Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations

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Introduction

A new class of backward stochastic differential equations has been studied by the authors in [3], and it has been used by the second author in [4], in order to give a probabilistic formula for the given solution of a system of parabolic partial differential equation.

The aim of the present paper is to study the regularity properties of the solution of the backward SDE (in short BSDE), and to deduce a converse of the results of [4], namely to show that a given function expressed in terms of the solution of the BSDE solves a certain system of parabolic PDEs. Our result generalizes the well-known Feynman-Kac formula (see Remark 3.3 below). It gives an existence and uniqueness result for a *system* of quasilinear (and *possibly degenerate*) parabolic equations. We also obtain an existence result for the viscosity solution of a quasilinear parabolic equation.

We shall extend our approach in a forthcoming publication, to the case of systems of quasilinear parabolic stochastic partial differential equations. Our approach may also prove useful for solving certain equations on manifolds.

The paper is organised as follows. In section 1, we shall state our assumptions, and recall some results from previous work. In section 2, we establish some estimates and regularity results for the solution of the BSDE, in section 3 we shall relate it to a system of quasilinear parabolic PDEs. Finally, in 4, we relate the solution of the one-dimensional BSDE to the viscosity solution of a quasilinear parabolic PDE, under much weaker assumptions.

1 Preliminaries

In all what follows, we shall work on a fixed finite time interval $[0,T]$. We suppose given on a probability space $(\Omega,\mathcal{F},P)$ a d-dimensional standard Wiener process $\{W_t;\ t\in[0,T]\}$. For $0\le t\le r\le T$, we define $\overset{\circ}{\mathcal{F}}{}^t_r=\sigma\{W_s-W_t;\ t\le s\le r\}$ and $\mathcal{F}^t_r$ denotes the completion of $\overset{\circ}{\mathcal{F}}{}^t_r$ with the P-null sets of $\mathcal{F}$. We shall write $\mathcal{F}_r$ for $\mathcal{F}^0_r$ and $\mathcal{F}^t$ for $\mathcal{F}^t_T$.

For any $0\le t\le r\le T$, $p\in\mathbb{N}$, we denote by $M^2(t,r;\mathbb{R}^p)$ the subset of $L^2(\Omega\times(t,r),\ dP\times ds,\ \mathbb{R}^p)$ consisting of $\mathcal{F}^t_s$-progresively measurable processes.

$C^k(\mathbb{R}^p,\mathbb{R}^q)$, $C^k_{l,b}(\mathbb{R}^p,\mathbb{R}^q)$, $C^k_p(\mathbb{R}^p,\mathbb{R}^q)$ will denote respectively the set of functions of class C^k from $\mathbb{R}^p$ into $\mathbb{R}^q$, the set of those functions of class C^k whose partial derivatives of order less than or equal to k are bounded (and hence the function itself growths at most like a linear function of the variable x at infinity), and the set of those functions of class C^k which, together with all their partial derivatives of order less than or equal to k, grow at most like a polynomial function of the variable x at infinity.

We are given $b \in C^3_{l,b}(\mathbb{R}^d;\mathbb{R}^d)$ and $\sigma \in C^3_{l,b}(\mathbb{R}^d;\mathbb{R}^{d\times d})$, and for each $t \in [0,T)$, $x \in \mathbb{R}^d$, we denote by $\{X^{t,x}_s,\ t \le s \le T\}$ the unique strong solution of the following SDE :

$$\begin{cases} dX^{t,x}_s &= b(X^{t,x}_s)ds + \sigma(X^{t,x}_s)dW_s,\ t \le s \le T \\ X^{t,x}_t &= x \end{cases} \tag{1}$$

It is well-known (see e.g. Stroock [8]) that the random field $\{X^{t,x}_s;\ 0 \le t \le s \le T,\ x \in \mathbb{R}^d\}$ has a version which is a.s. jointly continuous in (t,s,x), together which its x partial derivatives of order one and two.

Moreover, $\sup_{t\le s\le T}(|X^{t,x}_s| + |\nabla X^{t,x}_s| + |D^2X^{t,x}_s|) \in \cap_{p\ge 1}L^p(\Omega)$, for each t and x, where $\nabla X^{t,x}_s$ and $D^2X^{t,x}_s$ denote respectively the first and second order partial derivative of $X^{t,x}_s$ with respect to x.

Let us now recall the notion of derivation on Wiener space. We denote by S the set of random variables ξ of the form :

$$\xi = \varphi(W(h_1),\dots,W(h_n))$$

where $f \in C^\infty_b(\mathbb{R}^n)$, $h_1,\dots,h_n \in L^2(0,T;\mathbb{R}^d)$, and $W(h_1) = \int_0^T(h_i(s), dW_s)$ is the Wiener integral of h_i with respect to $\{W_s;\ 0 \le s \le T\}$ [$(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^d$]. To such a random variable ξ, we associate a "derivated process" $\{D_r\xi;\ r \in [0,T]\}$ defined as :

$$D_r\xi = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(W(h_1),\dots,W(h_n))h_i(r)$$

For $\xi \in$ S, we define its 1,2–norm by :

$$\|\xi\|^2_{1,2} = E(\xi^2) + E\int_0^T |D_r\xi|^2 dr$$

One can show (see e.g. Nualart–Pardoux [2]) that $D.\ : \mathrm{S} \to L^2(\Omega\times(0,T);\ \mathbb{R}^d)$ is closable, as an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega\times(0,T);\ \mathbb{R}^d)$, hence $D.$ can be extended as an operator from its domain which coincides with $\mathbb{D}^{1,2} \triangleq \overline{\mathrm{S}}^{\|\cdot\|_{1,2}}$ into $L^2(\Omega\times(0,T);\ \mathbb{R}^d)$. Note that if $\xi \in \mathbb{D}^{1,2}$ is $\mathcal{F}^t_s$ measurable, $D_r\xi = 0$ for $r \in [0,T]\setminus[t,s]$. We shall denote by $D^i_r\xi$, $1 \le i \le d$, the i–th component of $D_r\xi$.

It follows from the closedness property of $D.$ that the following results can be proved by approximation.

Lemma 1.1 *For any $0 \le t < s \le T$, $x \in \mathbb{R}^d$, $X^{t,x}_s \in (\mathbb{D}^{1,2})^d$, and a version of $\{D_rX^{t,x}_s;\ s,r \in [0,T]\}$ is given by :*

(i) $D_rX^{t,x}_s = 0$, $r \in [0,T]\setminus(t,s]$

(ii) For any $t < r \le T$, $\{D_rX^{t,x}_s;\ r \le s \le T\}$ is the unique solution of the linear SDE :

$$D_rX^{t,x}_s = \sigma(X^{t,x}_r) + \int_r^s b'(X^{t,x}_\alpha)D_rX^{t,x}_\alpha d\alpha + \int_r^s \sigma'_i(X^{t,x}_\alpha)D_rX^{t,x}_\alpha dW^i_\alpha$$

where we use the convention of summation over the repeated index i, from $i = 1$ to $i = d$, and σ_i denotes the i–th column of the matrix σ.

