

“Trees under attack”: a Ray-Knight representation of Feller’s branching diffusion with logistic growth

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Abstract

We obtain a representation of Feller’s branching diffusion with logistic growth in terms of the local times of a reflected Brownian motion H with a drift that is affine linear in the local time accumulated by H at its current level. As in the classical Ray-Knight representation, the excursions of H are the exploration paths of the trees of descendants of the ancestors at time $t = 0$, and the local time of H at height t measures the population size at time t (see e.g. [9]). We cope with the dependence in the reproduction by introducing a pecking order of individuals: an individual explored at time s and living at time $t = H_s$ is prone to be killed by any of its contemporaneans that have been explored so far. The proof of our main result relies on approximating H with a sequence of Harris paths H^N which figure in a Ray-Knight representation of the total mass of a branching particle system. We obtain a suitable joint convergence of H^N together with its local times and with the Girsanov densities that introduce the dependence in the reproduction.

1 Introduction

Feller’s branching diffusion with logistic growth is governed by the SDE

$$dZ_t = \sigma\sqrt{Z_t}dW_t + (\theta Z_t - \gamma Z_t^2)dt, \quad Z_0 = x > 0, \quad (1.1)$$

with positive constants σ , θ and γ . It has been studied in detail by Lambert [8], and models the evolution of the size of a large population with a near-critical reproduction. The diffusion term in (1.1) incorporates the individual offspring variance, and the drift term includes a supercriticality in the branching that is counteracted by a killing with a rate proportional to the “number of pairs of individuals”.

For $\theta = \gamma = 0$, equation (1.1) is the SDE of *Feller’s critical branching diffusion with variance parameter σ^2* . In this case, a celebrated theorem due to Ray and Knight says that Z has a representation in terms of the local times of reflected Brownian motion. To be specific, let $H = (H_s)_{s \geq 0}$ be a Brownian motion on \mathbb{R}_+ with variance parameter $4/\sigma^2$, reflected at the

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origin, and for $s, t \geq 0$ let $L_s(t, H)$ be the (semimartingale) local time accumulated by H at level t up to time s . Define

$$S_x := \inf\{s > 0 : (\sigma^2/4)L_s(0, H) \geq x\}. \quad (1.2)$$

Then $(\sigma^2/4)L_{S_x}(t, H)$, $t \geq 0$, is a weak solution of (1.1) with $\theta = \gamma = 0$, and is called the *Ray-Knight representation* of Feller's critical branching diffusion.

The Ray-Knight representation has a beautiful interpretation in an individual-based picture. Reflected Brownian motion $H = (H_s)_{s \geq 0}$ arises as a concatenation of excursions, and each of these excursions codes a *continuum random tree*, the genealogical tree of the progeny of an individual that was present at time $t = 0$. The size of this progeny at time $t > 0$ is $\sigma^2/4$ times the (total) local time spent by this excursion at level t . Starting with mass x at time $t = 0$ amounts to collecting a local time $(4/\sigma^2)x$ of H at level 0. The local time of H at level t then arises as a sum over the local time of the excursions, just as the state at time t of Feller's branching diffusion, Z_t , arises as a sum of the masses of countably many families, each of which belongs to the progeny of one ancestor that lived at time $t = 0$. The path $(H_s)_{0 \leq s \leq S_x}$ can be viewed as the *exploration path of the genealogical forest* arising from the ancestral mass x . We will briefly illustrate this in Section 3 along a discrete mass – continuous time approximation. For a more detailed explanation and some historical background we refer to the survey [13].

The motivation of the present paper was the question whether a similar picture is true also for (1.1) with strictly positive θ and γ , and whether also in this case a Ray-Knight representation is available for a suitably re-defined dynamics of an exploration process H . At first sight this seems prohibiting since the nonlinear term in (1.1) destroys the independence in the reproduction. However, it turns out that introducing an order among the individuals helps to overcome this hurdle. We will think of the individuals as being arranged “from left to right”, and decree that the pairwise fights are always won by the individual “to the left”, and lethal for the individual “to the right”. In this way we arrive at a population dynamics which leaves the evolution (1.1) of the total mass unchanged, see again the explanation in Section 3. The death rate coming from the pairwise fights leads in the exploration process of the genealogical forest to a downward drift which is proportional to $L_s(H_s, H)$, that is, proportional to the amount of mass seen to the left of the individual encountered at exploration time s (and living at real time H_s). In this way, those excursions of H which come later in the exploration tend to be smaller - the trees to the right are “under attack from those to the left”.

In quantitative terms, we will consider the stochastic differential equation

$$H_s = \frac{2}{\sigma}B_s + \frac{1}{2}L_s(0, H) + \frac{2\theta}{\sigma^2}s - \gamma \int_0^s L_r(H_r, H)dr, \quad s \geq 0, \quad (1.3)$$

where B is a standard Brownian motion. The last two terms are the above described components of the drift in the exploration process, and the term $L_s(0, H)/2$ takes care of the reflection of H at the origin. We will show

Proposition 1.1. *The SDE (1.3) has a unique weak solution.*

Our main result is the

Theorem 1. *Assume that H solves the SDE (1.3), and let, for $x > 0$, S_x be defined as in (1.2). Then $(\sigma^2/4)L_{S_x}(t, H)$, $t \geq 0$, solves (1.1).*

We will prove Proposition 1.1 by a Girsanov argument, and Theorem 1 along a discrete mass–continuous time approximation that is presented in Section 3. An important step in the proof of Theorem 1, and interesting in its own right, is Theorem 2 in Section 5, in which we obtain a convergence in distribution of processes that approximate reflected Brownian motion, together with their local times. A similar convergence result was proved in [14] for piecewise linear interpolation of discrete time random walks and their local time. The proof of Theorem 1 is completed in Section 6, using again Girsanov’s theorem. It remains a challenge to prove Theorem 1 directly by stochastic analysis. For this, one promising route is the approach of Norris, Rogers and Williams [11], about which we learned after completing this paper. The main result of [11] is a Ray-Knight theorem “of the first type” for Brownian motion with a drift depending on its local time, while ours is a Ray-Knight theorem “of the second type”, with reflection above the origin. Also, we think that our method involving an approximation by an interacting particle system is of independent interest.

2 Proof of Proposition 1.1

For abbreviation we write $L_s(t) := L_s(t, H)$. Following the Girsanov route, let B be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let H obey

$$H_s = \frac{2}{\sigma} B_s + \frac{1}{2} L_s(0), \quad (2.1)$$

with L being the semimartingale local time of H . Thus the process H is under the probability measure \mathbb{P} a reflected Brownian motion with variance parameter $4/\sigma^2$. In order to prove the existence part of Proposition 1.1 it suffices to construct, by a suitable reweighting of \mathbb{P} , a probability measure $\tilde{\mathbb{P}}$ under which

$$\tilde{B}_s := B_s - \int_0^s \left[\frac{\theta}{\sigma} - \frac{\sigma\gamma}{2} L_r(H_r) \right] dr, \quad s \geq 0, \quad (2.2)$$

is a standard Brownian motion. Indeed, (2.2) and (2.1) together imply that under $\tilde{\mathbb{P}}$ the process H solves (1.3) with \tilde{B} instead of B .

The *Girsanov condition* which makes the reweighting possible and whose validity we will check is

$$\mathbb{E} \exp \left(M_s - \frac{1}{2} \langle M \rangle_s \right) = 1, \quad s \geq 0, \quad (2.3)$$

with $M_s := \int_0^s \left[\frac{\theta}{\sigma} - \frac{\sigma\gamma}{2} L_r(H_r) \right] dB_r$.

Remark 2.1. If (2.3) holds, then there exists a measure $\tilde{\mathbb{P}}$ such that for all $s > 0$

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_s} = \exp \left(M_s - \frac{1}{2} \langle M \rangle_s \right),$$

where \mathcal{F}_s is the σ -field generated by $(H_r)_{0 \leq r \leq s}$. By Girsanov’s theorem, \tilde{B} defined by (2.2) then is a standard Brownian motion under $\tilde{\mathbb{P}}$. Thus, under $\tilde{\mathbb{P}}$, H solves (1.3) with \tilde{B} instead of B . This gives the existence part of Proposition 1.1, as soon as we have (2.3).

A sufficient condition for (2.3) is provided by the following lemma, which is Theorem 1.1, chapter 7 (page 152) in [5].

Lemma 2.2. *Assume that the quadratic variation of the continuous local martingale M is of the form $\langle M \rangle_s = \int_0^s R_r dr$, and that for all $s > 0$ there exist constants $a > 0$ and $c < \infty$ such that*

$$\mathbb{E} \exp(aR_r) \leq c, \quad 0 \leq r \leq s. \quad (2.4)$$

Then (2.3) is satisfied.

In our situation, $R_r = \left| \frac{\theta}{\sigma} - \frac{\sigma\gamma}{2} L_r(H_r) \right|^2$, for which (2.4) is implied by the following

Lemma 2.3. *Let H be a Brownian motion on \mathbb{R}_+ reflected at the origin, with variance parameter v^2 . Then for all $s > 0$ there exists $\alpha = \alpha(s, v) > 0$ and a constant $c < \infty$ such that*

$$\mathbb{E} \left(\exp(\alpha L_r(H_r)^2) \right) \leq c, \quad 0 \leq r \leq s.$$

Proof: Together with a simple scaling argument and a desintegration with respect to H_r , this is immediate from the following

Lemma 2.4. *Let β be a standard Brownian motion starting at 0, and denote by $L_1(t)$ the local time accumulated by $|\beta|$ at position t up to time 1. There exist constants $a > 0$ and $c > 0$ (not depending on t) such that*

$$\mathbb{E}[e^{aL_1(t)^2} | |\beta_1| = t] \leq c, \quad t \geq 0. \quad (2.5)$$

Proof: Denote by $K_1(x)$ the local time of β accumulated up to time 1 at position x . First observe that for $t \geq 0$

$$L_1(t) = K_1(t) + K_1(-t) \text{ a.s.} \quad (2.6)$$

For deriving (2.5), by symmetry it suffices to condition under the event $\{\beta_1 = t\}$. Writing \mathbb{P}^x for $\mathbb{P}[\cdot | \beta_1 = x]$ we conclude from (2.6) and the Cauchy-Schwarz inequality that for all $a > 0$

$$\mathbb{E}^t[e^{aL_1(t)^2}] \leq \left(\mathbb{E}^t[e^{4aK_1(t)^2}] \right)^{1/2} \left(\mathbb{E}^t[e^{4aK_1(-t)^2}] \right)^{1/2}. \quad (2.7)$$

For $u \leq 1$, the distribution of $K_1(t)$ under \mathbb{P}^t and conditioned under the event that β hits t first at time u , equals the distribution of $\sqrt{1-u} K_1(0)$ under \mathbb{P}^0 . Similarly, for $u_1, u_2 \leq 1$, the distribution of $K_1(-t)$ under \mathbb{P}^t and conditioned under the event that $(\beta_v)_{0 \leq v \leq 1}$ hits $-t$ first at time u_1 and last at time u_2 , equals the the distribution of $\sqrt{u_2 - u_1} K_1(0)$ under \mathbb{P}^0 . Consequently,

$$\mathbb{E}^t \left[e^{4aK_1(t)^2} \right] \leq \mathbb{E}^0 \left[e^{4aK_1(0)^2} \right], \quad \mathbb{E}^t \left[e^{4aK_1(-t)^2} \right] \leq \mathbb{E}^0 \left[e^{4aK_1(0)^2} \right]. \quad (2.8)$$

By a result due to Lévy (see formula (11) in [15]), $K_1(0)$ has under \mathbb{P}^0 a Raleigh distribution, i.e.

$$\mathbb{P}^0(K_1(0) > \ell) = e^{-\frac{1}{2}\ell^2}.$$

This means that $K_1^2(0)$ is exponentially distributed, and hence, for suitably small $\delta > 0$, $\mathbb{E}^0 \left[e^{\delta K_1(0)^2} \right]$ is finite. Now (2.5) follows from (2.7) and (2.8). \blacksquare

As stated above, Lemmas 2.2 and 2.3 give together with Remark 2.1 the existence part of Proposition 1.1. For its uniqueness part, assume that H is a weak solution of (1.3), governed by some measure \mathbb{P} . For $n \in \mathbb{N}$, we define

$$T_n := \inf\{r > 0 : L_r(H_r) > n\}. \quad (2.9)$$

By a Girsanov transformation we can change the measure \mathbb{P} into a measure $\bar{\mathbb{P}}$ under which, for all $n \in \mathbb{N}$,

$$\bar{B}_{s \wedge n \wedge T_n} = B_{s \wedge n \wedge T_n} + \int_0^{s \wedge n \wedge T_n} \left[\frac{\theta}{\sigma} - \gamma \frac{\sigma}{2} L_r(H_r) \right] dr, \quad s \geq 0,$$

is a standard Brownian motion stopped at $n \wedge T_n$. Then, under $\bar{\mathbb{P}}$, for all n , the process H satisfies

$$\begin{aligned} dH_s &= \frac{2}{\sigma} d\bar{B}_s + \frac{1}{2} dL_s(0), \quad 0 \leq s \leq n \wedge T_n, \\ H_0 &= 0. \end{aligned} \quad (2.10)$$

Because the weak solution of (2.10) is unique, the law of $(H_{s \wedge n \wedge T_n})_{s \geq 0}$ under $\bar{\mathbb{P}}$ is uniquely determined, and so is the law of $(H_{s \wedge n \wedge T_n})_{s \geq 0}$ under \mathbb{P} , since it can be recovered by the inverse Girsanov transformation (note that \mathbb{P} and $\bar{\mathbb{P}}$ are mutually absolutely continuous on $\mathcal{F}_{n \wedge T_n}$). From Theorem VI.1.7, page 225 of [16] one can infer that the local time $L = L_s(t)$ of H is jointly continuous on $[0, \infty) \times [0, \infty)$ in its two arguments \mathbb{P} -a.s. Consequently, $T_n \rightarrow \infty$ as $n \rightarrow \infty$, \mathbb{P} -a.s, which shows weak uniqueness for (1.3) and completes the proof of Proposition 1.1.

3 A discrete mass approximation

The aim of this section is to set up a “discrete mass - continuous time” approximation of (1.1) and (1.3). This will explain the intuition behind Theorem 1, and also will prepare for its proof.

