CIRM 2009

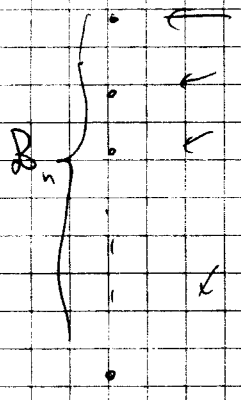
Speed of CDI for $\boxed{\Xi}$-coalescents

Assume we are given n blocks, B_n and perform coalescence operator

$p \in (0,1)$ coal$_p$

$1 - \varepsilon$ ~~4~~

think p "small"



draw i.i.d. Bernoulli(p)'s

collapse all blocks that have 1

into 1

interested in
$$\frac{\#\text{blocks}(\text{coal}_p(B_n))}{\#\text{blocks}(B_n)} = \frac{n - Y_n + 1(Y_n > 0)}{n}$$
$$= \frac{n - (Y_n - 1)^+}{n}$$

where $Y_n = \text{Bin}(n, p)$

$$\log\left(\frac{n - Y_n + 1(Y_n>0)}{n}\right) = \frac{-Y_n + 1(Y_n>0)}{n} + \mathcal{O}\left(\left[\frac{(Y_n-1)^+}{n}\right]^2\right)$$

if $\frac{Y_n}{n} < 1$ ← will be true with overwhelming probab. if $p \leq 1-\varepsilon$

generalisation

$\vec{p} \in \Delta = \{(x_1, x_2, \dots) : x_1 \geq x_2 \geq \dots \geq 0$
$\sum x_i \leq 1\}$

coal$_{\vec{p}}(B_n)$ = draw indep. color according to $\vec{p}$ for each block, simultaneously collapse all blocks with same color into 1 $\mathbb{P}(\text{color } i) = p_i$

$$\#\text{blocks}(\text{coal}_{\vec{p}}(B_n)) = \frac{n - \sum_{i=1}^{\infty} Y_n^i + 1(Y_n^i > 0)}{n}$$

where Y_n^i = # blocks with color i

again
$$\log \frac{\#\text{blocks}(\text{coal}_{\vec{p}}(B_n))}{n} = -\frac{\sum Y_n^i - 1(Y_n^i > 0)}{} - \frac{1}{2}\left(\frac{\sum (Y_n^i - 1)^+}{n}\right)^2$$
$$-\frac{1}{3}\left(\frac{\sum^n (Y_n^i - 1)^+}{n}\right)^3 - \dots -$$

Moreover if $\sum p_i \leq 1-\varepsilon < 1$, then uniformly over such $\vec{p}$

$$\log \frac{\#\text{blocks}\,(\text{coal}_{\vec{p}}(\mathcal{B}_n))}{n} = -\sum_i \frac{Y_n^i - \mathbb{1}_{(Y_n^i > 0)}}{n} + O_{\varepsilon}\left(\left[\frac{\sum (Y_n^i - 1)^+}{n}\right]^2\right)$$

$$\mathbb{E}\Big(\qquad \Big) = -\frac{\sum_{i=1}^{\infty} np_i - 1 + (1-p_i)^n}{n} + O_{\varepsilon}\left(\left(\sum p_i\right)^2\right)$$

Since $(1-p_i)^n = e^{-np_i} + \underset{\text{unif. in } p}{O}(np_i^2)$

$$= -\frac{\sum_{i=1}^{\infty} e^{-np_i} - 1 + np_i}{n} + O_{\varepsilon}\left(\left(\sum p_i\right)^2 + \sum p_i^2\right)$$

Ξ - coalescent

Ξ a probab. measure on Δ, $\Xi(\{\vec{0}\}) = 0$ (assume in this talk)

$\mathcal{B}_s$ = configuration of blocks at time s, $s > 0$

$\vec{p}$ arrives at rate $\frac{\Xi(d\vec{p})}{\sum p_i^2}$, "upon" each arrival perform $\text{coal}_{\vec{p}}(\mathcal{B}_{\text{current time}-})$

So if $(t, \vec{p})$ is an atom of PPP then $\mathcal{B}_t = \text{coal}_{\vec{p}}(\mathcal{B}_{t-})$

Notes: 1) $\frac{\Xi(dp)}{\sum p_i^2}$ may not be finite, but the construction still works (recall Lévy processes)

2) can start with $\mathcal{B}_{0+}$ = infinitely many blocks, standard version due to consistency: restriction to any b-tuple of blocks behaves as Ξ-coalescent started from b blocks