We now introduce the BSDE. Let $f\ :\ [0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k$ be such that for any $s\in[0,T]$, $(x,y,z)\to f(s,x,y,z)$ is of class C^3 and moreover :

(i) $f(s,\cdot,0,0)\in C^3_p(\mathbb{R}^d;\ \mathbb{R}^k)$,

(ii) the first order partial derivatives in y and z are bounded on $[0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$, as well as their derivatives of order one and two with respect to x,y,z.

Let $g\in C^3_p(\mathbb{R}^d)$. For any $t\in[0,T)$ and $x\in\mathbb{R}^d$, let $\{(Y^{t,x}_s,Z^{t,x}_s);\ t\le s\le T\}$ denote the unique element of $M^2(t,T;\mathbb{R}^k)\times M^2(t,T;\mathbb{R}^{k\times d})$ which solves the following BSDE (see [3]) :

$$Y^{t,x}_s=g(X^{t,x}_T)+\int_s^T f(X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)dr-\int_s^T Z^{t,x}_r dW_r,\ t\le s\le T\,. \tag{2}$$

For further reference, let us indicate the method of construction of the solution $(Y^{t,x}_\cdot,Z^{t,x}_\cdot)$. Dropping the superscript t,x for convenience, we construct the solution in three steps.

First, given arbitrary $\overline{Y}\in M^2(t,T;\ \mathbb{R}^k)$ and $\overline{Z}\in M^2(t,T;\ \mathbb{R}^{k\times d})$, we solve the equation

$$Y_s=g(X_T)+\int_s^T f(X_r,\overline{Y}_r,\overline{Z}_r)dr-\int_s^T Z_r dW_r,\ t\le s\le T\,, \tag{3}$$

whose unique solution is given explicitly by :

$$Y_s=E^{\mathcal{F}_s}[g(X_T)+\int_s^T f(X_r,\overline{Y}_r,\overline{Z}_r)dr]\,,$$

$\{Z_s,\ t\le s\le T\}$ is the unique element of $M^2(t,T;\mathbb{R}^{k\times d})$ which is given by Itô's representation theorem of Brownian martingales (see e.g. Karatzas-Shreve [1]), such that :

$$\int_t^T Z_s dW_s=g(X_T)+\int_t^T f(X_r,\overline{Y}_r,\overline{Z}_r)dr-E\left[g(X_T)+\int_t^T f(X_r,\overline{Y}_r,\overline{Z}_r)dr\right]$$

Next, given an arbitrary element $\overline{Y}\in M^2(t,T;\ \mathbb{R}^k)$, we solve the equation

$$Y_s=g(X_T)+\int_s^T f(X_r,\overline{Y}_r,Z_r)dr-\int_s^T Z_r dW_r,\ t\le s\le T\,, \tag{4}$$

by the iterative procedure :

$$\begin{aligned} Z^0_s &\equiv 0\,,\\ Y^{n+1}_s &= g(X_T)+\int_s^T f(X_r,\overline{Y}_r,Z^n_r)dr-\int_s^T Z^{n+1}_r dW_r,\ t\le s\le T,\ n\in\mathbb{N}\,, \end{aligned}$$

where the n–th equation is solved with the help of the first step.

We finally solve equation (2) by the iterative procedure :

$$\begin{aligned} Y^0_s &\equiv 0\text{ , and for } n\in\mathbb{N}\,,\\ Y^{n+1}_s &= g(X_T)+\int_s^T f(X_r,Y^n_r,Z^{n+1}_r)dr-\int_s^T Z^{n+1}_r dW_r,\ t\le s\le T, \end{aligned}$$

where the n–th equation is solved with the help of the second step.

The convergence of the two approximating sequences is proved respectively on page 57–58 and 59 of [3].

2 Regularity of the solution of the backward SDE

We first estimate higher order moments of (Y). We again suppress the superscript t, x for notational convenience. For $z \in \mathbb{R}^{k\times d}$, we denote $\|z\| = \sqrt{Tr(zz^*)}$.

Lemma 2.1 *For any $p \geq 1$,*

$$E\left(\sup_{t\le s\le T} |Y_s|^p\right) < \infty \qquad (5)$$

$$E\left[\left(\int_t^T |Z_s|^2\, ds\right)^{p/2}\right] < \infty \qquad (6)$$

Proof : From Itô's formula,

$$|g(X_T)|^2 = |Y_s|^2 - 2\int_s^T (f(X_r, Y_r, Z_r), Y_r)dr + 2\int_s^T (Y_r, Z_r dW_r) + \int_s^T \|Z_r\|^2 dr\ ,$$

and for any $p \geq 2$,

$$|g(X_T)|^{2p} \geq |Y_s|^{2p} - 2p\int_s^T |Y_r|^{2(p-1)}(f(X_r, Y_r, Z_r), Y_r)dr$$

$$+2p\int_s^T |Y_r|^{2(p-1)}(Y_r, Z_r dW_r) + p\int_s^T |Y_r|^{2(p-1)}\|Z_r\|^2 dr\ ,$$

hence for some constant C,

$$|Y_s|^{2p} + p\int_s^T |Y_r|^{2(p-1)}\|Z_r\|^2 dr \leq |g(X_T)|^{2p}$$

$$+2Cp\int_s^T |Y_r|^{2(p-1)}(|Y_r| + |Y_r|^2 + |Y_r|\,\|Z_r\|)dr - 2p\int_s^T |Y_r|^{2(p-1)}(Y_r, Z_r dW_r)$$

where we have used the assumption on f. Had we done the same calculation with the function $u \to u^p$ remplaced by

$$\varphi_{n,p}(u) = (u \wedge n)^p + pn^{p-1}(u-n)^+,\ n \in \mathbb{N}\ ,$$

we would have obtained :

$$\varphi_{n,p}(|Y_s|^2) + \int_s^T \varphi'_{n,p}(|Y_r|^2)\|Z_r\|^2 dr$$

$$\leq \varphi_{n,p}(|g(X_T)|^2) + 2C\int_s^T \varphi'_{n,p}(|Y_r|^2)(|Y_r| + |Y_r|^2 + |Y_r|\,\|Z_r\|)dr$$

$$-2\int_s^T \varphi'_{n,p}(|Y_r|^2)(Y_r, Z_r dW_r)$$

We can take the expectation in the last equation, and let $n \to +\infty$, in order to deduce by monotone convergence :

$$E(|Y_s|^{2p}) + pE\int_s^T |Y_r|^{2(p-1)}\|Z_r\|^2 dr \le E(|g(X_T)|^{2p})$$

$$+2CpE\int_s^T |Y_r|^{2(p-1)}(|Y_r| + |Y_r|^2 + |Y_r|\,\|Z_r\|)dr$$

It then follows by Hölder's inequality that there exists $C(p)$ such that :

$$E(|Y_s|^{2p}) + \frac{p}{2}E\int_s^T |Y_r|^{2(p-1)}\|Z_r\|^2 dr \le E(|g(X_T)|^{2p})$$

$$+C(p)E\int_s^T (1 + |Y_r|^{2p})dr\,, \quad t \le s \le T\,.$$

It follows from Gronwall's Lemma that

$$\sup_{t\le s\le T} E(|Y_s|^{2p}) < \infty \tag{7}$$

and hence

$$E\int_t^T |Y_r|^{2(p-1)}\|Z_r\|^2 dr < \infty \tag{8}$$

for an arbitrarily large p.