For $x > 0$ and $N \in \mathbb{N}$ the approximation of (1.1) will be given by the total mass $Z^{N,x}$ of a population of individuals, each of which has mass $1/N$. The initial mass is $Z_0^{N,x} = \lfloor Nx \rfloor / N$, and $Z^{N,x}$ follows a Markovian jump dynamics: from its current state k/N ,

$$Z^{N,x} \text{ jumps to } \begin{cases} (k+1)/N & \text{at rate } kN\sigma^2/2 + k\theta \\ (k-1)/N & \text{at rate } kN\sigma^2/2 + k(k-1)\gamma/N. \end{cases} \quad (3.1)$$

For $\gamma = 0$, this is (up to the mass factor $1/N$) as a Galton-Watson process in continuous time: each individual independently spawns a child at rate $N\sigma^2/2 + \theta$, and dies (childless) at rate $N\sigma^2/2$. For $\gamma \neq 0$, the additional quadratic death rate destroys the independence, and hence also the branching property. However, when viewing the individuals alive at time t as being arranged “from left to right”, and by decreeing that each of the pairwise fights (which happen at rate 2γ and always end lethal for one of the two involved individuals) is won by the individual to the left, we arrive at the additional death rate $2\gamma\mathcal{L}_i(t)/N$ for individual i , where $\mathcal{L}_i(t)$ denotes the number of individuals living at time t to the left of individual i .

The just described reproduction dynamics gives rise to a forest $F^{N,x}$ of trees of descent, drawn into the plane as sketched in Figure 1. At any branch point, we imagine the “new”

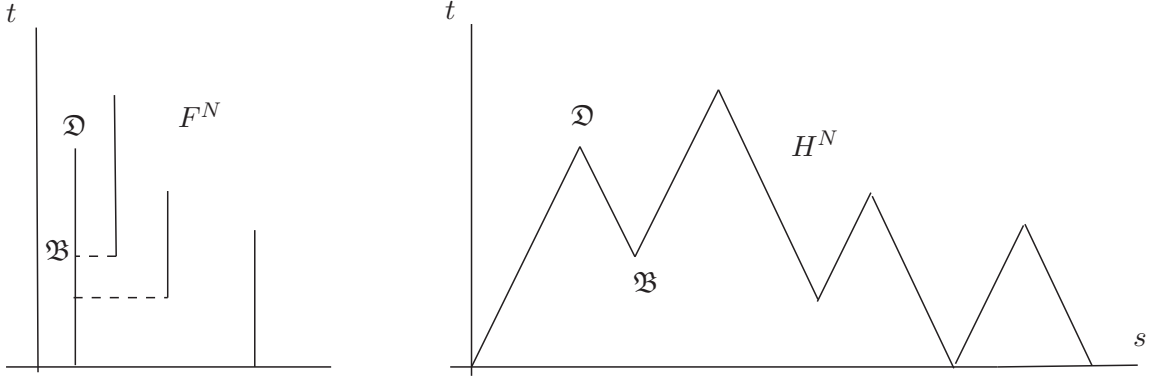


Figure 1: A realization of (the first two trees of) F^N and (the first two excursions of) its exploration path H^N . The t -axis is *real time* as well as *exploration height*, the s -axis is *exploration time*.

branch being placed to the right of the mother branch. Because of the asymmetric killing, the trees further to the right have a tendency to stay smaller: they are “under attack” by the trees to their left. Note also that, with the above described construction, the $F^{N,x}$, $x > 0$, are coupled: when x is increased by $1/N$, a new tree is added to the right. We denote the union of the $F^{N,x}$, $x > 0$, by F^N .

From F^N we read off a continuous and piecewise linear \mathbb{R}_+ -valued path $H^N = (H_s^N)$ (called the *exploration path* of F^N) in the following way:

Starting from the root of the leftmost tree, one goes upwards at speed $2N$ until one hits the top of the first mother branch (this is the leaf marked with \mathfrak{D} in Figure 1). There one turns and goes downwards, again at speed $2N$, until arriving at the next branch point (which is \mathfrak{B} in Figure 1). From there one goes upwards into the (yet unexplored) next branch, and proceeds in a similar fashion until being back at height 0, which means that the exploration of the leftmost tree is completed. Then explore the next tree, and so on. For $x > 0$ we denote by S_x^N the time at which the exploration of the forest $F^{N,x}$ is completed. Obviously, for each $t \geq 0$, the number of branches in $F^{N,x}$ that are alive at time t equals half the number of t -crossings of the exploration path of F^N stopped at S_x^N . Recalling that the slope of H^N is $\pm 2N$, we define

$$\Lambda_s^N(t) := \frac{1}{2N} \# \text{ of } t\text{-crossings of } H^N \text{ between exploration times } 0 \text{ and } s, \quad (3.2)$$

where we count a local minimum of H^N at t as two t -crossings, and a local maximum as none. Note that by our convention both $s \mapsto \Lambda_s^N(t)$ and $t \mapsto \Lambda_s^N(t)$ are right continuous, and in particular $\Lambda_0^N(0) = 0$. We call $\Lambda_s^N(t)$ the (*unscaled*) *local time* of H^N accumulated at height t up to time s . This name is justified also by the following *occupation times formula*, valid for all measurable $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$

$$\int_0^s f(H_r^N) dr = \int_0^\infty f(t) \Lambda_s^N(t) dt, \quad s \geq 0. \quad (3.3)$$

The exploration time S_x^N which it takes to traverse all of the $\lfloor Nx \rfloor$ trees in $F^{N,x}$ can be expressed as

$$S_x^N = \inf\{s : \Lambda_s^N(0) \geq \lfloor Nx \rfloor / N\}. \quad (3.4)$$

Proposition 3.1. *The exploration path $s \mapsto H_s^N$ obeys the following stochastic dynamics:*

- *At time $s = 0$, H^N starts at height 0 and with slope $2N$.*
- *While H^N moves upwards, its slope jumps from $2N$ to $-2N$ at rate $N^2\sigma^2 + 4\gamma N\ell$, where $\ell = \Lambda_s^N(H_s^N)$ is the local time accumulated by H^N at the current height H_s^N up to the current exploration time s .*
- *While H^N moves downwards, its slope jumps from $-2N$ to $2N$ at rate $N^2\sigma^2 + 2N\theta$.*
- *Whenever H^N reaches height 0, it is reflected above 0.*

PROOF: We give here an informal proof which contains the essential ideas. (A more formal proof can be carried out along the arguments of the proof of Theorem 2.4 in [1].)

Recall that the death rate of an individual i living at real time t is $N\sigma^2/2 + 2\gamma\mathcal{L}_i(t)/N$, where $\mathcal{L}_i(t)$ is the number of individuals living at time t to the left of individual i . Assume the individual i living at time t is explored first at time s , hence $H_s = t$, and H has slope $2N$ at time s . Because of (3.2), while H_s^N goes upward, we have $\mathcal{L}_i(t) = N\Lambda_s(H_s^N)$. The rate in t is the rate in s multiplied by the factor $2N$ which is the absolute value of the slope. This gives the claimed jump rate $2N(N\sigma^2/2 + 2\gamma\Lambda_s(H_s^N))$ from slope $2N$ to slope $-2N$, which can be seen as the rate at which the “death clock” rings (and leads to a downward jump of the slope) along the rising pieces of the exploration path H^N . On the other hand, the “birth clock” rings along the falling pieces of H^N , its rate being $N\sigma^2/2 + \theta$ in real time and $2N(N\sigma^2/2 + \theta)$ in exploration time, as claimed in the proposition. Note that the process of birth times along an individual’s lifetime is a homogeneous Poisson process which (in distribution) can as well be run backwards from the individual’s death time. Also note that, due to the “depth-first-search”-construction of H^N , along falling pieces of H^N always yet unexplored parts of the forest are visited as far as the birth points are concerned. ■

The next statement is a discrete version of Theorem 1, and will later be used for the proof of Theorem 1 by taking $N \rightarrow \infty$.

Corollary 3.2. *Let H^N be a stochastic process following the dynamics specified in Proposition 3.1, and Λ^N be its local time as defined by (3.2). For $x > 0$, let S_x^N be the stopping time defined by (3.4). Then $t \mapsto \Lambda_{S_x^N}^N(t)$ follows the jump dynamics (3.1).*

PROOF: By Proposition 3.1, H^N is equal in distribution to the exploration path of the random forest F^N . Hence $\Lambda_{S_x^N}^N(t)$ is equal in distribution to $Z_t^{N,x}$, where $NZ_t^{N,x}$ is the number of branches alive in $F^{N,x}$ at time t . Since $Z^{N,x}$ follows the dynamics (3.1), so does $\Lambda_{S_x^N}^N$. ■

The next lemma will also be important in the proof of Theorem 1.

Lemma 3.3. *Let H^N and S_x^N be as in Corollary 3.2. Then $S_x^N \rightarrow \infty$ a.s. as $x \rightarrow \infty$.*

PROOF: Consider $x = a/N$ for $a = 1, 2, \dots$. Applying (3.3) with $s = S_x^N$ and $f \equiv 1$ we obtain the equality $S_x^N = \int_0^\infty \Lambda_{S_x^N}^N(t) dt$.

According to Corollary 3.2, $\Lambda_{S_x^N}^N(t)$ follows the jump dynamics (3.1), with initial condition $\Lambda_{S_x^N}^N(t) = a/N$. By coupling $\Lambda_{S_x^N}^N(t)$ with a “pure death process” K^N that starts in a/N and jumps from k/N to $(k-1)/N$ at rate $k(k-1)(N\sigma^2/2 + \gamma/N)$, we see that $\int_0^\infty \Lambda_{S_x^N}^N(t) dt$ is stochastically bounded from below by $\int_0^{T_2} K_t^N dt$, where T_2 is the first time at which K^N

comes down to $2/N$. The latter integral equals a sum of independent exponentially distributed random variables with parameters $(j-1)(N^2\sigma^2/2 + \gamma)$, $j = 2, \dots, a$. This sum diverges as $a \rightarrow \infty$. ■

4 Convergence of the mass processes $Z^{N,x}$ as $N \rightarrow \infty$

The process $\{Z_t^{N,x}, t \geq 0\}$ with dynamics (3.1) is a Markov process with values in the set $E_N := \{k/N, k \geq 1\}$, starting from $\lfloor Nx \rfloor/N$, with generator A^N given by

$$\begin{aligned} A^N f(z) = Nz \left(N \frac{\sigma^2}{2} + \theta \right) \left[f \left(z + \frac{1}{N} \right) - f(z) \right] \\ + Nz \left(N \frac{\sigma^2}{2} + \gamma \left(z - \frac{1}{N} \right) \right) \left[f \left(z - \frac{1}{N} \right) - f(z) \right], \end{aligned} \quad (4.1)$$

for any $f : E_N \rightarrow \mathbb{R}$, $z \in E_N$. (Note that the distinction between symmetric and ordered killing is irrelevant here.) Applying successively the above formula to the cases $f(z) = z$ and $f(z) = z^2$, we get that

$$Z_t^{N,x} = Z_0^{N,x} + \int_0^t \left[\theta Z_r^{N,x} - \gamma Z_r^{N,x} \left(Z_r^{N,x} - \frac{1}{N} \right) \right] dr + M_t^{(1)}, \quad (4.2)$$

$$\begin{aligned} \left(Z_t^{N,x} \right)^2 = \left(Z_0^{N,x} \right)^2 + 2 \int_0^t Z_r^{N,x} \left[\theta Z_r^{N,x} - \gamma Z_r^{N,x} \left(Z_r^{N,x} - \frac{1}{N} \right) \right] dr \\ + \int_0^t \left[\sigma^2 Z_r^{N,x} + \frac{\theta}{N} Z_r^{N,x} + \frac{\gamma}{N} \left(Z_r^{N,x} - \frac{1}{N} \right) Z_r^{N,x} \right] dr + M_t^{(2)}, \end{aligned} \quad (4.3)$$

where $\{M_t^{(1)}, t \geq 0\}$ and $\{M_t^{(2)}, t \geq 0\}$ are local martingales. It follows from (4.2) and (4.3) that

$$\langle M^{(1)} \rangle_t = \int_0^t \left[\sigma^2 Z_r^{N,x} + \frac{\theta}{N} Z_r^{N,x} + \frac{\gamma}{N} \left(Z_r^{N,x} - \frac{1}{N} \right) Z_r^{N,x} \right] dr. \quad (4.4)$$

We now prove

Lemma 4.1. *For any $T > 0$,*

$$\sup_{N \geq 1} \sup_{0 \leq t \leq T} \mathbb{E} \left[\left(Z_t^{N,x} \right)^4 \right] < \infty.$$

An immediate Corollary of this Lemma is that $\{M_t^{(1)}\}$ and $\{M_t^{(2)}\}$ are in fact martingales.

PROOF: The same computation as above, but now with $f(z) = z^4$, gives

$$\left(Z_t^{N,x} \right)^4 = \left(Z_0^{N,x} \right)^4 + \int_0^t \Phi_N \left(Z_r^{N,x} \right) dr + M_t^{(4)}, \quad (4.5)$$

where $\{M_t^{(4)}, t \geq 0\}$ is a local martingale and for some $c > 0$ independent of N ,

$$\Phi_N(z) \leq c(1 + z^4). \quad (4.6)$$

We note that $NZ_t^{N,x}$ is bounded by the value at time t of a Yule process (which arises when suppressing the deaths), which is a finite sum of mutually independent geometric random

variables, hence has finite moments of any order. Hence $M^{(4)}$ is in fact a martingale. We then can take the expectation in (4.5), and deduce from (4.6) and Gronwall's Lemma that for $0 \leq t \leq T$,

$$\mathbb{E} \left[\left(Z_t^{N,x} \right)^4 \right] \leq \left[\left(Z_0^{N,x} \right)^4 + cT \right] e^{cT},$$

which implies the result. ■

We shall also need below the

Lemma 4.2. *For any $T > 0$,*

$$\sup_{N \geq 1} \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(Z_t^{N,x} \right)^2 \right] < \infty.$$

PROOF: It follows readily from (4.2) that

$$\sup_{r \leq t} |Z_r^{N,x}|^2 \leq 4|Z_0^{N,x}|^2 + 4t\theta^2 \int_0^t |Z_r^{N,x}|^2 dr + 4 \sup_{r \leq t} |M_r^{(1)}|^2.$$

This together with (4.4), Doob's L^2 -inequality for martingales and Lemma 4.1 implies the result. ■

Corollary 7.10 in the Appendix combined with (4.2), (4.4) and Lemma 4.1 guarantees that the tightness of $\{Z_0^N\}_{n \geq 1}$ implies that of $\{Z^N\}_{N \geq 1}$ in $D([0, +\infty))$.