3) Any pair of blocks coalesces at rate 1

$$\lambda_2 = \int \mathbb{P}\left(\substack{\text{1 and 2 colored} \\ \text{by same color}}\right) \frac{\Xi(d\vec{p})}{\sum p_i^2} = 1 \quad (\Xi \text{ is p.m.})$$

4) if $\Xi(\vec{p} : p_2 > 0) = 0$ $\quad \overset{\shortparallel}{\sum_i p_i^2} \quad \Lambda$-coalescent $\quad \text{coal}_{p_1 = p}(\mathcal{B}.)$

③

Suppose for some $\varepsilon > 0$ $\displaystyle\int_{\{\vec{p}\in\Delta\,;\,\sum p_i > 1-\varepsilon\}} \frac{\Xi(d\vec{p})}{\sum p_i^2} < \infty$, then if for some $\varepsilon > 0$ (C_ε)

$T_\varepsilon := \min\{$arrival time of $\vec{p}$ s.t. $\sum p_i > 1-\varepsilon\}$ have $\mathbb{P}(\underset{\text{continuous Exponent (rate } \varepsilon\text{) r.v.}}{T_\varepsilon} = 0) = 0$

Fix large n, interested in the evolution of

$\left(\#\text{blocks}(\mathcal{B}_s^{(n)}), s \geq 0\right)$ and

$\left(\#\text{blocks}(\underset{\mathcal{B}_s}{\underset{\parallel}{\mathcal{B}_s^{(\infty)}}}), s \geq 0\right)$ ← standard case

Def Say that the standard coalescent $\mathcal{B}$ CDI if

$$\mathbb{P}(\#\text{blocks}(\mathcal{B}_t) < \infty, \forall t > 0) = 1$$

Fact Schweinsberg

If $\Xi(\vec{p} : \sum_{i=1}^{n} p_i = 1, \exists \underset{n}{\text{finite}}) = 0$, then either standard Ξ-coal CDI or $\mathbb{P}(\#\text{block}(\mathcal{B}_t) = \infty, \forall t \geq 0) = 1$

From now on assume

$$\Xi(\vec{p} : \sum_{n}^{n} p_i = 1, \exists n < \infty) = 0$$

and moreover $\Xi(\vec{p} : \sum p_i \leq 1-\varepsilon) = 1$ for some $\varepsilon > 0$, (C_ε')

Consider $(\mathcal{B}_s^{(n_0)}, s \geq 0)$, n_0 large finite

$$N^{(n_0)}(s) = \#\text{blocks}(\mathcal{B}_s^{(n_0)}), \ s \geq 0$$

$$\mathbb{1}_{(N^{(n_0)}(t)=n)} \mathbb{E}\left(d \log N^{(n_0)}(t) \mid \mathcal{F}_t\right) = \left[-\int_\Delta \frac{\sum_{i=1}^{\infty}(e^{-np_i} - 1 + np_i)}{n} \frac{\Xi(d\vec{p})}{\sum p_i^2}\right.$$

$$\left. + \underbrace{\int_\Delta \Theta_\varepsilon\left(\sum p_i^2\right) \frac{\Xi(d\vec{p})}{\sum p_i^2}}_{\text{uniform in } n} + \int_\Delta \Theta_\varepsilon\left(\left(\sum_{i=1}^{\infty} p_i\right)^2\right) \frac{\Xi(d\vec{p})}{\sum p_i^2}\right] dt$$

[illegible]

So provided

$$(R) \quad \int_{\Delta} \underbrace{\frac{\left(\sum_{i=1}^{\infty} p_i\right)^2}{\sum_{i=1}^{\infty} p_i^2} \Theta(d\vec{p}) < \infty}$$

← Note: Always holds for Λ-coal with ~~Λ([illegible])~~

can write

$$\mathbb{E}\left(d \log N^{(n_0)}(t) \mid \mathcal{F}_t\right) = \left(- \frac{\Psi(N^{(n_0)}(t))}{N^{(n_0)}(t)} + \underbrace{H(t)}\right) dt$$

≤ bounded by constant

where

$$\Psi(q) = \int_{\Delta} \sum_{1}^{\infty} \left(e^{-q p_i} - 1 + q p_i\right) \frac{d\Theta(\vec{p})}{\sum_{i=1}^{\infty} p_i^2}$$