Now

$$\begin{aligned} |Y_s|^{2p} &\le |g(X_T)|^{2p} + C(p)\int_t^T (1 + |Y_r|^{2p})dr \\ &\quad -2p\int_s^T |Y_r|^{2(p-1)}(Y_r, Z_r dW_r) \\ E\left(\sup_{t\le s\le T}|Y_s|^{2p}\right) &\le E\left[|g(X_T)|^{2p} + C(p)\int_t^T (1 + |Y_r|^{2p})dr\right. \\ &\quad \left. -2p\int_t^T |Y_r|^{2(p-1)}(Y_r, Z_r dW_r)\right] \\ &\quad +2pE\left[\sup_{t\le s\le T}\int_t^s |Y_r|^{2(p-1)}(Y_r, Z_r dW_r)\right] \end{aligned}$$

Hence (5) follows from the Burkholder–Davis–Gundy inequality, (7) and (8) (which again holds with an arbitrarily large p).

Finally, for any $t \le a \le s \le b \le T$,

$$\int_a^s Z_r dW_r = Y_s - Y_a + \int_a^s f(r, X_r, Y_r, Z_r)dr$$

$$\sup_{a\le s\le b}\left|\int_a^s Z_r dW_r\right| \le 2\sup_{t\le s\le T}|Y_s| + \int_a^b |f(r, X_r, Y_r, Z_r)|dr$$

Hence, from Burkholder–Davis–Gundy's inequality, for any $p \geq 2$, $\exists\, c_p$ s.t.

$$\begin{aligned}\frac{1}{c_p}E\left[\left(\int_a^b \|Z_r\|^2 dr\right)^{p/2}\right] &\le E\left(\sup_{a\le s\le b}\left|\int_0^s Z_r dW_r\right|^p\right)\\ &\le c_p\left(1+E\left[\left(\int_a^b |Z_r| dr\right)^p\right]\right)\\ &\le c_p\left(1+(b-a)^{p/2}E\left[\left(\int_a^b |Z_r|^2 dr\right)^{p/2}\right]\right)\end{aligned}$$

Hence, provided $b-a\le c_p^{-4/p}$,

$$E\left[\left(\int_a^b \|Z_r\|^2 dr\right)^{p/2}\right]<\infty\,,$$

and (6) follows. □

Let us now express Z in terms of the Wiener space derivative of Y.

Proposition 2.2 *$Y, Z, \in L^2(t,T;\ \mathbb{D}^{1,2})$, and a version of $\{D_\theta Y_s,\ D_\theta Z_s;\ t\le\theta\le T,\ t\le s\le T\}$ is given by :*

(i) $D_\theta Y_s = 0,\ D_\theta Z_s = 0;\ t\le s<\theta\le T$

(ii) For any fixed $\theta\in[t,T]$ and $1\le i\le d$, $\{D^i_\theta Y_s,\ D^i_\theta Z_s;\ \theta\le s\le T\}$ is the unique solution of the BSDE :

$$D^i_\theta Y_s = g'(X_T)D^i_\theta X_T+\int_s^T F_i(r,\ D^i_\theta Y_r,\ D^i_\theta Z_r)dr-\int_s^T D^i_\theta Z_r dW_r\,, \tag{9}$$

where

$$F_i(r,u,v)=f'_x(X_r,Y_r,Z_r)D^i_\theta X_r+f'_y(X_r,Y_r,Z_r)u+f'_z(X_r,Y_r,Z_r)v\,.$$

Moreover, for any $1\le i\le d$, $\{D^i_s Y_s,\ t\le s\le T\}$ is a version of $\{(Z_s)_i,\ t\le s\le T\}$ (where $(Z_s)_i$ denote the i-th column ot he matrix Z_s).

Before proving the Proposition, let us establish the following simple but very useful Lemma, which is a particular case of a much more general result in Ustunel [9]. We nevertheless include a complete proof for the convenience of the reader.

Lemma 2.3 *Let $Z\in M^2(t,T;\ \mathbb{R}^d)$ be such that $\xi\triangleq\int_t^T(Z_s,dW_s)$ satisfies $\xi\in\mathbb{D}^{1,2}$. Then*

$$Z_i\in L^2(t,T;\ \mathbb{D}^{1,2}),\ 1\le i\le d\,, \tag{10}$$

and

$$D^i_s\xi=(Z_s)_i+\int_s^T D^i_s Z_r dW_r,\ ds\times dP\ a.e. \tag{11}$$

Proof : The fact that (10) implies $\xi \in \mathbb{D}^{1,2}$ and (11) is well known (see e.g. Nualart–Pardoux [2], Proposition 3.4). Hence we only need to prove (10). Let us assume that $d = 1$ for notational convenience.

Note that if (10) holds, then

$$\|\xi\|_{1,2}^2 = 2E\int_t^T Z_s^2 ds + E\int_t^T\int_t^T \|D_s Z_r\|^2 ds dr .$$

Hence the result follows if we show that the set

$$\{\xi = \int_t^T Z_s dW_s ;\ Z \in L^2(t,T;\ \mathbb{D}^{1,2})\}$$

is dense in $\mathbb{D}^{1,2} \cap L^2(\Omega, \mathcal{F}_T^t, P)$. But that follows from the fact that the above set contains $\{\xi \in \mathbb{S} \cap L^2(\Omega, \mathcal{F}_T^t, P);\ E\xi = 0\}$, which can be seen from Ocone's formula (see e.g. [2] Corollary A2)

$$\xi = \int_t^T E(D_s\xi/\mathcal{F}_s) dW_s ,$$

which applies to such ξ's. □

We can now proceed to the :

Proof of Proposition 2.2 We restrict ourselves to the case $d = 1$. We first remark that the fact that equation (9) has a unique solution follows easily from the results of [3], since $f_x'(X_r, Y_r, Z_r)D_\theta X_r$ is bounded in $L^p(\Omega)$, $p \geq 1$, and $f_y'(X_r, Y_r, Z_r)$, $f_z'(X_r, Y_r, Z_r)$ are bounded.