Standard arguments exploiting (4.2) and (4.3) now allow us to deduce the convergence of the mass processes (for a detailed proof, see Méléard [12], page 21).

Proposition 4.3. *As $N \rightarrow \infty$, $Z^{N,x} \Rightarrow Z^x$, where Z^x is the unique solution of the SDE (1.1) and thus is a Feller diffusion with logistic growth.*

5 The exploration path in the case $\theta = \gamma = 0$.

Let H^N be a stochastic process as in Proposition 3.1 with $\theta = \gamma = 0$. The aim of this section is to provide a version of the joint convergence (as $N \rightarrow \infty$) of H^N and its local time which is suitable for the change of measure that will be carried through in Section 6. This is achieved in Theorem 2 and its Corollary 5.1. The proof of Theorem 2 is carried out in two major parts. The first part (Proposition 5.3) provides a refined version of the joint convergence of H^N and its local time at level 0, the second part (starting from Lemma 5.5) extends this to the other levels as well.

We define the (scaled) local time accumulated by H^N at level t up to time s as

$$L_s^N(t) := \frac{4}{\sigma^2} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t < H_u^N < t + \varepsilon\}} du$$

Note that this process is neither right- nor left-continuous as a function of s . However since the jumps are of size $O(1/N)$, the limit of L^N as $N \rightarrow \infty$ will turn out to be continuous. In fact, we will show that L^N converges as $N \rightarrow \infty$ to the semimartingale local time of the limiting process H , hence the scaling factor $4/\sigma^2$. It is readily checked that $L_{0+}^N(0) = \frac{2}{N\sigma^2}$, and

$$L_s^N(t) = \frac{4}{\sigma^2} \Lambda_s^N(t), \quad \forall s \geq 0, t \geq 0, \quad (5.1)$$

where Λ was defined in (3.2). Then with S_x^N defined in (3.4), we may rewrite

$$S_x^N = \inf\{s > 0 : L_s^N(0) \geq \frac{4}{\sigma^2}[Nx]/N\}. \quad (5.2)$$

From (5.1) and Corollary 3.2 we see that

$$Z_t^{N,x} := \frac{\sigma^2}{4} L_{S_x^N}^N(t)$$

follows the jump dynamics (3.1) in the case $\theta = \gamma = 0$.

Let $\{V_s^N, s \geq 0\}$ be the càdlàg $\{-1, 1\}$ -valued process which is such that for a. a. $s > 0$,

$$\frac{dH_s^N}{ds} = 2NV_s^N.$$

We can express L^N in terms of H^N and V^N as

$$L_s^N(t) = \frac{4}{\sigma^2} \frac{1}{2N} \left[\sum_{0 \leq r < s} \mathbf{1}_{\{H_r^N=t\}} \left(1 + \frac{1}{2}(V_r^N - V_{r-}^N) \right) + \mathbf{1}_{\{H_s^N=t\}} \mathbf{1}_{\{V_{s-}^N=-1\}} \right]$$

where we put $V_{0-} = +1$. Note that any $r < s$ at which t is a local minimum of H^N counts twice in the sum of the last line, while any $r < s$ at which t is a local maximum of H^N is not counted in the sum.

Conversely, we have

$$\begin{aligned} H_s^N &= 2N \int_0^s V_r^N dr, \\ V_s^N &= 1 + 2 \int_0^s \mathbf{1}_{\{V_{r-}^N=-1\}} dP_r^N - 2 \int_0^s \mathbf{1}_{\{V_{r-}^N=1\}} dP_r^N + \frac{N\sigma^2}{2}(L_s^N(0) - L_{0+}^N(0)), \end{aligned} \quad (5.3)$$

where $\{P_s^N, s \geq 0\}$ is a Poisson process with intensity $N^2\sigma^2$. Note that $M_s^N = P_s^N - N^2\sigma^2 s$, $s \geq 0$, is a martingale. It is easily deduced that

$$H_s^N + \frac{V_s^N}{N\sigma^2} = \frac{1}{2}L_s^N(0) + M_s^{1,N} - M_s^{2,N}, \quad (5.4)$$

where

$$M_s^{1,N} = \frac{2}{N\sigma^2} \int_0^s \mathbf{1}_{\{V_{r-}^N=-1\}} dM_r^N \quad \text{and} \quad M_s^{2,N} = \frac{2}{N\sigma^2} \int_0^s \mathbf{1}_{\{V_{r-}^N=1\}} dM_r^N \quad (5.5)$$

are two mutually orthogonal martingales. Thanks to an averaging property of the V^N (see step 2 in the proof of Proposition 5.3 below) these two martingales will converge as $N \rightarrow \infty$ to two independent Brownian motions with variance parameter $2/\sigma^2$ each. Together with the appropriate convergence of $L^N(0)$, (5.4) then gives the required convergence of H^N , see Proposition 5.3. We are now ready to state the main result of this section.

Theorem 2. *For any $x > 0$, as $N \rightarrow \infty$,*

$$(\{H_s^N, M_s^{1,N}, M_s^{2,N}, s \geq 0\}, \{L_s^N(t), s, t \geq 0\}, S_x^N) \Rightarrow (\{H_s, \frac{2}{\sigma}B_s^1, \frac{2}{\sigma}B_s^2, s \geq 0\}, \{L_s(t), s, t \geq 0\}, S_x)$$

for the topology of locally uniform convergence in s and t . Here H solves the SDE (2.1) whose driving Brownian motion B is given as

$$B_s = \frac{1}{\sqrt{2}}(B_s^1 - B_s^2),$$

L is the semimartingale local time of H , and S_x has been defined in (1.2).

An immediate consequence of this result is

Corollary 5.1. *For any $x > 0$, as $N \rightarrow \infty$,*

$$(\{H_s^N, M_s^{1,N}, M_s^{2,N}, s \geq 0\}, \{L_{S_x^N}^N(t), t \geq 0\}) \Rightarrow (\{H_s, \frac{2}{\sigma}B_s^1, \frac{2}{\sigma}B_s^2, s \geq 0\}, \{L_{S_x}(t), t \geq 0\})$$

in $C([0, \infty)) \times (D([0, \infty)))^3$.

Recall that convergence in $D([0, \infty))$ is equivalent to locally uniform convergence, provided the limit is continuous.

A first preparation for the proof of Theorem 2 is

Lemma 5.2. *The sequence $\{H^N\}$ is tight in $C([0, \infty))$.*

PROOF: To get rid of the local time term in (5.4), we consider a new pair of processes (R^N, W^N) , which is $\mathbb{R} \times \{-1, 1\}$ -valued and satisfies

$$\begin{aligned} R_s^N &= 2N \int_0^s W_r^N dr, \\ W_s^N &= 1 + 2 \int_0^s \text{sign}(R_r^N) \mathbf{1}_{\{W_{r-}^N = -\text{sign}(R_r^N)\}} dP_r^N - 2 \int_0^s \text{sign}(R_r^N) \mathbf{1}_{\{W_{r-}^N = \text{sign}(R_r^N)\}} dP_r^N. \end{aligned}$$

We observe that

$$(H^N, V^N) \equiv (|R^N|, \text{sign}(R^N)W^N).$$

Clearly tightness of $\{R^N\}$ will imply that of $\{H^N\}$, since

$$\forall s, t \quad |H_s^N - H_t^N| \leq |R_s^N - R_t^N|$$

Now we have

$$R_s^N + \frac{W_s^N}{N\sigma^2} = \frac{1}{N\sigma^2} + \frac{2}{N\sigma^2} \int_0^s (-W_{r-}^N) dM_r^N.$$

By Proposition 7.8 in the Appendix, the sequence $\{R^N\}_{N \geq 1}$ is tight, and so is $\{H^N\}_{N \geq 1}$. ■

Proposition 5.3. *Fix $x > 0$. As $N \rightarrow \infty$,*

$$\begin{aligned} (H^N, M^{1,N}, M^{2,N}, L^N(0), S_x^N) &\Rightarrow \left(H, \frac{\sqrt{2}}{\sigma}B^1, \frac{\sqrt{2}}{\sigma}B^2, L(0), S_x\right) \\ &\text{in } C([0, \infty]) \times (D([0, \infty]))^3 \times [0, \infty], \end{aligned}$$

where B^1 and B^2 are two mutually independent standard Brownian motions, and H solves the SDE

$$H_s = \frac{2}{\sigma}B_s + \frac{1}{2}L_s(0), \quad s \geq 0, \quad (5.6)$$

with $B_s := (1/\sqrt{2})(B_s^1 - B_s^2)$, and $L(0)$ denoting the local time at level 0 of H . (Note that B is again a standard Brownian motion.)

PROOF: STEP 1. Note that

- i) from Lemma 5.2, the sequence $\{H^N, N \geq 1\}$ is tight in $C([0, \infty])$;
- ii) $\sup_{s \geq 0} \frac{|V_s^N|}{N\sigma^2} \rightarrow 0$ in probability as $N \rightarrow \infty$;
- iii) from Proposition 7.8, $\{M^{1,N}, N \geq 1\}$ and $\{M^{2,N}, N \geq 1\}$ are tight in $D([0, \infty])$, any limiting martingales M^1 and M^2 being continuous;
- iv) it follows from the first 3 items, (5.4) and Proposition 7.7 that $\{L_s^N(0), N \geq 1\}$ is tight in $D([0, \infty])$, the limit K of any converging subsequence being continuous and increasing.

Working along a diagonal subsequence we can extract a subsequence, still denoted as an abuse like the original sequence, such that

$$(H^N, M^{1,N}, M^{2,N}, L^N(0)) \Rightarrow (H, M^1, M^2, K).$$

STEP 2. We claim that for any $s > 0$,

$$\int_0^s \mathbf{1}_{\{V_{r-}^N=1\}} dr \rightarrow \frac{s}{2}, \quad \int_0^s \mathbf{1}_{\{V_{r-}^N=-1\}} dr \rightarrow \frac{s}{2}$$

in probability, as $N \rightarrow \infty$. To see this, let A_s^N (resp. I_s^N) denote the number of local maxima (resp. minima) of the process H^N on the interval $]0, s]$. We have

$$I_s^N \leq A_s^N \leq I_s^N + 1 \tag{5.7}$$

and

$$P_s^N \leq A_s^N + I_s^N \leq P_s^N + NL_s^N(0),$$

where $\{P_s^N, s \geq 0\}$ is the Poisson processes with intensity $N^2\sigma^2$ appearing in (5.3). We deduce from this system of inequalities, the law of large numbers for Poisson processes, and the tightness of $\{L^N(0), N \geq 1\}$ that

$$\frac{A_s^N + I_s^N}{\sigma^2 N^2} \rightarrow s \quad \text{in probability, as } N \rightarrow \infty. \tag{5.8}$$

From (5.7) and (5.8) we have

$$\frac{A_s^N}{\sigma^2 N^2} \rightarrow \frac{s}{2}, \quad \frac{I_s^N}{\sigma^2 N^2} \rightarrow \frac{s}{2} \quad \text{in probability, as } N \rightarrow \infty.$$

Now

$$\begin{aligned} A_s^N &= \int_0^s \mathbf{1}_{\{V_{r-}^N=1\}} dP_r^N \\ &= \sigma^2 N^2 \int_0^s \mathbf{1}_{\{V_r^N=1\}} dr + \int_0^s \mathbf{1}_{\{V_{r-}^N=1\}} dM_r^N. \end{aligned}$$

Consequently

$$\begin{aligned} \int_0^s \mathbf{1}_{\{V_r^N=1\}} dr &= \frac{A_s^N}{\sigma^2 N^2} - (\sigma^2 N^2)^{-1} \int_0^s \mathbf{1}_{\{V_{r-}^N=1\}} dM_r^N \\ &\rightarrow \frac{s}{2} \quad \text{in probability, as } N \rightarrow \infty. \end{aligned}$$

Indeed

$$\begin{aligned} \mathbb{P} \left(\left| \int_0^s \mathbf{1}_{\{V_{r^-}^N = -1\}} dM_r^N \right| > \varepsilon N^2 \right) &\leq \frac{\sqrt{\mathbb{E}(|M_s^N|^2)}}{\varepsilon N^2} \\ &\leq \frac{\sqrt{\sigma^2 N^2 s}}{\varepsilon N^2} \\ &\rightarrow 0, \quad \text{as } N \rightarrow \infty. \end{aligned}$$

The second part of the claim follows, since

$$\int_0^s \mathbf{1}_{\{V_r^N = -1\}} dr = s - \int_0^s \mathbf{1}_{\{V_r^N = 1\}} dr.$$

STEP 3. By Step 1 iii), M^1 and M^2 are continuous martingales. In order to identify them, we first introduce some useful notation. We write $M_t^{\otimes 2}$ for the 2×2 matrix whose (i, j) -entry equals $M_t^i \times M_t^j$, where M_t^i is the i -th coordinate of the vector $M_t := \begin{pmatrix} M_t^1 \\ M_t^2 \end{pmatrix}$. We denote by $\langle\langle M \rangle\rangle$ the $\mathbb{R}^{2 \times 2}$ -valued predictable process which is such that

$$M^{\otimes 2} - \langle\langle M \rangle\rangle$$

is a martingale, and note that the (i, j) -entry of the matrix $\langle\langle M \rangle\rangle_t$ equals $\langle M^i, M^j \rangle_t$. Likewise we proceed with $M^{1,N}$ and $M^{2,N}$.

