Now, let

$$\begin{aligned}Y_t^N = X_0^N &+ \int_{[0,t]\times]0,1]^4} \psi^N(X_{s^-}^N, u, p, v, w) \bar{M}(ds, du, dp, dv, dw) \\ &- \frac{1}{N} \int_{[0,t]\times[0,1]^2} \mathbf{1}_{u \le X_{s^-}^N} \mathbf{1}_{v \ge X_{s^-}^{N+1}} M_1^N(ds, du, dv).\end{aligned}$$

We have

$$X_t^N = \mathcal{K}_t^N + Y_t^N, \quad \forall t \ge 0 \tag{5.4}$$

From lemma 5.1, we have $\forall T \ge 0$

$$\sup_{0 \le t \le T} \sup_{N \ge 1} \mid \varphi^N(s) \mid \le C \quad a.s.$$

Using the last identity, we deduce by Aldous' tightness criterion (see Aldous [1]) that

$$\{\mathcal{K}_t^N, t \ge 0, N \ge 1\} \text{ is tight in } D([0, \infty)).$$

Since $\mathcal{K}^N$ is tight, there exists a subsequence of the sequence $\mathcal{K}^N$ such that

$$\mathcal{K}^N \Rightarrow \mathcal{K} \text{ weakly in } D([0, \infty)),$$

where $\mathcal{K}$ is a continuous martingale (since the jumps of $\mathcal{K}^N$ are of size $\pm \frac{1}{N}$) such that

$$< \mathcal{K} >_t = \int_0^t c X_s (1 - X_s) ds, \tag{5.5}$$

where $c = \Lambda(0)$. The main result of this section is

Theorem 5.2. *Suppose that $X_0^N \to x$ a.s, as $N \to \infty$. Then the $[0,1]-$valued process $\{X_t, t \geq 0\}$ defined by (5.3) is the (unique in law) solution of the stochastic differential equation*

$$\begin{aligned} X_t = x - \alpha \int_0^t X_s(1-X_s)ds \\ + \int_0^t \sqrt{\Lambda(0) X_s(1-X_s)} dB_s \\ + \int_{[0,t]\times]0,1[^2} p(\mathbf{1}_{u \leq X_{s-}} - X_{s-}) \bar{M}(ds, du, dp), \end{aligned} \tag{5.6}$$

where $\bar{M}$ is the compensated measure M defined in section 3.2, and B is a standard Brownian motion.

The identification of the limiting equation is done similarly as in the proof of Theorem 3.6. Strong uniqueness of the solution to (5.6) follows again from Dawson and Li [8], and weak uniqueness could also be proved by a duality argument.

Since Kingman's coalescent comes down from infinity, we have fixation in our new model in finite time as soon as $\Lambda(0) > 0$.

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