We first consider equation (3) with

$$\overline{Y}, \overline{Z} \in M^2(t,T;\ \mathbb{R}^k) \cap L^2(t,T;\ (\mathbb{D}^{1,2})^k).$$

Hence $g(X_T) + \int_s^T f(X_s, \overline{Y}_s, \overline{Z}_s) ds \in \mathbb{D}^{1,2}$, and it follows from Lemma 2.3 that $Z \in L^2(t,T;\ (\mathbb{D}^{1,2})^k)$, and consequently $Y_s \in \mathbb{D}^{1,2}$, $t \leq s \leq T$, and for $t \leq \theta \leq s$, $1 \leq i \leq d$

$$\begin{aligned} D_\theta^i Y_s &= g'(X_T) D_\theta^i X_T + \\ &\int_s^T [f_x'(X_r, \overline{Y}_r, \overline{Z}_r) D_\theta^i X_r + f_y'(X_r, \overline{Y}_r, \overline{Z}_r) D_\theta^i \overline{Y}_r + f_z'(X_r, \overline{Y}_r, \overline{Z}_r) D_\theta^i \overline{Z}_r] dr \\ &- \int_s^T D_\theta^i Z_r dW_r \end{aligned}$$

We next consider equation (4) with

$$\overline{Y} \in M^2(t,T;\ \mathbb{R}^k) \cap L^2(t,T;\ (\mathbb{D}^{1,2})^k).$$

From the last result, the corresponding approximating sequence satisfies $Y^n, Z^n \in L^2(t,T;\ (\mathbb{D}^{1,2})^k)$, $n \in \mathbb{N}$, and for $t \leq \theta \leq s$, $1 \leq i \leq d$, $n \in \mathbb{N}$,

$$\begin{aligned} D_\theta^i Y^{n+1} &= g'(X_T) D_\theta^i X_T \\ &+ \int_s^T [f_x'(X_r, \overline{Y}_r, Z_r^n) D_\theta^i X_r + f_y'(X_r, \overline{Y}_r, Z_r^n) D_\theta^i \overline{Y}_r + f_z'(X_r, \overline{Y}_r, Z_r^n) D_\theta^i Z_r^n] dr \\ &- \int_s^T D_\theta^i Z_r^{n+1} dW_r,\ \theta \leq s \leq T \end{aligned}$$

Using estimates very similar to those on pages 57–58 of [3], we first show that $E\int_\theta^T |D_\theta^i Z_r^n|^2 dr \le C$ and $E\int_\theta^T |D_\theta^i Y_r^n|^4 dr \le C$. Hence

$$E\int_s^T (D_\theta^i Y_r^{n+2} - D_\theta^i Y_r^{n+1})[(f_x'(X_r,\overline{Y}_r,Z_r^{n+1}) - (f_x'(X_r,\overline{Y}_r,Z_r^n))D_\theta^i X_r$$

$$+(f_y'(X_r,\overline{Y}_r,Z_r^{n+1}) - (f_y'(X_r,\overline{Y}_r,Z_r^n))D_\theta^i \overline{Y}_r$$

$$+(f_z'(X_r,\overline{Y}_r,Z_r^{n+1}) - (f_y'(X_r,\overline{Y}_r,Z_r^n))D_\theta^i Z_r^n]dr$$

tends to zero, as $n \to \infty$. One can then show that $D_\theta^i Y^n$, $D_\theta^i Z^n$ are Cauchy in $L^2(\theta,T\,;\,(\mathbb{D}^{1,2})^k)$, and the limit satisfies, for $t \le \theta \le s \le T$, $1 \le i \le d$,

$$D_\theta^i Y_s = g'(X_T)D_\theta^i X_T$$

$$\int_s^T [f_x'(X_r,\overline{Y}_r,Z_r)D_\theta^i X_r + f_y'(X_r,\overline{Y}_r,Z_r)D_\theta^i \overline{Y}_r + f_z'(X_r,\overline{Y}_r,Z_r)D_\theta^i Z_r]dr$$

$$-\int_s^T D_\theta^i Z_r dW_r$$

The same kind of procedure applies to the second approximating sequence, which proves all but the last statement of the Proposition.

Finally, since for $t < \theta \le s \le T$,

$$Y_s = Y_t - \int_t^s f(X_r,Y_r,Z_r)dr + \int_t^s Z_r dW_r\,,$$

$$D_\theta^i Y_s = (Z_\theta)_i - \int_\theta^s [f'(X_r,Y_r,Z_r)D_\theta^i X_r + f_y'(X_r,Y_r,Z_r)D_\theta^i Y_r$$

$$+f_z'(X_r,Y_r,Z_r)D_\theta^i Z_r]dr$$

$$-\int_\theta^s D_\theta^i Z_r dW_r\,,$$

for a.e. s, the jump of $D_\theta Y_s$ at $\theta = s$ equals Z_s. With the version of $D_\theta Y_s$ that we have choosen above, that means exactly that

$$D_s Y_s = Z_s\,,\ s \text{ a.e.}$$

□

We next want to show that $\{D_s Y_s\,;\ t \le s \le T\}$ processes an a.s. continuous version. For that sake, we first recall that the matrix valued process $\{\nabla X_s = \left(\frac{\partial X_s^i}{\partial x^j}\right)_{1\le i,j\le d};\ t \le s \le T\}$ solves the following SDE :

$$\nabla X_s = I + \int_t^s b'(X_r)\nabla X_r dr + \int_t^s \sigma_l'(X_r)\nabla X_r dW_r^l$$

The next formula is an immediate consequence of the uniqueness of the solution of the SDE satisfied by $D_\theta X_s$:

$$D_\theta X_s = \nabla X_s (\nabla X_\theta)^{-1} \sigma(X_\theta)\,,\ t \le \theta \le s \le T\,. \tag{12}$$

Let $\{\nabla Y_s, \nabla Z_s\,;\ t \le s \le T\}$ be the unique element of $M^2(t,T;\ \mathbb{R}^{k\times d}) \times M^2(t,T;\ \mathbb{R}^{k\times d\times d})$ which solves :

$$\nabla Y_s = g'(X_T)\nabla X_T \tag{13}$$

$$+ \int_s^T [f'_x(X_r, Y_r, Z_r)\nabla X_r + f'_y(X_r, Y_r, Z_r)\nabla Y_r + f'_z(X_r, Y_r, Z_r)\nabla Z_r]dr$$

$$- \int_s^T \nabla Z_r dW_r,\ t \le s \le T\,.$$

We shall later interpret ∇Y_s (resp. ∇Z_s) as the matrix of first order partial derivatives of Y_s (resp. Z_s) with respect to x (x denoting again the initial condition for X_t). For the time being, let us establish the :

Lemma 2.4 *For $t \le \theta \le s \le T$,*

$$D_\theta Y_s = \nabla Y_s (\nabla X_\theta)^{-1} \sigma(X_\theta)\,,$$

and the process $\{D_s Y_s\,;\ t \le s \le T\}$ as defined by Proposition 2.2 is a.s. continuous.