From Step 2 we see that

$$\begin{aligned} \langle\langle \begin{pmatrix} M^{1,N} \\ M^{2,N} \end{pmatrix} \rangle\rangle_s &= \frac{4}{\sigma^2} \int_0^s \begin{pmatrix} \mathbf{1}_{\{V_r^N = -1\}} & 0 \\ 0 & \mathbf{1}_{\{V_r^N = 1\}} \end{pmatrix} dr \\ &\rightarrow \frac{2}{\sigma^2} s I \end{aligned}$$

in probability, locally uniformly in s , where I denotes the 2×2 identity matrix. Consequently

$$\begin{pmatrix} M_s^{1,N} \\ M_s^{2,N} \end{pmatrix}^{\otimes 2} - \langle\langle \begin{pmatrix} M^{1,N} \\ M^{2,N} \end{pmatrix} \rangle\rangle_s \Rightarrow \begin{pmatrix} M_s^1 \\ M_s^2 \end{pmatrix}^{\otimes 2} - \frac{2}{\sigma^2} s I$$

in $D([0, \infty); \mathbb{R}^4)$ and since weak limits of martingales are local martingales, there exist two mutually independent standard Brownian motions B^1 and B^2 such that

$$M_s^1 = \frac{\sqrt{2}}{\sigma} B_s^1, \quad M_s^2 = \frac{\sqrt{2}}{\sigma} B_s^2, \quad s \geq 0.$$

Taking the weak limit in (5.4) we deduce that

$$\begin{aligned} H_s &= \frac{\sqrt{2}}{\sigma} (B_s^1 - B_s^2) + \frac{1}{2} K_s \\ &= \frac{\sqrt{2}}{\sigma} B_s + \frac{1}{2} K_s, \end{aligned}$$

where $B_s = (B_s^1 - B_s^2)/\sqrt{2}$ is also a standard Brownian motion. STEP 4. For each $\ell \geq 1$, we

define the function $f_\ell : \mathbb{R}_+ \rightarrow [0, 1]$ by $f_\ell(x) = (1 - \ell x)^+$. We have that for each $N, \ell \geq 1, s > 0$, since $L^N(0)$ increases only when $H^N = 0$,

$$\mathbb{E}\left(\int_0^s f_\ell(H_r^N) dL_r^N(0) - L_s^N(0)\right) \geq 0.$$

Thanks to Lemma 7.3 in the Appendix we can take the limit in this last inequality as $N \rightarrow \infty$, yielding

$$\mathbb{E}\left(\int_0^s f_\ell(H_r) dK_r - K_s\right) \geq 0.$$

Then taking the limit as $\ell \rightarrow \infty$ yields

$$\mathbb{E}\left(\int_0^s \mathbf{1}_{\{H_r=0\}} dK_r - K_s\right) \geq 0.$$

But the random variable under the expectation is clearly nonpositive, hence it is zero a.s., in other words

$$K_s = \int_0^s \mathbf{1}_{\{H_r=0\}} dK_r, \quad \forall s \geq 0,$$

which means that the process K increases only when $H_r = 0$.

From the occupation times formula

$$\frac{4}{\sigma^2} \int_0^s g(H_r) dr = \int_0^\infty g(t) L_s(t) dt$$

applied to the function $g(h) = \mathbf{1}_{\{h=0\}}$, we deduce that the time spent by the process H at 0 has a.s. zero Lebesgue measure. Consequently

$$\int_0^s \mathbf{1}_{\{H_r=0\}} dB_r \equiv 0 \quad a.s.$$

hence a.s.

$$B_s = \int_0^s \mathbf{1}_{\{H_r>0\}} dB_r \quad \forall s \geq 0.$$

It then follows from Tanaka's formula applied to the process H and the function $h \rightarrow h^+$ that $K = L(0)$. STEP 5. We have proved so far that $Q^N \Rightarrow Q$, where $Q^N = (H^N, M^{1,N}, M^{2,N}, L^N(0))$,

$Q = (H, \frac{\sqrt{2}}{\sigma} B^1, \frac{\sqrt{2}}{\sigma} B^2, L(0))$. Note that not only subsequences but the entire sequence Q^1, Q^2, Q^3, \dots converges, since the limit law is uniquely characterized. STEP 6. It remains

to check that for any $x > 0$

$$(Q^N, S_x^N) \Rightarrow (Q, S_x) \quad \text{in } C([0, \infty]) \times (D([0, \infty]))^3 \times [0, \infty].$$

To this end, let us define the function Φ from $\mathbb{R}_+ \times C_\uparrow(\mathbb{R}_+)$ into \mathbb{R}_+ by

$$\Phi(x, y) = \inf\{s > 0 : y(s) > \frac{4}{\sigma^2} x\}.$$

For any x fixed, the function $\Phi(x, \cdot)$ is continuous in the neighborhood of a function y which is strictly increasing at the time when it first reaches the value x . Clearly $S_x = \Phi(x, L(\cdot))$. Define

$$S_x'^N := \Phi(x, L^N(\cdot)).$$

We note that for any $x > 0, s \mapsto L_s(0)$ is a.s. strictly increasing at time S_x , which is a stopping time. This fact follows from the strong Markov property, the fact that $H_{S_x} = 0$, and $L_\varepsilon(0) > 0$, for all $\varepsilon > 0$. Consequently S_x is a.s. a continuous function of the trajectory $L(\cdot)$, then also of Q , and

$$(Q^N, S_x'^N) \Rightarrow (Q, S_x).$$

It remains to prove that $S_x'^N - S_x^N \rightarrow 0$ in probability. For any $y < x$ and N large enough

$$0 \leq S_x'^N - S_x^N \leq S_x'^N - S_y'^N.$$

Clearly $S_x'^N - S_y'^N \Rightarrow S_x - S_y$, hence for any $\varepsilon > 0$,

$$0 \leq \limsup_N \mathbb{P}(S_x'^N - S_x^N \geq \varepsilon) \leq \limsup_N \mathbb{P}(S_x'^N - S_y'^N \geq \varepsilon) \leq \mathbb{P}(S_x - S_y \geq \varepsilon).$$

The result follows, since $S_y \rightarrow S_{x-}$ as $y \rightarrow x, y < x$, and $S_{x-} = S_x$ a.s. ■

For the proof of Theorem 2 we will need the following lemmata:

Lemma 5.4. *For any $s > 0, t > 0$, the following identities hold a.s.*

$$\begin{aligned} (H_s^N - t)^+ &= 2N \int_0^s V_r^N \mathbf{1}_{\{H_r^N > t\}} dr, \\ V_s^N \mathbf{1}_{\{H_s^N > t\}} &= \frac{\sigma^2 N}{2} L_s^N(t) + \int_0^s \mathbf{1}_{\{H_r^N > t\}} dV_r^N. \end{aligned}$$

PROOF: The first identity is elementary, and is true along any piecewise linear, continuous trajectory $\{H_r^N\}$ satisfying $dH_s^N/ds = 2NV_s^N$ for almost all s , with $V_s^N \in \{-1, 1\}$. The other identities which we will state in this proof are true a.s. In these identities we exclude the trajectories of H^N which have a local maximum or minimum at the level t . This implies that the two processes $s \rightarrow V_s^N$ and $s \rightarrow \mathbf{1}_{\{H_s^N > t\}}$ do not jump at the same time. Hence from

$$\begin{aligned} \mathbf{1}_{\{H_s^N > t\}} &= \sum_{0 < r < s} \mathbf{1}_{\{H_r^N = t\}} V_r^N - \mathbf{1}_{\{V_s^N = -1\}} \mathbf{1}_{\{H_s^N = t\}} \\ &= \frac{\sigma^2 N}{2} \int_0^s V_r^N dL_r^N(t), \end{aligned}$$

we deduce by differentiating the product that

$$V_s^N \mathbf{1}_{\{H_s^N > t\}} = \frac{\sigma^2 N}{2} \int_0^s (V_r^N)^2 dL_r^N(t) + \int_0^s \mathbf{1}_{\{H_r^N > t\}} dV_r^N.$$

Since $(V_r^N)^2 = 1$, this is the second identity in the lemma. ■

Lemma 5.5. *Denote by $L_s(t)$ the local time at level t up to time s of H . Then with probability one $(s, t) \mapsto L_s(t)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R} .*

PROOF: This is Theorem VI.1.7 page 225 of Revuz, Yor [16]. ■

Proposition 5.6. For each $d \geq 1$, $0 \leq t_1 < t_2 < \dots < t_d$,

$$\{(H_s^N, L_s^N(t_1), L_s^N(t_2), \dots, L_s^N(t_d)), s \geq 0\} \Rightarrow \{(H_s, L_s(t_1), L_s(t_2), \dots, L_s(t_d)), s \geq 0\}$$

in $C([0, \infty)) \times (D([0, \infty)))^d$.

PROOF: We prove the result in case $d = 1$ only, the proof of the general case being very similar. From (5.3) and Lemma 5.4 we deduce that for any $t \geq 0$, a. s.

$$L_s^N(t) = 2(H_s^N - t)^+ + \frac{2}{N\sigma^2} V_s^N \mathbf{1}_{\{H_s^N > t\}} - 2 \int_0^s \mathbf{1}_{\{H_r^N > t\}} (dM_r^{1,N} - dM_r^{2,N}), \quad s \geq 0. \quad (5.9)$$

Let

$$U_s^N = \int_0^s \mathbf{1}_{\{H_r^N > t\}} (dM_r^{1,N} - dM_r^{2,N}).$$

By Proposition 7.8 we have that $\{U^N\}_{N \geq 1}$ is tight in $D([0, \infty))$. Moreover,

$$\langle M^{1,N} - M^{2,N} \rangle_s = \frac{4}{\sigma^2} s \quad (5.10)$$

$$\langle U^N \rangle_s = \langle U^N, M^{1,N} - M^{2,N} \rangle_s = \frac{4}{\sigma^2} \int_0^s \mathbf{1}_{\{H_r^N > t\}} dr. \quad (5.11)$$

From the occupation times formula

$$\int_0^s \mathbf{1}_{\{H_r = t\}} dr = \frac{\sigma^2}{4} \int_0^\infty \mathbf{1}_{\{r=t\}} L_s(r) dr = 0 \quad a.s.$$

Then by Lemma 7.2 from the Appendix we deduce that along an appropriate sequence

$$\left\{ \int_0^s \mathbf{1}_{\{H_r^N > t\}} dr, s \geq 0 \right\} \Rightarrow \left\{ \int_0^s \mathbf{1}_{\{H_r > t\}} dr, s \geq 0 \right\} \quad (5.12)$$

From (5.10), (5.11) and (5.12), we have again along an appropriate subsequence

$$(U_s^N, M_s^{1,N} - M_s^{2,N}) \Rightarrow \left(\frac{2}{\sigma} \int_0^s \mathbf{1}_{\{H_r > t\}} dB_r, \frac{2}{\sigma} B_s \right) \quad \text{in} \quad (D([0, \infty)))^2.$$

Moreover, arguments similar to that used in the proof of Proposition 5.3 establish that

$$(H_s^N, U_s^N) \Rightarrow \left(H_s, \frac{2}{\sigma} \int_0^s \mathbf{1}_{\{H_r > t\}} dB_r \right) \quad \text{in} \quad (D([0, \infty)))^2.$$

Now from any subsequence, we can extract a subsequence along which we can take the weak limit in (5.9). But Tanaka's formula gives us the identity

$$L_s(t) = 2(H_s - t)^+ - \frac{4}{\sigma} \int_0^s \mathbf{1}_{\{H_r > t\}} dB_r,$$

which characterizes the limit of L^N as the local time of H . Since the law of H is uniquely characterized, the whole sequence converges. \blacksquare

Proposition 5.7. For each $s \geq 0$ fixed, $\{L_s^N(t), t \geq 0\}_{N \geq 1}$ is tight in $D([0, \infty))$.

PROOF: We have

$$\begin{aligned} L_s^N(t) &= 2(H_s^N - t)^+ + \frac{2}{N\sigma^2} V_s^N \mathbf{1}_{\{H_s^N > t\}} + \frac{4}{N\sigma^2} \int_0^s V_{r-}^N \mathbf{1}_{\{H_r^N > t\}} dM_r^N \\ &= K_t^N + G_t^N, \end{aligned} \quad (5.13)$$

where

$$\begin{aligned} K_t^N &= 2(H_s^N - t)^+ + \frac{2}{N\sigma^2} V_s^N \mathbf{1}_{\{H_s^N > t\}}, \\ G_t^N &= \frac{4}{N\sigma^2} \int_0^s V_{r-}^N \mathbf{1}_{\{H_r^N > t\}} dM_r^N. \end{aligned}$$

We shall apply Proposition 7.7. Since

$$\begin{aligned} \{K_0^N = 2H_s^N + \frac{2}{N\sigma^2} V_s^N \mathbf{1}_{\{H_s^N > 0\}}, N \geq 1\} &\text{ is tight and} \\ \limsup_{N \rightarrow \infty} |K_t^N - K_{t'}^N| &\leq 2|t - t'|, \end{aligned}$$

the sequence $\{K_t^N\}_{N \geq 1}$ satisfies the conditions of Proposition 7.5, and we now show that the sequence $\{G_t^N\}_{N \geq 1}$ satisfies the conditions of Proposition 7.6.