Note

(B) $(R) \Rightarrow (C_\varepsilon)$ △

1) (R) is true for all Λ-coalescents ~~[illegible]~~

2) (R) is not true for all Θ-coalescents

exple $\quad \Theta\left(\left(\underbrace{\frac{1}{2^n}, \frac{1}{2^n}, \dots, \frac{1}{2^n}}_{2^{n+1},\ \text{or}\ 2^n - 2,\ \dots}, 0, 0, \dots\right)\right) = \frac{1}{2^n}, \; n \geq 1$

$$\sum p_i^2 = \sum \frac{1}{2^{2n}} = \frac{1}{2^n} \qquad \Theta(\vec{p}) = \frac{1}{2^n}$$

$$\sum p_i \approx 1$$

3) probabilistic interpretation of (R):

- if in the PPP graphical construction using $\mathrm{coal}_{\vec{p}}(\mathcal{B}^n_\cdot)$ one becomes "color blind" - ~~all~~ as blocks ~~in~~ colored by any $p_1, p_2, \dots$ ~~are~~ ~~merged~~ together in one block, the corresponding (standard) Λ-coalesc "exists" in the sense that $\#\text{blocks}(\text{collapse}(\mathcal{B}))_{0+} = \infty$ iff (R), and if not (R) $\#\text{blocks}(\text{collapse}(\mathcal{B})) = 1 \; \forall t$

4) Could have CDI + (R)

CDI + ¬(R)

non CDI + (R)

non CDI + ¬(R)

no precise (suff. - nec. criterion for CDI) but

Theorem of Schweinsberg says that if (C_ε) then

$$\mathrm{CDI} \iff \int \frac{dq}{\psi(q)} < \infty$$

(in fact his criterion in terms of quantities that behave asympt. the same as ψ, also summability)

And ~~also~~ $\int \frac{dq}{\psi(q)} < \infty \Rightarrow \mathrm{CDI}$

but there are examples of CDI + $\int_{(0,\infty)} \frac{dq}{\psi(q)} = \infty$, of course not satisfying (C_ε), $\forall \varepsilon \geq 0$

new

<u>Theorem</u> If (R) then the standard Ξ coalescent has speed of coming down from infinity

$$\lim_{t \to 0} \frac{N^{\Xi}(t)}{v_{\Xi}(t)} = 1 \quad \text{a.s.,}$$

where $v_{\Xi}(t)$ solves $\int_{v_{\Xi}(t)}^{\infty} \frac{dq}{\psi(q)} < \infty$

Why? Similar to the work Berestycki × 2, ~~L~~ for Λ-coalescents

use

$$\log \frac{N^{\Xi}(t)}{v_{X_z}(t)} - \log \frac{N^{\Xi}(z)}{v_{X_z}(z)} + \int_z^t \Big[\underbrace{\frac{\psi(N(u))}{N(u)} - \frac{\psi(v_{X_z}(u))}{v_{X_z}(u)}}_{\text{a MG}} + \underbrace{H(u)}_{O(1)} \Big] du$$

$$\mathrm{var}(d \log N(t) \mid \mathcal{F}_t) = \underbrace{H_1(t)}_{} dt$$

again $\leq C$ uniformly in $N(t) \geq n_0$

and $q \mapsto \frac{\psi(q)}{q}$ ↗ function

X_z chosen so that $v_{X_z}(z) = N(z) = v(X_z + z)$

or equivalently (R) will guarantee that

$\log \frac{N^{(n)}(t)}{v_n(t)}$ stays close to 0 for some ~~order~~ 1 amount of time uniformly over n

where $v_n(t) = v_{\boxminus}(t_n + t)$ and $v_{\boxminus}(t_n) = n$

Why $\partial_t \log v_n(t) = (\log v_n(t))' = \frac{1}{v_n(t)} v_n'(t) = -\frac{\psi(v_n(t))}{v_n(t)}$

$v_{\boxminus}(t_n+t)$ s.t. $\int_{v_{\boxminus}(t_n+t)}^{\infty} \frac{dq}{\psi(q)} = t_n + t$

$$\boxed{-\frac{1}{\psi(v_{\boxminus}(t_n+t))} v_{\boxminus}'(t_n+t) = 1}$$

$\int \frac{dq}{\psi(q)} \not< \infty$	(D)	(R)	
0	0	0	✓
0	0	1	✓ [illegible]-cool
0	1	0	✓
0	1	1	✗
1	0	0	✗
1	0	1	✗
1	1	0	✓
1	1	1	✓