Proof : We deduce from the uniqueness of the solution of equation (9) and formula (12) that :

$$D^i_\theta Y_s = \nabla Y_s (\nabla X_\theta)^{-1} \sigma_i(X_\theta)\,,\ t \le \theta \le T\,.$$

The first of the statements follows, hence

$$D_s Y_s = \nabla Y_s (\nabla X_s)^{-1} \sigma(X_s)$$

and the continuity of $D_s Y_s$ follows from that of $\nabla Y_s, \nabla X_s$ and X_s □

From Proposition 2.2 and Lemma 2.4, we deduce that $\{Z_s\,;\ t \le s \le T\}$ has an a.s. continuous version, and we shall from now on identify $\{Z_s\}$ with its continuous version. An immediate consequence of Proposition 2.2 and Lemma 2.4 is now :

Lemma 2.5 *For any $0 \le t \le s \le T$, $x \in \mathbb{R}^d$,*

$$Z^{t,x}_s = \nabla Y^{t,x}_s (\nabla X^{t,x}_s)^{-1} \sigma(X^{t,x}_s)$$

and in particular

$$Z^{t,x}_t = \nabla Y^{t,x}_t \sigma(x)\,.$$

□

Since one can establish $L^p(\Omega)$ estimates for $\sup_s |\nabla Y_s|$ as was done for $\sup_s |Y_s|$ in Lemma 2.1, we deduce from the last Lemma :

Lemma 2.6 *For any $p \ge 1$,*

$$E\left(\sup_{t \le s \le T} \|Z^{t,x}_s\|^p\right) < \infty$$

□

We now study the regularity with respect to x of $Y_t^{t,x}$. Our proof is an adaptation of that of Stroock [8] for the usual SDEs, but we include the continuity with respect to t of $Y_t^{t,x}$ and its x-derivatives. Let us first introduce some notations. If g is a function of $x \in \mathbb{R}^d$, for $h \in \mathbb{R} \setminus \{0\}$, let $\Delta_h^i g(x) \triangleq h^{-1}[g(x+he_i) - g(x)]$, $1 \le i \le d$, where e_i denotes the i-th vector of an arbitrary orthonormal basis of $\mathbb{R}^d$.

Let us state the main technical steps for the process $X^{t,x}$ for further reference. We omit the proofs which are adaptations of those in [8]. Note that we define $X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}$ for any $(s,t) \in [0,T]^2$, $x \in \mathbb{R}^d$ by letting $X_s^{t,x} = X_{s\vee t}^{t,x}$ and similarly for $Y_s^{t,x}$, while $Z_s^{t,x} = 0$ for $s < t$.

Lemma 2.7 *For any $p \ge 2$, there exists a constant c_p such that for any $t, t' \in [0,T]$, $x, x' \in \mathbb{R}^d$, $i \in \{1,\dots,d\}$, $h, h' \in \mathbb{R} \setminus \{0\}$,*

$$E\left(\sup_{0\le s\le T} |X_s^{t,x}|^p\right) \le c_p(1+|x|^p) \tag{14}$$

$$E\left(\sup_{0\le s\le T} |X_s^{t,x} - X_s^{t',x'}|^p\right) \le c_p(1+|x|^p)\left(|x-x'|^p + |t-t'|^{\frac{p}{2}}\right) \tag{15}$$

$$E\left(\sup_{0\le s\le T} |\Delta_h^i X_h^{t,x}|^p\right) \le c_p \tag{16}$$

$$E\left(\sup_{0\le s\le T} |\Delta_h^i X_s^{t,x} - \Delta_{h'}^i X_s^{t',x'}|^p\right) \tag{17}$$

$$\le c_p(1+|x|^p)\left(|x-x'|^p + |h-h'|^p + |t-t'|^{\frac{p}{2}}\right)$$

□

It then follows immediately, using Kolmogorov's Lemma :

Corollary 2.8 *For any $t \in [0,T]$, $x \in \mathbb{R}^d$, the mapping $x \to X_s^{t,x}$ is a.s. differentiable, and the matrix of partial derivatives $\nabla X_s^{t,x}$ possesses a version which is a.s. continuous in (s,t,x).*

□

Iterating the argument, we obtain the existence of jointly continuous second derivatives.

We now follow the same procedure to establish :

Theorem 2.9 *$\{Y_s^{t,x};\ (s,t) \in [0,T]^2,\ x \in \mathbb{R}^d\}$ has a version whose trajectories belong to $C^{0,0,2}([0,T]^2 \times \mathbb{R}^d)$.*

Before proceeding to the proof, let us state the Corollary which we shall need in the next section :

Corollary 2.10 *For any $t \in [0,T]$, the mapping $x \to Y_t^{t,x}$ is of class C^2, the function and its partial derivatives of order one and two being continuous in (t,x)*

□

Proof of Theorem 2.9 We shall only prove the analog of Lemma 2.7. Going back to the proof of Lemma 2.1, and using (14) we deduce that for any $p \geq 2$, there exist c_p and q such that :

$$E\left(\sup_{0\leq s\leq T}|Y_s^{t,x}|^p\right) \leq c_p(1+|x|^q) \tag{18}$$

and moreover

$$E\left[\left(\int_t^T \|Z_s^{t,x}\|^2 ds\right)^{p/2}\right] \leq c_p(1+|x|^q)\,. \tag{19}$$

Next for $t \vee t' \leq s \leq T$,

$$\begin{aligned}
Y_s^{t,x} - Y_s^{t',x'} &= \left(\int_0^1 g'\left(X_T^{t',x'} + \lambda(X_T^{t,x} - X_T^{t',x'})\right)d\lambda\right)[X_T^{t,x} - X_T^{t',x'}] \\
&\quad + \int_s^T \Big(\varphi_r(t,x\,;\,t',x')[X_r^{t,x} - X_r^{t',x'}] + \psi_r(t,x\,;\,t',x')[Y_r^{t,x} - Y_r^{t',x'}] \\
&\qquad + \chi_r(t,x\,;\,t',x')[Z_r^{t,x} - Z_r^{t',x'}]\Big)\,dr \\
&\quad - \int_s^T (Z_r^{t,x} - Z_r^{t',x'})dW_r
\end{aligned}$$

where

$$\begin{aligned}
\varphi_r(t,x\,;\,t',x') &= \int_0^1 f_x'(\Sigma_{r,\lambda}^{t,x;t',x'})d\lambda \\
\psi_r(t,x\,;\,t',x') &= \int_0^1 f_y'(\Sigma_{r,\lambda}^{t,x;t',x'})d\lambda \\
\chi_r(t,x\,;\,t',x') &= \int_0^1 f_z'(\Sigma_{r,\lambda}^{t,x;t',x'})d\lambda
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_{r,\lambda}^{t,x;t',x'} &= \Big(r, X_r^{t',x'} + \lambda(X_r^{t,x} - X_r^{t',x'}),\ Y_r^{t',x'} + \lambda(Y_r^{t,x} - Y_r^{t',x'}), \\
&\qquad Z_r^{t',x'} + \lambda(Z_r^{t,x} - Z_r^{t',x'})\Big)\,.
\end{aligned}$$

Combining the argument of Lemma 2.1 with (15), we obtain :

$$E\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x} - Y_s^{t',x'}\right|^p\right) \leq c_p(1+|x|^q)\times\left(|x-x'|^p + |t-t'|^{p/2}\right) \tag{20}$$

In fact we should restrict the sup to $t \vee t' \leq s \leq T$, but (20) follows then easily from that restricted result. We have moreover :

$$E\left[\left(\int_{t\wedge t'}^T \|Z_s^{t,x} - Z_s^{t',x'}\|^2 ds\right)^{p/2}\right] \leq c_q(1+|x|^q)\left(|x-x'|^p + |t-t'|^{p/2}\right) \tag{21}$$

We next have :

$$\Delta_h^i Y_s^{t,x} = \int_0^1 g'(X_T^{t,x} + \lambda h \Delta_h^i X_T^{t,x}) \Delta_h^i X_T^{t,x} d\lambda$$

$$+ \int_s^T \int_0^1 \left[f_x'(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i X_r^{t,x} + f_y'(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Y_r^{t,x} + f_z'(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Z_r^{t,x} \right] d\lambda dr$$

$$- \int_s^T \Delta_h^i Z_r^{t,x} dW_r ,$$

where $\Xi_{r,\lambda}^{t,x,h} = (r, X_r^{t,x} + \lambda h \Delta_h^i X_r^{t,x},\ Y_r^{t,x} + \lambda h \Delta_h^i Y_h^{t,x},\ Z_r^{t,x} + \lambda h \Delta_h^i Z_r^{t,x})$.