Condition (1) follows easily from the fact that $\mathbb{E}(|G_t^N|^2) \leq 16s/\sigma^2$. In order to verify condition (2), we will show that for any $T > 0$, there exists $C > 0$ such that for any $0 < t < T, \varepsilon > 0$,

$$\mathbb{E}[(G_{t+\varepsilon}^N - G_t^N)^2 (G_t^N - G_{t-\varepsilon}^N)^2] \leq C(\varepsilon^{3/2} + \varepsilon^2).$$

In order to simplify the notations below we let

$$\begin{aligned} \varphi_r^N &:= V_{r-}^N \mathbf{1}_{\{t-\varepsilon < H_r^N \leq t\}}, \\ \psi_r^N &:= V_{r-}^N \mathbf{1}_{\{t < H_r^N \leq t+\varepsilon\}}. \end{aligned}$$

An essential property, which will be crucial below, is that $\varphi_r^N \psi_r^N = 0$. Also $(\varphi_r^N)^2 = |\varphi_r^N|$, and similarly for ψ_r^N , since those functions take their values in the set $\{-1, 0, 1\}$. The quantity we want to compute equals up to a fixed multiplicative constant

$$N^{-4} \mathbb{E} \left[\left(\int_0^s \varphi_r^N dM_r^N \right)^2 \left(\int_0^s \psi_r^N dM_r^N \right)^2 \right].$$

We note that we have the identity

$$\left(\int_0^s \varphi_r^N dM_r^N \right)^2 = 2 \int_0^s \int_0^{r-} \varphi_u^N dM_u^N \varphi_r^N dM_r^N + \int_0^s |\varphi_r^N| dM_r^N + \sigma^2 N^2 \int_0^s |\varphi_r^N| dr,$$

and similarly with φ^N replace by ψ^N . Because $\varphi_r^N \psi_r^N = 0$, the expectation of the product of

$$\int_0^s \int_0^{r-} \varphi_u^N dM_u^N \varphi_r^N dM_r^N \quad \text{or} \quad \int_0^s |\varphi_r^N| dM_r^N$$

with

$$\int_0^s \int_0^{r-} \psi_u^N dM_u^N \psi_r^N dM_r^N \quad \text{or} \quad \int_0^s |\psi_r^N| dM_r^N$$

vanishes. We only need to estimate the expectations

$$\mathbb{E}\left(\int_0^s \int_0^{r^-} \varphi_u^N dM_u^N \varphi_r^N dM_r^N \int_0^s |\psi_r^N| dr\right), \mathbb{E}\left(\int_0^s |\varphi_r^N| dM_r^N \int_0^s |\psi_r^N| dr\right),$$

and $\mathbb{E}\left(\int_0^s |\varphi_r^N| dr \int_0^s |\psi_r^N| dr\right),$

together with similar quantities with φ^N and ψ^N interchanged. The estimates of the first two expectations are very similar. We estimate the second one as follows, using the Cauchy-Schwarz inequality, and Lemma 5.8 below :

$$\begin{aligned} \mathbb{E}\left(\int_0^s |\varphi_r^N| dM_r^N \int_0^s |\psi_r^N| dr\right) &\leq \sqrt{\mathbb{E} \int_0^s |\varphi_r^N|^2 d\langle M^N \rangle_r} \sqrt{\mathbb{E} \left[\left(\int_0^s |\psi_r^N| dr\right)^2\right]} \\ &\leq CN\varepsilon^{3/2} \end{aligned}$$

Finally, again from Lemma 5.8,

$$\mathbb{E}\left(\int_0^s |\varphi_r^N| dr \int_0^s |\psi_r^N| dr\right) \leq C\varepsilon^2.$$

The first quantity should be multiplied by N^2 , and the second by N^4 , and then both should be divided by N^4 . The proposition is established. \blacksquare

Lemma 5.8. *Let $s, \varepsilon, T > 0$. Then there exists a constant C such that for all $N \geq 1$ and $0 < t, t' < T$,*

$$\begin{aligned} \mathbb{E}\left(\int_0^s \mathbf{1}_{\{t-\varepsilon < H_r^N \leq t\}} dr\right) &\leq C\varepsilon, \\ \mathbb{E}\left(\int_0^s \mathbf{1}_{\{t-\varepsilon < H_r^N \leq t\}} dr \int_0^s \mathbf{1}_{\{t'-\varepsilon < H_r^N \leq t'\}} dr\right) &\leq C\varepsilon^2. \end{aligned}$$

PROOF: We will prove the second inequality, the first one follows from the second one with $t = t'$ and the Cauchy-Schwarz inequality.

For $s, t > 0$ define $F_s^N(t) := \int_0^s \mathbf{1}_{\{0 \leq H_r^N \leq t\}} dr$. It follows readily from the definition of L^N that

$$\frac{\partial F_s^N}{\partial t}(t) = \frac{\sigma^2}{4} L_s^N(t).$$

Hence

$$\begin{aligned} \mathbb{E}\left(\int_0^s \mathbf{1}_{\{t-\varepsilon < H_r^N \leq t\}} dr \int_0^s \mathbf{1}_{\{t'-\varepsilon < H_r^N \leq t'\}} dr\right) &= \frac{\sigma^4}{16} \mathbb{E}\left(\int_{t-\varepsilon}^t L_s^N(r) dr \int_{t'-\varepsilon}^{t'} L_s^N(u) du\right) \\ &= \frac{\sigma^4}{16} \mathbb{E}\left(\int_{t-\varepsilon}^t \int_{t'-\varepsilon}^{t'} L_s^N(r) L_s^N(u) dr du\right) \\ &= \frac{\sigma^4}{16} \int_{t-\varepsilon}^t \int_{t'-\varepsilon}^{t'} \mathbb{E}(L_s^N(r) L_s^N(u)) dr du \\ &\leq \frac{\sigma^4}{16} \varepsilon^2 \sup_{0 \leq r, u \leq T} \mathbb{E}(L_s^N(r) L_s^N(u)) \\ &= \frac{\sigma^4}{16} \varepsilon^2 \sup_{0 \leq r \leq T} \mathbb{E}((L_s^N(r))^2). \end{aligned}$$

On the other hand, since by Itô's formula there exists a martingale \bar{M}_s^N such that

$$(H_s^N)^2 + \frac{2}{N\sigma^2} H_s^N V_s^N = \frac{4}{\sigma^2} s + \bar{M}_s^N.$$

we conclude that

$$\sup_{N \geq 1} \mathbb{E}((H_s^N)^2) < \infty.$$

The second inequality now follows from (5.13). \blacksquare

Proposition 5.9. *For all $d > 1$, $0 \leq s_1 < s_2 < \dots < s_d$,*

$$(H^N, L_{s_1}^N, \dots, L_{s_d}^N) \Rightarrow (H, L_{s_1}, \dots, L_{s_d}) \quad \text{in } C([0, \infty)) \times (D([0, \infty)))^d.$$

PROOF: We prove the result in the case $d = 1$ only, the proof in the general case being very similar. From Proposition 5.6 there follows in particular that for all $k \geq 1$, $0 \leq t_1 < t_2 < \dots < t_k$, we have

$$(H^N, L_s^N(t_1), L_s^N(t_2), \dots, L_s^N(t_k)) \Rightarrow (H, L_s(t_1), L_s(t_2), \dots, L_s(t_k))$$

That is, $\{L_s^N\}$ converges in finite-dimensional distributions to $\{L_s\}$, jointly with H^N . By Proposition 5.7, $\{L_s^N(t), t \geq 0\}_{N \geq 1}$ is tight. The result follows. \blacksquare

We are now prepared to complete the

PROOF OF THEOREM 2: The main task is to combine the assertions of Propositions 5.6 and 5.9, which means to turn the ‘‘partial’’ convergences asserted for L^N in these propositions into a convergence that is joint in s and t . We will also combine this result with Proposition 5.3 in order to get joint convergence of all our processes. To facilitate the reading, we will divide the proof into several steps. STEP 1. Let $\{s_n, n \geq 1\}$ denote a countable dense subset

of \mathbb{R}_+ . Our first claim is that for all $n \in \mathbb{N}$,

$$(H^N, M^{1,N}, M^{2,N}, L_{s_1}^N, \dots, L_{s_n}^N, S_x^N) \Rightarrow (H, \frac{2}{\sigma} B^1, \frac{2}{\sigma} B^2, L_{s_1}, \dots, L_{s_n}, S_x) \quad (5.14)$$

in $C(\mathbb{R}_+) \times D(\mathbb{R}_+)^{n+2} \times \mathbb{R}_+$.

To make the core of the argument clear, let us write just for the moment

$$Y^N := (M^{1,N}, M^{2,N}, S_x^N), Y := (\frac{2}{\sigma} B^1, \frac{2}{\sigma} B^2, S_x), \Lambda^N := (L_{s_1}^N, \dots, L_{s_n}^N), \Lambda := (L_{s_1}, \dots, L_{s_n}).$$

Then (5.14) translates into

$$(H^N, Y^N, \Lambda^N) \Rightarrow (H, Y, \Lambda). \quad (5.15)$$

By Proposition 5.3, $(H^N, Y^N) \Rightarrow (H, Y)$, and by Proposition 5.9, $(H^N, \Lambda^N) \Rightarrow (H, \Lambda)$. Because in our situation Λ is a.s. a function of H , these two convergences imply (5.15). (More generally, this implication would be true if Y and Λ would be conditionally independent given H .) STEP 2. Now having established (5.14), it follows from a well known theorem due to

Skorohod that all the processes appearing there can be constructed on a joint probability space, such that there exists an event \mathcal{N} with $\mathbb{P}(\mathcal{N}) = 0$ and for all $\omega \notin \mathcal{N}$,

$$S_x^N(\omega) \rightarrow S_x(\omega), \quad (5.16)$$

$$(H_s^N(\omega), M_s^{1,N}(\omega), M_s^{2,N}(\omega)) \rightarrow (H_s(\omega), \frac{2}{\sigma}B_s^1(\omega), \frac{2}{\sigma}B_s^2(\omega)) \quad \text{locally uniformly in } s \geq 0, \quad (5.17)$$

and for all $n \geq 1$,

$$L_{s_n}^N(t)(\omega) \rightarrow L_{s_n}(t)(\omega) \quad \text{locally uniformly in } t \geq 0, \quad (5.18)$$

as $N \rightarrow \infty$. Here we have made use of Lemma 5.5, which allows us to assume that $(s, t) \mapsto L_s(t)(\omega)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R} for all $\omega \notin \mathcal{N}$, possibly at the price of enlarging the null set \mathcal{N} , and of Lemma 7.1 from the Appendix. STEP 3. We claim that in the situation

described in the previous step one even has for all $C, T > 0, \omega \notin \mathcal{N}$,

$$\sup_{0 \leq s \leq C, 0 \leq t \leq T} |L_s^N(t, \omega) - L_s(t, \omega)| \rightarrow 0, \quad (5.19)$$

as $N \rightarrow \infty$. In other words, in Skorokhod's construction there is a.s. convergence of $L_s^N(t)$ to $L_s(t)$, locally uniform in s and t . To prove (5.19), we will make use of the fact that for any $\omega \notin \mathcal{N}$, and all N, t , the mapping $s \mapsto L_s^N(t)(\omega)$ is increasing and the mapping $s \mapsto L_s(t)(\omega)$ is continuous and increasing. Moreover, since the mapping $(s, t) \mapsto L_s(t, \omega)$ is continuous from the compact set $[0, C] \times [0, T]$ into \mathbb{R}_+ , for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 \leq s < s' \leq C, 0 \leq t \leq T$ and $s' - s \leq \delta$ implies that

$$L_{s'}(t, \omega) - L_s(t, \omega) \leq \varepsilon.$$

Hence there exists $k \geq 1$ and $0 =: s_0 < r_1 < \dots < r_k := C$ such that $\{r_i, 0 \leq i < k\} \subset \{s_n, n \geq 1\}$ and moreover, $r_i - r_{i-1} \leq \delta$ for all $1 \leq i \leq k$. We have

$$\sup_{0 \leq s \leq C, 0 \leq t \leq T} |L_s^N(t, \omega) - L_s(t, \omega)| \leq \sup_{1 \leq i \leq k} [A_{N,i} + B_{N,i}]$$

where

$$A_{N,i} = \sup_{r_{i-1} \leq s \leq r_i, 0 \leq t \leq T} (L_s^N(t, \omega) - L_s(t, \omega))^+ \\ B_{N,i} = \sup_{r_{i-1} \leq s \leq r_i, 0 \leq t \leq T} (L_s^N(t, \omega) - L_s(t, \omega))^-.$$

For $r_{i-1} \leq s \leq r_i$,

$$(L_s^N(t, \omega) - L_s(t, \omega))^+ \leq (L_{r_i}^N(t, \omega) - L_s(t, \omega))^+ \\ \leq (L_{r_i}^N(t, \omega) - L_{r_i}(t, \omega))^+ + \varepsilon, \\ (L_s^N(t, \omega) - L_s(t, \omega))^- \leq (L_{r_{i-1}}^N(t, \omega) - L_s(t, \omega))^- \\ \leq (L_{r_{i-1}}^N(t, \omega) - L_{r_{i-1}}(t, \omega))^- + \varepsilon.$$

Finally,

$$\sup_{0 \leq s \leq C, 0 \leq t \leq T} |L_s^N(t, \omega) - L_s(t, \omega)| \leq 2 \sup_{0 \leq i \leq k} \sup_{0 \leq t \leq T} |L_{r_i}^N(t, \omega) - L_{r_i}(t, \omega)| + 2\varepsilon,$$

while from (5.18),

$$\limsup_{N \rightarrow \infty} \sup_{0 \leq s \leq C, 0 \leq t \leq T} |L_s^N(t, \omega) - L_s(t, \omega)| \leq 2\varepsilon.$$

This implies (5.19), since $\varepsilon > 0$ is arbitrary. The assertion of Theorem 2 is now immediate by combining (5.16), (5.17) and (5.19). \blacksquare

6 Change of measure and proof of Theorem 1

As in the previous section, let, for fixed $N \in \mathbb{N}$, H^N be a process that follows the dynamics described in Proposition 3.1 for $\theta = \gamma = 0$. We denote the underlying probability measure by \mathbb{P} , and the filtration by $\mathcal{F} = (\mathcal{F}_s)$. Our first aim is to construct, by a Girsanov reweighting of the restrictions $\mathbb{P}|_{\mathcal{F}_s}$, a measure $\tilde{\mathbb{P}}^N$ under which H^N follows the dynamics from Proposition 3.1 for a prescribed $\theta \geq 0$ and $\gamma > 0$.

Here, a crucial role is played by the point process P^N of the successive local maxima and minima of H^N , excluding the minima at height 0. Under \mathbb{P} , this is a Poisson process with intensity $\sigma^2 N^2$. More precisely, the process $Q^{1,N}$ which counts the successive local minima of H^N (except those at height 0) is a point process with predictable intensity $\lambda_s^{1,N} := N^2 \sigma^2 \mathbf{1}_{\{V_{s-}^N = -1\}}$, and the process $Q^{2,N}$ which counts the successive local maxima of H^N is a point process with predictable intensity $\lambda_s^{2,N} := N^2 \sigma^2 \mathbf{1}_{\{V_{s-}^N = +1\}}$. (Recall that the process V^N is the (càdlàg) sign of the slope of H^N .)

For the rest of this section we fix $\theta \geq 0$ and $\gamma > 0$. In view of Proposition 3.1 we want to change the rate $\lambda_s^{1,N}$ to $\lambda_s^{1,N} (1 + \frac{2\theta}{N\sigma^2})$ and the rate $\lambda_s^{2,N}$ to $\lambda_s^{2,N} (1 + \frac{4\gamma L_s^N(H_s)}{N\sigma^2})$.