Using again arguments similar to those in Lemma 2.1 and (16), we obtain that there exists c_p and q such that

$$E\left(\sup_{t \le s \le T} |\Delta_h^i Y_s^{t,x}|^p \right) \le c_p (1 + |x|^q + |h|^q) , \tag{22}$$

$$E\left[\left(\int_t^T \|\Delta_h^i Z_s^{t,x}\|^2 \right)^{p/2} \right] \le c_p (1 + |x|^q + |h|^q) \tag{23}$$

Finally,

$$\Delta_h^i Y_s^{t,x} - \Delta_{h'}^i Y_s^{t',x'} = \left(\int_0^1 g'(X_T^{t,x} + \lambda h \Delta_h^i X_T^{t,x}) d\lambda \right) \Delta_h^i X_T^{t,x}$$

$$- \left(\int_0^1 g'(X_T^{t',x'} + \lambda h \Delta_{h'}^i X_T^{t',x'}) d\lambda \right) \Delta_{h'}^i X_T^{t',x'}$$

$$+ \int_s^T \int_0^1 [f_x'(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i X_r^{t,x} - f_x'(\Xi_{r,\lambda}^{t',x',h'}) \Delta_{h'}^i X_r^{t',x'}] d\lambda dr$$

$$+ \int_s^T \int_0^1 [f_y'(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Y_r^{t,x} - f_y'(\Xi_{r,\lambda}^{t',x',h'}) \Delta_{h'}^i Y_r^{t',x'}] d\lambda dr$$

$$+ \int_s^T \int_0^1 [f_z'(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Z_r^{t,x} - f_z'(\Xi_{r,\lambda}^{t',x',h'}) \Delta_{h'}^i Z_r^{t',x'}] d\lambda dr$$

$$- \int_s^T [\Delta_h^i Z_r^{t,x} - \Delta_{h'}^i Z_r^{t',x'}] dW_r$$

We claim that, again by the procedure of Lemma 2.1, using the properties of f and (17), (20), (21), we can deduce :

$$E\left(\sup_{t \wedge t' \le s \le T} |\Delta_h^i Y_s^{t,x} - \Delta_{h'}^i Y_s^{t',x'}|^p \right) \le c_p (1 + |x|^q + |x'|^q + |h|^q + |h'|^q) \times (|x - x'|^p + |h - h'|^p + |t - t'|^{p/2})$$

$$E\left[\left(\int_{t \wedge t'}^T \|\Delta_h^i Z_s^{t,x} - \Delta_{h'}^i Z_s^{t',x'}\|^2 ds \right)^{p/2} \right] \le c_p (1 + |x|^q + |x'|^q + |h|^q + |h'|^q) \times (|x - x'|^p + |h - h'|^p + |t - t'|^{p/2})$$

Let us only indicate how we can treat the "hardest" term :

$$\Big|E\int_a^b \Big(\int_0^1 [f'_z(\Xi_{r,\lambda}^{t,x,h})\Delta_h^i Z_r^{t,x} - f'_z(\Xi_{r,\lambda}^{t',x',h'})\Delta_{h'}^i Z_r^{t',x'}]d\lambda\,,$$

$$\Delta_h^i Y_r^{t,x} - \Delta_{h'}^i Y_r^{t',x'}\Big)\,|\Delta_h^i Y_r^{t,x} - \Delta_{h'}^i Y_{r'}^{t',x'}|^{p-2}dr\Big|$$

$$\le cE\int_a^b \|\Delta_h^i Z_r^{t,x} - \Delta_{h'}^i Z_r^{t',x'}\| \times |\Delta_h^i Y_r^{t,x} - \Delta_{h'}^i Y_r^{t',x'}|^{p-1}dr$$

$$+cE\int_a^b \|\Delta_h^i Z_r^{t,x}\| \Big(\int_0^1 |\Xi_{r,\lambda}^{t,x,h} - \Xi_{r,\lambda}^{t',x',h'}|d\lambda\Big)\,|\Delta_h^i Y_r^{t,x} - \Delta_{h'}^i Y_r^{t',x'}|^{p-1}dr$$

$$\le \frac{1}{2}E\Big(\sup_{a\le r\le b}|\Delta_h^i Y_r^{t,x} - \Delta_{h'}^i Y_r^{t',x'}|^p\Big)$$

$$+\bar{c}(b-a)E\left[\Big(\int_a^b \|\Delta_h^i Z_r^{t,x} - \Delta_{h'}^i Z_r^{t',x'}\|^2 dr\Big)^{p/2}\right]$$

$$+\bar{c}\sqrt{E\left[\Big(\int_a^b \|\Delta_h^i Z_r^{t,x}\|^2 dr\Big)^{p/2}\right]}\sqrt{E\left[\Big(\int_a^b\int_0^1 |\Xi_{r,\lambda}^{t,x,h} - \Xi_{r,\lambda}^{t',x',h'}|^2 d\lambda dr\Big)^p\right]}$$

We note that the two first terms on the right are substracted from the left terms of the full inequality, with $(b-a)$ small enough, and the last term is estimated with the help of (23), (15), (20) and (21). Note also that we choose first $b = T$, $a = T - \alpha$, then $b = T - \alpha$, $a = T - 2\alpha$, etc... □

As a by-product of the above proof, we obtain :

Corollary 2.11 *$\{(\nabla Y_s^{t,x},\ \nabla Z_s^{t,x}),\ t \le s \le T\}$, the unique solution of the BSDE (13), is the gradient of $\{(Y_s^{t,x},\ Z_s^{t,x}),\ t \le s \le T\}$ with respect to x.*

Proof : This follows easily from the fact that (17) holds true with $h, h' \in \mathbb{R}$ if we define $\Delta_0^i X_s^{t,x} = \frac{\partial X_s^{t,x}}{\partial x_i}$, and that by definition of the partial derivatives,

$$\frac{\partial Y_s^{t,x}}{\partial x_i} = \lim_{h\to 0}\Delta_h^i Y_s^{t,x},\ \frac{\partial Z_s^{t,x}}{\partial x_i} = \lim_{h\to 0}\Delta_h^i Z_s^{t,x}\,.$$