As in Section 5 we will use the process $M_s^N = P_s^N - N\sigma^2 s$, $s \geq 0$, which is a martingale under \mathbb{P} . Taking the route designed by Proposition 7.15, we consider the local martingales

$$X_s^{N,1} := \int_0^s \frac{2\theta}{N\sigma^2} \mathbf{1}_{\{V_{r-}^N = -1\}} dM_r^N, \quad X_s^{N,2} := \int_0^s \frac{\gamma L_r^N(H_r^N)}{N} \mathbf{1}_{\{V_{r-}^N = 1\}} dM_r^N, \quad X^N := X^{N,1} + X^{N,2}.$$

Let $Y^N := \mathcal{E}(X^N)$ denote the Doléans exponential of X^N , which is equivalent to Y^N being the solution of

$$Y_s^N = 1 + \int_0^s Y_{r-}^N \left(\frac{2\theta}{N\sigma^2} \mathbf{1}_{\{V_{r-}^N = -1\}} + \frac{\gamma L_r^N(H_r^N)}{N} \mathbf{1}_{\{V_{r-}^N = 1\}} \right) dM_r^N, \quad s \geq 0, \quad (6.1)$$

see Proposition 7.11. We will show that Y^N is a martingale, which will directly render the required change of measure.

Proposition 6.1. *Y^N is a $(\mathcal{F}, \mathbb{P})$ -martingale.*

PROOF: The proof amounts to showing that, for all $s > 0$,

$$\mathbb{E}[Y_s^N] = 1. \quad (6.2)$$

A key idea is to work along the excursions of H^N , that is, along the sequence of stopping times $\tau_a^{N,s} := S_{a/N}^N \wedge s$, $a = 0, 1, 2, \dots$. Since N and s are fixed, we will suppress the superscripts N and s for brevity and write τ_a instead of $\tau_a^{N,s}$.

STEP 1. We want to show that

$$\mathbb{E}[Y_{\tau_a}^N] = 1, \quad a = 1, 2, \dots \quad (6.3)$$

It follows readily from the identity (6.1) that for all $\tau_{a-1} \leq r \leq \tau_a$,

$$\frac{Y_r^N}{Y_{\tau_{a-1}}^N} \leq \prod_{k \geq 1: \tau_{a-1} \leq T_k \leq \tau_a} \left(1 + \frac{2\theta}{N\sigma^2} \mathbf{1}_{\{V_{T_k-}^N = -1\}} + \frac{\gamma L_{T_k}^N(H_{T_k}^N)}{N} \mathbf{1}_{\{V_{T_k-}^N = 1\}} \right)$$

where T_k is the k^{th} -jump time of the Poisson process $(P_s^N, s \geq 0)$. Within the excursion of H^N between the times τ_{a-1} and τ_a , these jump times coincide with the times of the local maxima and minima of H^N in the time interval (τ_{a-1}, τ_a) . Since for $a > 1$ there are reflections of X^N at 0 in the time interval $(0, \tau_{a-1})$, the parity of those k for which $V_{T_k-}^N = 1$, $\tau_{a-1} \leq T_k \leq \tau_a$, depends on a . However, noting that

$$L_{T_k}^N(H_{T_k}^N) \leq \frac{4}{N\sigma^2}k, \quad (6.4)$$

we infer the existence of a constant $c > 0$ such that

$$\frac{Y_r^N}{Y_{\tau_{a-1}}^N} \leq c^{P_{\tau_a}^N} \times (P_{\tau_a}^N + 1)!!, \quad \tau_{a-1} \leq r \leq \tau_a, \quad (6.5)$$

where we define for $k \in \mathbb{N}$, $k!! = 1 \cdot 3 \cdot 5 \cdots k$ if k is odd, and $k!! = 1 \cdot 3 \cdot 5 \cdots (k-1)$ if k is even.

Now

$$Y_{\tau_a}^N = Y_{\tau_{a-1}}^N \left(1 + \int_{\tau_{a-1}}^{\tau_a} \frac{Y_{r-}^N}{Y_{\tau_{a-1}}^N} [p(r) + L_{r-}^N(H_r^N)q(r)] dM_r^N \right), \quad (6.6)$$

where $0 \leq p(r) \leq \frac{2}{\theta}N^2$, $0 \leq q(r) \leq \frac{\gamma}{N}$, p and q are predictable.

The claimed equalities $\mathbb{E}[Y_{\tau_a}^N] = 1$, $a = 1, 2, \dots$, follow by induction on a from (6.6), provided the process

$$\mathcal{M}_s^N = \int_0^s \mathbf{1}_{] \tau_{a-1}, \tau_a]}(r) \frac{Y_{r-}^N}{Y_{\tau_{a-1}}^N} [p(r) + L_{r-}^N(H_r^N)q(r)] dM_r^N, \quad s \geq 0,$$

is a martingale. From Theorem T8 in Brémaud [3] page 27, this is a consequence of the fact that

$$\mathbb{E} \int_0^s \mathbf{1}_{] \tau_{a-1}, \tau_a]}(r) \frac{Y_{r-}^N}{Y_{\tau_{a-1}}^N} [p(r) + L_{r-}^N(H_r^N)q(r)] dr < \infty.$$

In order to verify the latter inequality, we compute

$$\begin{aligned} \mathbb{E} \left[\int_{\tau_{a-1}}^{\tau_a} \frac{Y_{r-}^N}{Y_{\tau_{a-1}}^N} [p(r) + L_{r-}^N(H_r^N)q(r)] dr \right] &\leq C_N s \mathbb{E} \left[c^{P_{\tau_a}^N} \times (P_{\tau_a}^N + 1)!! (1 + P_{\tau_a}^N) \right] \\ &\leq C_N s \mathbb{E} \left[c^{P_s^N} \times (P_s^N + 1)!! (1 + P_s^N) \right] \\ &\leq C_N s C_{N,s}, \end{aligned}$$

where we have used (6.5), (6.4) and $\tau_a \leq s$, and where C_N and $C_{N,s}$ are constants which depend only on N and (N, s) , respectively. The fact that $C_{N,s} < \infty$ follows from

$$\mathbb{E}[c_2^{P_s^N} (P_s^N + 1)!! P_s^N] = \exp\{-N^2 \sigma^2 s\} \sum_{k=0}^{\infty} c_2^k (k+1)!! k \frac{(N^2 \sigma^2 s)^k}{k!}$$

Since

$$\begin{aligned} \frac{(k+1)!! k}{k!} &= \frac{k(k+1)}{2 \cdot 4 \cdots (2[\frac{k}{2}])} = \frac{k(k+1)}{2^{[\frac{k}{2}]} [\frac{k}{2}]!} \cdot \frac{1}{[\frac{k}{2}]!} \\ &< \frac{1}{[\frac{k}{2}]!} \quad \forall k \geq 20, \end{aligned}$$

we deduce that $\mathbb{E}[c_2^{P_s^N} (P_s^N + 1)!! P_s^N] < \infty$. This completes the proof of (6.3).

STEP 2. We can now define a consistent family of probability measures $\tilde{\mathbb{P}}^{N,s,a}$ on \mathcal{F}_{τ_a} , $a = 1, 2, \dots$ by putting

$$\frac{d\tilde{\mathbb{P}}^{N,s,a}}{d\mathbb{P}|_{\mathcal{F}_{\tau_a}}} = Y_{\tau_a}^N, \quad a \in \mathbb{N}.$$

We write $\tilde{\mathbb{P}}^{N,s}$ for the probability measure on the σ -field generated by union of the σ -fields \mathcal{F}_{τ_a} , $a = 1, 2, \dots$, whose restriction to \mathcal{F}_{τ_a} is $\tilde{\mathbb{P}}^{N,s,a}$ for all $a = 1, 2, \dots$, and put

$$A := \inf\{a \in \mathbb{N} : \tau_a = s\}.$$

We will now show that

1. (i) $A < \infty$ $\tilde{\mathbb{P}}^{N,s}$ - a.s. (and consequently $\tau_A = s$ $\tilde{\mathbb{P}}^{N,s}$ - a.s.),
2. (ii) under $\tilde{\mathbb{P}}^{N,s}$, $(H_r^N)_{0 \leq r \leq \tau_A} = (H_r^N)_{0 \leq r \leq s}$ is a stochastic process following the dynamics specified in Proposition 3.1.

Indeed, applying Girsanov's theorem (Proposition 7.15 in the Appendix) to the 2-variate point process

$$(Q_r^{1,N}, Q_r^{2,N}) = \left(\int_0^r \mathbf{1}_{\{V_{u^-}^N = -1\}} dP_u^N, \int_0^r \mathbf{1}_{\{V_{u^-}^N = 1\}} dP_u^N \right), \quad 0 \leq r \leq \tau_a, \quad (6.7)$$

we have that under $\tilde{\mathbb{P}}^{N,s,a}$

$$\begin{aligned} Q_r^{1,N} & \text{ has intensity } (N^2 \sigma^2 + 2\theta N) \mathbf{1}_{\{V_{r^-}^N = -1\}} dr \\ Q_r^{2,N} & \text{ has intensity } \sigma^2 [N^2 + \gamma N L_r^N(H_r^N)] \mathbf{1}_{\{V_{r^-}^N = 1\}} dr. \end{aligned}$$

Thus, for all $a \in \mathbb{N}$, $(H_r^N)_{0 \leq r \leq \tau_a}$ is, under $\tilde{\mathbb{P}}^{N,s,a}$, a stochastic process following the dynamics from Proposition 3.1 up to the stopping time τ_a . Considering the sequence of excursions $(H_r^N)_{\tau_{a-1} \leq r \leq \tau_a}$, $a = 1, 2, \dots$ under $\tilde{\mathbb{P}}^{N,s}$, we infer from Lemma 3.3 the validity of the claims (i) and (ii).

STEP 3. We now prove (6.2). For this we observe that

$$\mathbb{E}[Y_s^N] = \sum_{a \geq 1} \mathbb{E}[Y_s^N; A = a] = \sum_{a \geq 1} \mathbb{E}[Y_{\tau_a}^N; A = a] = \sum_{a \geq 1} \tilde{\mathbb{P}}^{N,s}(A = a) = \tilde{\mathbb{P}}^{N,s}(A < \infty) = 1. \quad \blacksquare$$

Corollary 6.2. *Let $\tilde{\mathbb{P}}^N$ be the probability measure on \mathcal{F} whose restriction to \mathcal{F}_s , $s > 0$, has density Y_s^N (given by (6.1)) with respect to $\mathbb{P}|_{\mathcal{F}_s}$. Then under $\tilde{\mathbb{P}}^N$ the process H^N follows the dynamics from Proposition 3.1 for the prescribed θ and γ .*

PROOF: This is immediate from Proposition 6.1 and the discussion preceding it, combined with Proposition 7.15 in the Appendix applied to the process defined in (6.7), now with $0 \leq r < \infty$. \blacksquare

Next we will analyze the behaviour of the Girsanov densities as $N \rightarrow \infty$. For this we use the two martingales $M^{1,N}$ and $M^{2,N}$ defined in (5.5), and note that (6.1) can be rewritten as

$$Y_s^N = 1 + \int_0^s Y_{r-}^N \left\{ \theta dM_r^{1,N} + \frac{\gamma \sigma^2 L_r^N(H_r^N)}{2} dM_r^{2,N} \right\}, \quad s \geq 0.$$

The two (pure jump) martingales $M^{1,N}$ and $M^{2,N}$ have jump sizes $2/(N\sigma^2)$, hence the random variable under the expectation in formula (7.7) vanishes for suitably large N . Thus (see Definition 7.12 in the Appendix), the sequences $\{M^{1,N}\}_{N \geq 1}$ and $\{M^{2,N}\}_{N \geq 1}$ have uniformly controlled variations, and because of Proposition 7.13 (1) they are good. Hence

$$X^N \Rightarrow \int_0^\cdot \left\{ \frac{\sqrt{2}\theta}{\sigma} dB_r^1 + \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2} dB_r^2 \right\} := X..$$

Moreover, by Proposition 7.13 (3), $\{X_s^N\}_{N \geq 1}$ is also a good sequence, hence by Proposition 7.13 (2)

$$Y^N = \mathcal{E}(X^N) \Rightarrow \mathcal{E}(X) =: Y.$$

Combining these facts with Corollary 5.1, we deduce, again from Proposition 7.13 (3), that

$$(H^N, L_{S_x^N}, Y^N) \Rightarrow (H, L_{S_x}, Y). \quad (6.8)$$

Since B^1 and B^2 are mutually orthogonal, by Proposition 7.11 we have

$$\begin{aligned} Y_s &= \mathcal{E} \left(\int_0^\cdot \left\{ \frac{\sqrt{2}\theta}{\sigma} dB_r^1 + \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2} dB_r^2 \right\} \right)_s \\ &= \mathcal{E} \left(\frac{\sqrt{2}\theta}{\sigma} B^1 \right)_s \mathcal{E} \left(\int_0^\cdot \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2} dB_r^2 \right)_s \\ &= \exp \left\{ \frac{\sqrt{2}\theta}{\sigma} B_s^1 + \int_0^s \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2} dB_r^2 - \int_0^s \left[\frac{\theta^2}{\sigma^2} + \frac{\gamma^2\sigma^2}{4} L_r(H_r)^2 \right] dr \right\} \end{aligned}$$

Applying Lemma 2.2 and Lemma 2.3 again, we deduce that Y is a martingale. In particular $\mathbb{E}[Y_s] = 1$ for all $s \geq 0$. Define the probability measure $\tilde{\mathbb{P}}$ by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_s} = Y_s, \quad \forall s \geq 0,$$

then H , under $\tilde{\mathbb{P}}$, solves the SDE (1.3) with B_s there replaced by

$$\tilde{B}_s := \frac{1}{\sqrt{2}}(B_s^1 - B_s^2) - \frac{\theta}{\sigma}s + \frac{\gamma\sigma}{2} \int_0^s L_r(H_r) dr,$$

which is a standard Brownian motion under $\tilde{\mathbb{P}}$ due to Proposition 7.14.

The following general and elementary Lemma will allow us to conclude the required convergence under the transformed measures.

Lemma 6.3. *Let (ξ_N, η_N) , (ξ, η) be random pairs defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with η_N, η nonnegative scalar random variables, and ξ_N, ξ taking values in some complete separable metric space \mathcal{X} . Assume that $\mathbb{E}[\eta_N] = \mathbb{E}[\eta] = 1$. Write $(\tilde{\xi}_N, \tilde{\eta}_N)$ for the random pair (ξ_N, η_N) defined under the probability measure \mathbb{P}^N which has density η_N with respect to \mathbb{P} , and $(\tilde{\eta}, \tilde{\xi})$ for the random pair (η, ξ) defined under the probability measure $\tilde{\mathbb{P}}$ which has density η with respect to \mathbb{P} . Then $(\tilde{\xi}_N, \tilde{\eta}_N)$ converges in distribution to $(\tilde{\eta}, \tilde{\xi})$, provided that (ξ_N, η_N) converges in distribution to (ξ, η) .*

PROOF: Due to the equality $\mathbb{E}[\eta_N] = \mathbb{E}[\eta] = 1$ and a variant of Scheffé's lemma, the sequence η_N is uniformly integrable. Hence for all bounded continuous $F : \mathcal{X} \times \mathbb{R}_+ \rightarrow \mathbb{R}$,

$$\mathbb{E}[F(\tilde{\xi}_N, \tilde{\eta}_N)] = \mathbb{E}[F(\xi_N, \eta_N)\eta_N] \rightarrow \mathbb{E}[F(\xi, \eta)\eta] = \mathbb{E}[F(\tilde{\xi}, \tilde{\eta})].$$

■

Combining (6.8) with Lemma 6.3 yields the

Theorem 3. *Let H^N be a stochastic process following the dynamics specified in Proposition 3.1, and let H be the unique weak solution of the SDE (1.3). We have*

$$(H^N, L_{S_x^N}^N) \Rightarrow (H, L_{S_x}) \quad \text{in } C([0, \infty]) \times D([0, \infty]), \quad (6.9)$$

where S_x^N and S_x are defined in (5.2) and (1.2).

We can now proceed with the

COMPLETION OF THE PROOF OF THEOREM 1 : Define $Z_t^{N,x} := \frac{\sigma^2}{4} L_{S_x^N}^N(t)$. By Corollary 3.2, $Z^{N,x}$ follows the dynamics (3.1). From (6.9), $\frac{\sigma^2}{4} L_{S_x}$ is the limit in distribution of $Z^{N,x}$ as $N \rightarrow \infty$. Hence by Proposition 4.3, $t \mapsto \frac{\sigma^2}{4} L_{S_x}(t)$ is a weak solution of the SDE (1.1), which completes the proof of Theorem 1. ■

Remark 6.4. Theorem 1 establishes a correspondence between the solution H of the SDE (1.3) and the logistic Feller process, i.e. the solution of (1.1). This connection can be expressed in particular through the occupation times formula for H , which states that for any Borel measurable and positive valued function f ,

$$\int_0^{S_x} f(H_s) ds = \int_0^\infty f(t) Z_t^x dt.$$

This formula in the particular case $f \equiv 1$ states that

$$S_x = \int_0^\infty Z_t^x dt.$$

The quantity on the right is the area under the trajectory Z^x . It is the limit of the properly scaled total branch length of the approximating forests F^N defined in Section 3. We now establish another identity concerning this same quantity, with the help of a time change introduced by Lambert in [8]. Consider the additive functional

$$A_t = \int_0^t Z_r^x dr,$$

and the associated time change

$$\alpha_t = \inf\{r > 0, A_r > t\}.$$

As noted in [8], the process $U_t^x := Z_{\alpha_t}^x$ is an Ornstein–Uhlenbeck process, solution of the SDE

$$dU_t^x = (\theta - \gamma U_t^x) dt + \sigma dB_t, \quad U_0^x = x.$$

Of course this identification is valid only for $0 \leq t \leq \tau_x$, where $\tau_x := \inf\{t > 0, U_t = 0\}$. Let T_x be the extinction time of the logistic Feller process Z_t^x . We clearly have $\alpha_{\tau_x} = T_x$, and consequently

$$\tau_x = \int_0^\infty Z_r^x dr.$$

We have identified the time at which the local time at 0 of the exploration process H reaches x with the area under the logistic Feller trajectory starting from x , and with the time taken by the Ornstein–Uhlenbeck process U^x to reach 0. The reader may notice that in the particular case $\gamma = 0$, the identity $S_x = \tau_x$ is not a surprise, see also the discussion and the references in [13] Section 6.

7 Appendix

In this section we collect some known results from the theory of weak convergence of stochastic processes, and the theory of martingales. All proofs which are not given here may be found in the references.

We denote by $D([0, \infty))$ the space of functions from $[0, \infty)$ into \mathbb{R} which are right continuous and have left limits at any $t > 0$ (as usual such a function is called càdlàg). We briefly write \mathbb{D} for the space of adapted, càdlàg stochastic processes. We shall always equip the space $D([0, \infty))$ with the Skorohod topology, for the definition of which we refer the reader to Billingsley [2] or Joffe, Métivier [6]. The results collected in the following statement as well as in Proposition 7.4 can be found in [2].

Lemma 7.1. *Suppose $\{x_n, n \geq 1\} \subset D([0, \infty))$ and $x_n \rightarrow x$ in the Skorohod topology.*

- (i) *If x is continuous, then x_n converges to x locally uniformly.*
- (ii) *If each x_n is continuous, then so is x , and x_n converges to x locally uniformly.*

In particular, the space $C([0, \infty))$ is closed in $D([0, \infty))$ equipped with the Skorohod topology.

The following two lemmata are used in the proofs of Propositions 5.6 and 5.3:

Lemma 7.2. *Fix $t > 0$. Let $x_n, x \in C([0, \infty))$, $n \geq 1$ be such that*

1. *$x_n(s) \rightarrow x(s)$ locally uniformly.*
2. *for each $s > 0$,*

$$\int_0^s \mathbf{1}_{\{x(r)=t\}} dr = 0.$$

Then

$$\int_0^s \mathbf{1}_{\{x_n(r)>t\}} dr \rightarrow \int_0^s \mathbf{1}_{\{x(r)>t\}} dr \quad \text{locally uniformly in } s \geq 0.$$

PROOF: We prove convergence for each $s > 0$. The local uniformity is then easy. Given $\varepsilon > 0$, then there exists N_0 such that

$$\sup_{0 \leq r \leq s} |x_n(r) - x(r)| < \varepsilon \quad \forall n \geq N_0.$$

Then for all $n \geq N_0$,

$$\begin{aligned} & \left| \mathbf{1}_{\{x_n(r) > t\}} - \mathbf{1}_{\{x(r) > t\}} \right| \leq \mathbf{1}_{\{t - \varepsilon < x(r) < t + \varepsilon\}} \\ & \left| \int_0^s \mathbf{1}_{\{x_n(r) > t\}} dr - \int_0^s \mathbf{1}_{\{x(r) > t\}} dr \right| \leq \int_0^s \mathbf{1}_{\{t - \varepsilon < x(r) < t + \varepsilon\}} dr. \end{aligned}$$

The result follows from

$$\lim_{\varepsilon \rightarrow 0} \int_0^s \mathbf{1}_{\{t - \varepsilon < x(r) < t + \varepsilon\}} dr = \int_0^s \mathbf{1}_{\{x(r) = t\}} dr = 0.$$

■

Lemma 7.3. *Let $x_n, y_n \in D([0, \infty))$, $n \geq 1$ and $x, y \in C([0, \infty))$ be such that*

1. y_n is increasing, $\forall n \geq 1$;
2. $x_n(t) \rightarrow x(t), y_n(t) \rightarrow y(t)$, locally uniformly.

Then y is increasing and

$$\int_0^t x_n(s) dy_n(s) \rightarrow \int_0^t x(s) dy(s), \quad \text{locally uniformly in } t \geq 0.$$

PROOF: We prove convergence for each $t > 0$. The local uniformity is then easy.

$$\begin{aligned} & \left| \int_0^t x(s) dy(s) - \int_0^t x_n(s) dy_n(s) \right| \\ & \leq \left| \int_0^t [x(s) - x_n(s)] dy_n(s) \right| + \left| \int_0^t x(s) [dy(s) - dy_n(s)] \right| \\ & \leq \sup_{0 \leq s \leq t} |x(s) - x_n(s)| y_n(t) + \int_0^t |x(s) - \xi_\varepsilon(s)| [dy(s) - dy_n(s)] + \int_0^t |\xi_\varepsilon(s)| [dy(s) - dy_n(s)], \end{aligned}$$

where ξ_ε is a step function which is such that $\sup_{0 \leq s \leq t} |x(s) - \xi_\varepsilon(s)| \leq \varepsilon$. The first and last term of the above right hand side clearly tend to 0 as $n \rightarrow \infty$. Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left| \int_0^t x(s) dy(s) - \int_0^t x_n(s) dy_n(s) \right| & \leq \varepsilon \limsup_{n \rightarrow \infty} [y_n(t) + y(t)] \\ & \leq 2y(t) \times \varepsilon. \end{aligned}$$

It remains to let $\varepsilon \rightarrow 0$.

■

Proposition 7.4. *The sequence $\{X^n, n \geq 1\} \subset \mathbb{D}$ is tight in $D([0, \infty))$ iff the two following conditions hold*

- (1) for each $t \geq 0$, $\{X_t^n - X_{t-}^n, n \geq 1\}$ is tight in \mathbb{R} .
- (2) $\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(w^T(X^n, \delta) > \eta) = 0, \quad \forall T, \delta, \eta > 0$

where

$$w^T(x, \delta) := \inf_{\pi \in \Pi_\delta^T} \max_{t_i \in \pi} \sup_{t_i \leq s < t_{i+1}} |x(t) - x(s)|,$$

if $x \in D([0, \infty))$ and Π_δ^T is the set of all increasing sequences $0 = t_0 < t_1 < \dots < t_n = T$ with the property that $\inf_{0 \leq i < n} |t_{i+1} - t_i| \geq \delta$.

An approach to checking tightness and convergence to a continuous process of a sequence of càdlàg processes is to use Theorem 15.5 from [2], which we recall here for the convenience of the reader:

Proposition 7.5. *Let $\{X_t^n, t \geq 0\}_{n \geq 1}$ be a sequence of random elements of $D([0, \infty))$. A sufficient condition for $\{X^n\}$ to be tight is that the two conditions (i) and (ii) be satisfied :*

- (i) $\{X_0^n, n \geq 1\}$ is tight in \mathbb{R} ;
- (ii) for any $T, \varepsilon, \eta > 0$, there exists $\delta > 0$ such that

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{0 \leq s \leq t \leq T; t-s \leq \delta} |X_s^n - X_t^n| \geq \varepsilon \right) \leq \eta.$$

Moreover, if (i) and (ii) are satisfied, then any limit of a converging subsequence is a. s. continuous.

We now state yet another tightness criterion, which is Theorem 8.8 from Ethier and Kurtz [4] :

Proposition 7.6. *Let $\{X_t^n, t \geq 0\}_{n \geq 1}$ be a sequence of random elements of $D([0, \infty))$. A sufficient condition for $\{X^n\}$ to be tight is that the two conditions (i) and (ii) be satisfied :*

- (i) For each $t \geq 0$, the sequence of random variables $\{X_t^n, n \geq 1\}$ is tight in \mathbb{R} ;
- (ii) for each $T > 0$, there exists $\beta, C > 0$ and $\theta > 1$ such that

$$\mathbb{E} \left(|X_{t+h}^n - X_t^n|^\beta |X_t^n - X_{t-h}^n|^\beta \right) \leq Ch^\theta,$$

for all $0 \leq t \leq T, 0 \leq h \leq t, n \geq 1$.

Note that convergence in $D([0, \infty))$ is not additive : $x_n \rightarrow x$ and $y_n \rightarrow y$ in $D([0, \infty))$ does not imply that $x_n + y_n \rightarrow x + y$ in $D([0, \infty))$. This is due to the fact that to the sequence x_n is attached a sequence of time changes, and to the sequence y_n is attached another sequence of time changes, such that the time changed x_n and y_n converge uniformly. But there may not exist a sequence of time changes which makes $x_n + y_n$ converge. If now $\{X_t^n, t \geq 0\}_{n \geq 1}$ and $\{Y_t^n, t \geq 0\}_{n \geq 1}$ are two tight sequences of random elements of $D([0, \infty))$, we cannot conclude that $\{X_t^n + Y_t^n, t \geq 0\}_{n \geq 1}$ is tight. However, if $x_n \rightarrow x$ and $y_n \rightarrow y$ in $D([0, \infty))$ and x is continuous, then we deduce easily from Lemma 7.1 that $x_n + y_n \rightarrow x + y$ in $D([0, \infty))$. We have similarly the

Proposition 7.7. *If $\{X_t^n, t \geq 0\}_{n \geq 1}$ and $\{Y_t^n, t \geq 0\}_{n \geq 1}$ are two tight sequences of random elements of $D([0, \infty))$ such that any limit of a weakly converging subsequence of the sequence $\{X_t^n, t \geq 0\}_{n \geq 1}$ is a. s. continuous (for instance that sequence could satisfy the conditions of Proposition 7.5), then $\{X_t^n + Y_t^n, t \geq 0\}_{n \geq 1}$ is tight in $D([0, \infty))$.*

Consider a sequence $\{X_t^n, t \geq 0\}_{n \geq 1}$ of one-dimensional semimartingales, which is such that for each $n \geq 1$,

$$\begin{aligned} X_t^n &= X_0^n + \int_0^t \varphi_n(X_s^n) ds + M_t^n, & t \geq 0; \\ \langle M^n \rangle_t &= \int_0^t \psi_n(X_s^n) ds, & t \geq 0; \end{aligned}$$

where for each $n \geq 1$, M^n is a locally square-integrable martingale, φ_n and ψ_n are Borel measurable functions from \mathbb{R} into \mathbb{R} and \mathbb{R}_+ respectively.

We now establish a general tightness criterion adapted to this situation.