□

3 Backward SDEs and systems of quasilinear parabolic partial differential equations

We now relate our BSDE to the following system of quasilinear parabolic differential equations :

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),(\nabla u\sigma)(t,x)) &= 0 \\ u(T,x) &= g(x) \end{cases} \qquad (24)$$

where $u\ :\ \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^k$, and

$$\mathcal{L}u = \begin{pmatrix} Lu_1 \\ \vdots \\ Lu_k \end{pmatrix},$$

$$L = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i}.$$

Let us first recall a result from [4] :

Theorem 3.1 *If $u \in C^{1,2}([0,T]) \times \mathbb{R}^d)$ solves equation (24), then $u(t,x) = Y_t^{t,x}$, $t \geq 0$, $x \in \mathbb{R}^d$, where $\{(Y_s^{t,x}, Z_s^{t,x});\ t \leq s \leq T\}_{t\geq 0, x\in\mathbb{R}^d}$ is the unique solution of the BSDE (2).*

Proof : Use Itô's formula applied to $u(s, X_s^{t,x})$ between $s = t$ and $s = T$, and note that $\{u(s, X_s^{t,x}),$ $(\nabla u\sigma)(s, X_s^{t,x})\}$ solves the BSDE (2). □

We are now in a position to prove the converse to the above result :

Theorem 3.2 *Under the assumptions stated in section 1,*

$$u(t,x) \triangleq Y_t^{t,x};\ t \geq 0,\ x \in \mathbb{R}^d$$

is of class $C^{1,2}([0,T] \times \mathbb{R}^d)$, and solves equation (24).

Proof : From Theorem 2.9, $u \in C^{0,2}([0,T] \times \mathbb{R}^d)$ Let $h > 0$ be such that $t + h \leq T$. Clearly, $Y_{t+h}^{t,x} = Y_{t+h}^{t+h, X_{t+h}^{t,x}}$. Hence

$$\begin{aligned} u(t+h,x) - u(t,x) &= u(t+h,x) - u(t+h, X_{t+h}^{t,x}) + u(t+h, X_{t+h}^{t,x}) - u(t,x) \\ &= -\int_t^{t+h} Lu(t+h, X_r^{t,x})dr - \int_t^{t+h} (\nabla u\sigma)(t+h, X_r^{t,x})dW_r \\ &\quad - \int_t^{t+h} f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_t^{t+h} Z_r^{t,x} dW_r \end{aligned}$$

where we have used Itô's formula (and the fact that $u(t,\cdot) \in C^2(\mathbb{R}^d)$) and the BSDE. Let now $t = t_0 < t_1 < \ldots < t_n = T$. We have

$$\begin{aligned} g(x) - u(t,x) &= -\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} [Lu(t_{i+1}, X_r^{t_i,x}) + f(X_r^{t_i,x}, Y_r^{t_i,x}, Z_r^{t_i,x})]dr \\ &\quad + \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} [Z_r^{t_i,x} - (\nabla u\sigma)(t_{i+1}, X_r^{t_i,x})]dW_r \end{aligned}$$

It now follows from Theorem 2.9 and Lemma 2.5 that, if we take a sequence of meshes $t = t_0^n < t_1^n < \cdots < t_n^n = T$ such that $\lim_{n\to\infty}\sup_{i\leq n-1}(t_{i+1}^n - t_i^n) = 0$, we obtain in the limit :

$$u(t,x) = g(x) + \int_t^T [Lu(s,x) + f(s,x,u(s,x),\ (\nabla u\sigma)(s,x))]ds$$

hence $u \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and satisfies the equation (24). □

Remark 3.3 In the classical case where $k = 1$ and

$$f(t,x,y,z) = c(t,x)y \ ,$$

our result reduces to the classical Feynman–Kac formula. Indeed, in that case the BSDE (2) has the explicit solution :

$$Y_s^{t,x} = e^{\int_s^T c(r,X_r^{t,x})dr} g(X_T) - \int_s^T e^{\int_s^r c(\sigma,X_\sigma^{t,x})d\sigma} Z_r^{t,x} dW_r$$

and

$$\begin{aligned} Y_t^{t,x} &= E(Y_t^{t,x}) \\ &= E\left[e^{\int_t^T c(r,X_r^{t,x})dr} g(X_T)\right] . \end{aligned}$$

□

4 Backward SDEs and viscosity solutions of quasilinear parabolic PDEs

We now restrict ourselves to the case $k = 1$, and we shall show that when the coefficients f and g are Lipschitz continuous, the BSDE provides the unique viscosity solution of a quasilinear parabolic PDE. The results of this section are particular cases of results in Peng [5]. However, we present then for the sake of completeness, and because the argument here is simpler than that in [5].

We first recall a technical Lemma. Let $f = \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d^d} \to \mathbb{R}^d$ be $\mathcal{P} \otimes \mathcal{B} \otimes \mathcal{B}_d/\mathcal{B}$ measurable, where $\mathcal{P}$ denotes the σ-algebra of $\{\mathcal{F}_t\}$-progressively measurable subsets of $\Omega \times [0,T]$. as usual, we shall write $f(t,y,z)$ instead of $f(\omega,t,y,z)$. We assume that

$$f(.,0,0) \in M^2(0,T) \tag{25}$$

and that there exists $c > 0$ such that

$$|f(t,y,z) - f(t,y',z')| \le c(|y-y'| + |z-z'|) \tag{26}$$

Given $Q, \overline{Q} \in L^2(\Omega, \mathcal{F}_T, P)$ and $F \in M^2(0,T)$, let $\{(Y_t, Z_t), t \ge 0\}$ (resp. $\{(\overline{Y}_t, \overline{Z}_t), t \ge 0\}$) denote the unique solution of the BSDE

$$Y_t = Q + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \tag{27}$$

(resp. of the BSDE

$$\overline{Y}_t = \overline{Q} + \int_t^T [f(s, \overline{Y}_s, \overline{Z}_s) + F_s]ds - \int_t^T \overline{Z}_s dW_s). \tag{28}$$

We have the following comparison result :

Lemma 4.1 *Let $\overline{Q} \ge 0$ a.s., $F_t \ge 0$ a.s., t a.e. Then $\overline{Y}_t \ge Y_t$ a.s., t a.e.*

Proof : This result is proved in Proposition 2.4 of [5] under the additional assumption that f is C^1 in (y,z). The present result follows by a standard approximation argument. □

We now use again the notations from section 2, assuming only that b, σ, f, g are globally Lipschitz with respect to (x,y,z), uniformly in t -this last precision concerns only f. We define again, as in Theorem 3.1,

$$u(t,x) \triangleq Y_t^{t,x} \tag{29}$$

and note that the estimate (20) still applies, and hence u is locally Lipschitz in x and Hölder continuous in t, and therefore u is Hölder continuous in (t,x).

However, we do not expect that u is differentiable in (t,x). We are going to show that u is the viscosity solution of the backward parabolic PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) + Lv(t,x) + f(t,x,v(t,x),(\nabla v\sigma)(t,x)) = 0 \\ v(T,x) = g(x) \end{cases} \tag{30}$$

Definition 4.2 *Let $u \in C([0,T] \times \mathbb{R}^d)$ satisfy $u(T,x) = g(x), x \in \mathbb{R}^d$. u is said to be a viscosity sub-solution (resp. super-solution) of equation (4.6) if in addition for any $(t,x) \in (0,T) \times \mathbb{R}^d$ and $\varphi \in C^{1,2}((0,T) \times \mathbb{R}^d)$ such that $\varphi(t,x) = u(t,x)$ and (t,x) is a minimum (resp. maximum) of $\varphi - u$,*

$$\frac{\partial \varphi}{\partial t}(t,x) + L\varphi(t,x) + f(t,x,\varphi(t,x),(\nabla\varphi\sigma)(t,x)) \geq 0$$

$$(\text{resp.} \frac{\partial \varphi}{\partial t}(t,x) + L\varphi(t,x) + f(t,x,\varphi(t,x),(\nabla\varphi\sigma)(t,x)) \leq 0).$$

u is said to be a viscosity solution of (30) if it is both a viscosity sub-and super-solution of (30). □

We can now establish the main result of this section

Theorem 4.3 *The function u defined by (4.5) is the unique viscosity solution of the backward parabolic PDE (30).*

Proof : Uniqueness follows from Ishii-Lions [7]. We shall show that u is a viscosity sub-solution of (30). The property of being a super-solution could be proved analogously.

We first note that for $0 < t \leq t+h < T$ and $x \in \mathbb{R}^d$,

$$u(t,x) = u(t,h,X_{t+h}^{t,x}) + \int_t^{t+h} f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds - \int_t^{t+h} Z_s^{t,x}dW_s$$

Let now $\varphi \in C^{1,2}((0,T) \times \mathbb{R}^d)$ satisfy $\varphi(t,x) = u(t,x)$ and $\varphi \geq u$ on $(0,T) \times \mathbb{R}^d$.
We can without loss of generality assume that φ has bounded derivatives.
We then have

$$\begin{aligned} \varphi(t,h,X_{t+h}^{t,x}) - \varphi(t,x) \quad &+ \int_t^{t+h} f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds \\ &- \int_t^{t+h} Z_s^{t,x}dW_s \geq 0 \end{aligned}$$

Let $\{(\overline{Y}_s,\overline{Z}_s), t \leq s \leq t+h\}$ denote the solution of the BSDE

$$\overline{Y}_s = \varphi(t+h,X_{t+h}^{t,x}) + \int_s^{t+h} f(r,X_r^{t,x},\overline{Y}_r,\overline{Z}_r)dr - \int_s^{t+h} \overline{Z}_r dW_r \tag{31}$$

From Lemma 4.1,

$$\varphi(t,x) \le \varphi(t+h, X^{t,x}_{t+h}) + \int_t^{t+h} f(s, X^{t,x}_s, \overline{Y}_s, \overline{Z}_s)ds - \int_t^{t+h} \overline{Z}_s dW_s$$

and from Itô's formula

$$\int_t^{t+h} [\frac{\partial \varphi}{\partial s}(s, X^{t,x}_s) \; + \; L\varphi(s, X^{t,x}_s) + f(s, X^{t,x}_s, \overline{Y}_s, \overline{Z}_s)]ds$$
$$+ \int_t^{t+h} [\nabla\varphi\sigma(s, X^{t,x}_s) - \overline{Z}_s]dW_s \ge 0.$$

We note $\hat{Y}_s = \overline{Y}_s - \varphi(s, X^{t,x}_s), \hat{Z}_s = \overline{Z}_s - \nabla\varphi\sigma(s, X^{t,x}_s)$.
The last inequality can be rewritten as :

$$\int_t^{t+h} [(\tfrac{\partial \varphi}{\partial s} + L\varphi)(s, X^{t,x}_s) \; + f(s, X^{t,x}_s, \varphi(s, X^{t,x}_s) + \hat{Y}_s, (\nabla\varphi\sigma)(s, X^{t,x}_s) + \hat{Z}_s)]ds$$
$$- \int_t^{t+h} \hat{Z}_s dW_s \ge 0 \tag{32}$$

We deduce from (31) and Itô's formula that $\{(\hat{Y}_s, \hat{Z}_s), t \le s \le t+h\}$ is the unique solution of the BSDE

$$\hat{Y}_s = \int_s^{t+h} [(\frac{\partial \varphi}{\partial s} + L\varphi)(r, X^{t,x}_r) \; + f(r, X^{t,x}_r, \varphi(r, X^{t,x}_r) + \hat{Y}_r, (\nabla\varphi\sigma)(r, X^{t,x}_r) + \hat{Z}_r)]dr$$
$$- \int_s^{t+h} \hat{Z}_r dW_r.$$

We want to compare $\{(\hat{Y}_s, \hat{Z}_s)\}$ with the solution $\{(\tilde{Y}_s, 0), (t \le s \le t+h\}$ of the BSDE

$$\tilde{Y}_s = \int_s^{t+h} [(\frac{\partial \varphi}{\partial s} + L\varphi)(r,x) + f(r, x, \varphi(r,x) + \tilde{Y}_r, (\nabla\varphi\sigma)(r,x))]dr \tag{33}$$

We note that since φ has bounded derivatives,

$$\sup_{t \le r \le t+h} E(|\varphi(r, X^{t,x}_r) - \varphi(r,x)|^2) \to 0 \text{ as } h \to 0,$$

$$\sup_{t \le r \le t+h} E(|(\sigma\nabla\varphi)(r, X^{t,x}_r) - (\sigma\nabla\varphi)(r,x)|^2) \to \text{ as } h \to 0.$$

It is then easy to deduce from the techniques of section 1 :

$$E\left(\sup_{t \le s \le t+h} |\hat{Y}_s - \tilde{Y}_s|^2 + \int_t^{t+h} |\hat{Z}_s|^2 ds\right) = 0(h)$$

Now from (32) $\hat{Y}_s \ge 0, t \le s \le t+h$, hence

$$h^{-1}\tilde{Y}_s \ge -\varepsilon(h)$$

where $\varepsilon(h) \to 0$ as $h \to 0$. Hence

$$\frac{1}{h}\int_t^{t+h} [(\frac{\partial \varphi}{\partial s} + L\varphi)(r,x) + f(r, x, \varphi(r,x) + \tilde{Y}_r, (\nabla\varphi\sigma)(r,x))]dr \ge -\varepsilon(h)$$

Moreover it follows readily from (33) that there exists a constant $\overline{c}$ such that

$$|\tilde{Y}_s| \le \overline{c}h, t \le s \le t+h.$$

We remark that $\tilde{Y}$ clearly depends on h, also this was not made explicit. We finally conclude that

$$\frac{\partial \varphi}{\partial s}(t,x) + L\varphi(t,x) + f(t, x, \varphi(t,x), (\nabla\varphi\sigma)(t,x)) \ge 0$$

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Remark 4.4 We note that, with the help of the techniques in Peng [6], it is possible to relax the assumption of uniform Lipschitz continuity of f with respect to y. □

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