Proposition 7.8. *A sufficient condition for the above sequence $\{X_t^n, t \geq 0\}_{n \geq 1}$ of semi-martingales to be tight in $D([0, \infty))$ is that both*

$$\text{the sequence of r.v.'s } \{X_0^n, n \geq 1\} \text{ is tight;} \quad (7.1)$$

and for some $p > 1$,

$$\forall T > 0, \text{ the sequence of r.v.'s } \left\{ \int_0^T [|\varphi_n(X_t^n)| + \psi_n(X_t^n)]^p dt, n \geq 1 \right\} \text{ is tight.} \quad (7.2)$$

Those conditions imply that both the bounded variation parts $\{V^n, n \geq 1\}$ and the martingale parts $\{M^n, n \geq 1\}$ are tight, and that the limit of any converging subsequence of $\{V^n\}$ is a.s. absolutely continuous with respect to Lebesgue's measure.

If moreover, for any $T > 0$, as $n \rightarrow \infty$,

$$\sup_{0 \leq t \leq T} |M_t^n - M_{t-}^n| \rightarrow 0 \quad \text{in probability,}$$

then any limit X of a converging subsequence of the original sequence $\{X^n\}_{n \geq 1}$ is a.s. continuous.

We will introduce in the proof the Sobolev space $W^{1,p}(0, T)$, which can be defined as the space of those continuous functions $x : [0, T] \rightarrow \mathbb{R}$ for which there exists $y \in L^p(0, T)$ satisfying

$$x(t) = x(0) + \int_0^t y(s) ds, \quad 0 \leq t \leq T.$$

Here, p is the exponent which appears in the formulation of condition (7.2). Since $p > 1$, $W^{1,p}(0, T)$ is a reflexive Banach space, and consequently its bounded subsets are relatively compact for the weak topology (the topology induced by the pairing with elements of the dual space). We use here “weak topology” in the functional analytic sense, which should be well distinguished from “weak convergence” of probability measures. **PROOF OF PROPOSITION**

7.8 It follows from Corollary 2.3.3 and section 2.1.4 in Joffe, Métivier [6] that given (7.1), a sufficient condition for tightness of the sequence $\{X^n\}_{n \geq 1}$ is that for each $T > 0, \varepsilon, \eta > 0$, there exists $\delta > 0$ and $n_0 \geq 1$ such that for any sequence $\{\tau_n\}_{n \geq 1}$ of $[0, T]$ -valued stopping times,

$$\sup_{n \geq n_0} \sup_{\theta \leq \delta} \mathbb{P} \left(\int_{\tau_n}^{(\tau_n + \theta) \wedge T} [|\varphi_n(X_t^n)| + \psi_n(X_t^n)] dt \geq \eta \right) \leq \varepsilon. \quad (7.3)$$

But from Hölder's inequality, if $1/p + 1/q = 1$,

$$\int_{\tau_n}^{(\tau_n + \theta) \wedge T} [|\varphi_n(X_s^n)| + \psi_n(X_s^n)] ds \leq \theta^{1/q} \left(\int_0^T [|\varphi_n(X_t^n)| + \psi_n(X_t^n)]^p dt \right)^{1/p},$$

hence (7.3) follows easily from (7.2).

Clearly our criterion implies tightness of both the bounded variation part and the martingal part of X^n . Consider the bounded variation part

$$V_t^n := \int_0^t \varphi_n(X_s^n) ds.$$

Our condition implies in particular that for any $T > 0$, the $W^{1,p}(0, T)$ norm of V^n is tight, which implies that from any subsequence of the sequence $\{V^n, n \geq 1\}$, we can extract a further subsequence which converges in law both for the uniform topology (this follows from Lemma 7.1) and the weak topology of $W^{1,p}(0, T)$. Hence the trajectories of any limit of a converging subsequence belong a. s. to $W^{1,p}(0, T)$, i. e. they are absolutely continuous.

Finally the second statement in the Lemma follows from Theorem III.10.2 in Ethier and Kurtz [4]. \blacksquare

The two following corollaries follow from the above proposition.

Corollary 7.9. *A sufficient condition for (7.2) to hold is that for all $T > 0$,*

$$\left\{ \sup_{0 \leq t \leq T} [|\varphi_n(X_t^n)| + \psi_n(X_t^n)], n \geq 1 \right\} \text{ is tight.} \quad (7.4)$$

Corollary 7.10. *A sufficient condition for (7.2) to hold is that for all $T > 0$*

$$\limsup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[\varphi_n^2(X_t^n) + \psi_n^2(X_t^n)] < \infty. \quad (7.5)$$

PROOF: We show that (7.2) follows from (7.5). Indeed a sufficient condition for (7.2) is that

$$\limsup_n \mathbb{E} \int_0^T [|\varphi_n(X_t^n)| + \psi_n(X_t^n)]^2 dt < \infty,$$

which clearly follows from (7.5). \blacksquare

In the remaining part of this Appendix, we collect some results from stochastic calculus. For a semimartingale $X = (X_t, t \geq 0)$, consider the stochastic linear equation of Doléans

$$Y_t = 1 + \int_0^t Y_{r-} dX_r. \quad (7.6)$$

The following proposition follows from Theorem 1 and Theorem 2 in [10], page 122.

Proposition 7.11. (1) *Equation (7.6) has a unique solution (up to indistinguishability) within the class of semimartingales. This solution is denoted by $\mathcal{E}(X)$ and is called the stochastic exponential of X . It has the following representation*

$$\mathcal{E}(X)_t = \exp \left\{ X_t - X_0 - \frac{1}{2} \langle X^c \rangle_t \right\} \prod_{r \leq t} (1 + \Delta X_r) e^{-\Delta X_r}.$$

(2) *If U and X are two semimartingales, then*

$$\mathcal{E}(U)_t \mathcal{E}(X)_t = \mathcal{E}(U + X + [U, X])_t.$$

(3) *If X is a local martingale, then $\mathcal{E}(X)$ is a nonnegative local martingale and a supermartingale.*

For $\delta > 0$ we define $h_\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $h_\delta(r) = (1 - \delta/r)^+$. For $x \in D([0, \infty))$, we define $x^\delta \in D([0, \infty))$ by

$$x_t^\delta := x_t - \sum_{0 < s \leq t} h_\delta(|\Delta x_s|) \Delta x_s.$$

Definition 7.12. (1) Let G, G^n in \mathbb{D} , $\{G^n, n \geq 1\}$ be a sequence of semimartingales adapted to a given filtration (\mathcal{F}_t) and assume $G^n \Rightarrow G$. The sequence (G^n) is called good if for any sequence $\{I^n, n \geq 1\}$ of (\mathcal{F}_t) -progressively measurable processes in \mathbb{D} such that $(I^n, G^n) \Rightarrow (I, G)$, then G is a semimartingale for a filtration with respect to which I is adapted, and $(I^n, G^n, \int I_{s-}^n dG_s^n) \Rightarrow (I, G, \int I_{s-} dG_s)$.

(2) A sequence of semimartingales $\{G^n\}_{n \geq 1}$ is said to have uniformly controlled variations if there exists $\delta > 0$, and for each $\alpha > 0, n \geq 1$, there exists a semimartingale decomposition $G^{n,\delta} = M^{n,\delta} + A^{n,\delta}$ and a stopping time $T^{n,\alpha}$ such that $\mathbb{P}(\{T^{n,\alpha} \leq \alpha\}) \leq \frac{1}{\alpha}$ and furthermore

$$\sup_n \mathbb{E} \left\{ [M^{n,\delta}, M^{n,\delta}]_{t \wedge T^{n,\alpha}} + \int_0^{t \wedge T^{n,\alpha}} |dA^{n,\delta}| \right\} < \infty. \quad (7.7)$$

Proposition 7.13. Let G, G^n in \mathbb{D} , $\{G^n, n \geq 1\}$ be a sequence of semimartingales and assume $G^n \Rightarrow G$.

- (1) The sequence $\{G^n\}$ is good if and only if it has uniformly controlled variations.
- (2) If $\{G^n\}$ is good, then $(G^n, \mathcal{E}(G^n)) \Rightarrow (G, \mathcal{E}(G))$.
- (3) Suppose $(I^n, G^n) \Rightarrow (I, G)$, and $\{G^n\}$ is good. Then $J^n = \int I_{s-}^n dG_s^n$, $n = 1, 2, \dots$, is also a good sequence of semimartingales. Moreover under the same conditions, $(I^n, G^n, \mathcal{E}(G^n)) \Rightarrow (I, G, \mathcal{E}(G))$.

PROOF: See [7], page 32 ff. ■

Finally, we state two multivariate versions of the Girsanov theorem, one for the Brownian and one for the point process case. The second one combines Theorems T2 and T3 from [3], pages 165-166.

Proposition 7.14. Let $\{(B_s^{(1)}, \dots, B_s^{(d)}), s \geq 0\}$ be a d -dimensional standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, let $\phi = (\phi_1, \dots, \phi_d)$ be an \mathcal{F} -progressively measurable process with $\int_0^s \phi_i(r)^2 dr < \infty$ for all $1 \leq i \leq d$ and $s \geq 0$. Let $X_s^{(i)} := \int_0^s \phi_i(r) dB_r^{(i)}$ and put $Y := \mathcal{E}(X^{(1)} + \dots + X^{(d)})$, or in other words

$$Y_s = \exp \left\{ \int_0^s \langle \phi(r), dB_r \rangle - \frac{1}{2} \int_0^s |\phi(r)|^2 dr \right\}.$$

If $\mathbb{E}[Y_s] = 1$, $s \geq 0$, then $\tilde{B}_s := B_s - \int_0^s \phi(r) dr$, $s \geq 0$, is a d -dimensional standard Brownian motion under the probability measure $\tilde{\mathbb{P}}$ defined by $d\tilde{\mathbb{P}}|_{\mathcal{F}_s}/d\mathbb{P}|_{\mathcal{F}_s} = Y_s$, $s \geq 0$.

Proposition 7.15. Let $\{(Q_s^{(1)}, \dots, Q_s^{(d)}), s \geq 0\}$ be a d -variate point process adapted to some filtration \mathcal{F} , and let $\{\lambda_s^{(i)}, s \geq 0\}$ be the predictable $(\mathbb{P}, \mathcal{F})$ -intensity of $Q^{(i)}$, $1 \leq i \leq d$. Assume

that none of the $Q^{(i)}$, $Q^{(j)}$, $i \neq j$, jump simultaneously. Let $\{\mu_r^{(i)}, r \geq 0\}$, $1 \leq i \leq d$, be nonnegative \mathcal{F} -predictable processes such that for all $s \geq 0$ and all $1 \leq i \leq d$

$$\int_0^s \mu_r^{(i)} \lambda_r^{(i)} dr < \infty \quad \mathbb{P} \text{-a.s.}$$

For $i = 1, \dots, d$ and $s \geq 0$ define

$$X_s^{(i)} := \int_0^s (\mu_r^{(i)} - 1) dM_r^{(i)}, \quad Y^{(i)} := \mathcal{E}(X^{(i)}), \quad Y = \mathcal{E}(X^{(1)} + \dots + X^{(d)}).$$

Then, with $\{T_k^i, k = 1, 2, \dots\}$ denoting the jump times of $Q^{(i)}$,

$$Y_s^{(i)} = \left(\prod_{k \geq 1: T_k^i \leq s} \mu_{T_k^i}^{(i)} \right) \exp \left\{ \int_0^s (1 - \mu_r^{(i)}) \lambda_r^{(i)} dr \right\} \quad \text{and} \quad Y_s = \prod_{j=1}^d Y_s^{(j)}, \quad s \geq 0.$$

If $\mathbb{E}[Y_s] = 1$, $s \geq 0$, then, for each $1 \leq i \leq d$, the process $Q^{(i)}$ has the $(\tilde{\mathbb{P}}, \mathcal{F})$ -intensity $\tilde{\lambda}_r^{(i)} = \mu_r^{(i)} \lambda_r^{(i)}$, $r \geq 0$, where the probability measure $\tilde{\mathbb{P}}$ is defined by $d\tilde{\mathbb{P}}|_{\mathcal{F}_s} / d\mathbb{P}|_{\mathcal{F}_s} = Y_s$, $s \geq 0$.

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References

- [1] M. Ba, E. Pardoux, A. B. Sow, Binary trees, exploration processes, and an extended Ray–Knight Theorem, Preprint, 2009.
- [2] P. Billingsley, *Convergence of Probability Measures*, 3rd ed., Wiley, New York 1995.
- [3] P. Brémaud, *Point processes and queues, martingale dynamics*, Springer-Verlag New York, 1981.
- [4] S. Ethier, Th. Kurtz, *Markov processes, Characterisation and convergence*, John Wiley and Sons Inc., New York, 1986.
- [5] A. Friedman, *Stochastic differential equations and applications, vol 1*, Academic Press, 1975.
- [6] A. Joffe, M. Métivier, Weak convergence of sequences of semimartingales with applications to multitype branching processes, *Adv. Appl. Prob.* **18** (1986), 20–65.
- [7] Th. Kurtz and Ph. Protter, Weak convergence of stochastic integrals and differential equations, *Probabilistic models for nonlinear partial differential equations, Lecture Notes in Math* **1627**, 1–41, 1996.
- [8] A. Lambert, The branching process with logistic growth, *Ann. Appl. Probab.* **15** (2005), 1506–1535.
- [9] J-F. Le Gall, Itô’s excursion theory and random trees, *Stochastic Process. Appl.* **120** (2010), 721–749.

- [10] R. S. Liptser and A. N. Shiriyayev, *Theory of martingales*, Kluwer Academic Publishers, 1989.
- [11] J. R. Norris, L.C.G. Rogers and D. Williams, Self-avoiding random walk: a Brownian motion model with local time drift, *Probab. Th. Rel. Fields* **74** (1987), 271-287.
- [12] S. Méléard, Quasi-stationary distributions for population processes, *Lecture at CIMPA school, St Louis, Sénégal*, 2010.
- [13] E. Pardoux and A Wakolbinger, From exploration paths to mass excursions - variations on a theme of Ray and Knight, to appear in: *Surveys in Stochastic Processes*, Proceedings of the 33rd SPA Conference in Berlin, 2009, J. Blath, P. Imkeller, S. Roelly (eds.), EMS 2010.
- [14] E. Perkins, Weak invariance principles for local time, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **60** (1982), 437–451.
- [15] J. Pitman, The distribution of local times of a Brownian bridge, *Séminaire de probabilités (Strasbourg)* **33**, Lecture Notes in Math. **1709**, pp. 388–394, 1999.
- [16] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, 3rd ed., Springer Verlag, New York 1999.