Homogenization of a singular random one dimensional PDE with time varying coefficients

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Abstract

In this paper we study the homogenization of a non-autonomous parabolic equation with a large random rapidly oscillating potential in the case of one dimensional spatial variable. We show that if the potential is a statistically homogeneous rapidly oscillating function of both temporal and spatial variables then, under proper mixing assumptions, the limit equation is deterministic and convergence in probability holds. To the contrary, for the potential having a microstructure only in one of these variables, the limit problem is stochastic and we only have convergence in law.

1   Introduction

Our goal is to study the limit, as $\varepsilon \to 0$, of the solution of the linear parabolic PDE

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial x^2}(t,x) + \varepsilon^{-\gamma} c\left(\dfrac{t}{\varepsilon^\alpha}, \dfrac{x}{\varepsilon^\beta}\right) u^\varepsilon(t,x), \quad t \geq 0,\ x \in \mathbb{R}; \\ u^\varepsilon(0,x) = g(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $g \in L^2(\mathbb{R}) \cap C_b(\mathbb{R})$, $\{c(t,x),\, t \in \mathbb{R}_+,\, x \in \mathbb{R}\}$ is a stationary random field defined on a probability space $(\mathcal{S}, \mathcal{A}, P)$, such that

$$E\, c(t,x) = 0, \quad t \in \mathbb{R}_+,\, x \in \mathbb{R}, \tag{1.2}$$

where E denotes expectation with respect to the probability measure P. In all this paper, we will assume that the random field c is uniformly bounded, i. e.

$$\sup_{t \geq 0,\ x \in \mathbb{R},\ s \in \mathcal{S}} |c(t,x,s)| < \infty.$$

We define the correlation function of the random field c as follows :

$$\Phi(t,x) := E\left[c(s,y)c(s+t, y+x)\right]. \tag{1.3}$$

We assume that $\Phi \in L^1(\mathbb{R} \times \mathbb{R})$. Additional mixing conditions, specific to each particular case, are formulated separately in each section.

We will consider various possible values for the parameters $\alpha,\ \beta \geq 0$, and we will see that the correct value for γ, such that the limiting effect of the highly oscillating term is non trivial, is

$$\gamma = \left(\frac{\alpha}{4} + \frac{\beta}{2}\right) \vee \frac{\alpha}{2},$$

and that the highly oscillating term can have three types of limit. If $\alpha = 0$, the result is similar to that obtained in [8], that is the limiting PDE is a type of SPDE driven by a noise which is white in space, and correlated in time. If $\beta = 0$, the limit is an SPDE driven by a noise which is white in time and correlated in space. We believe that in all cases where $\alpha > 0$ and $\beta > 0$, the limiting PDE is deterministic. One intuitive explanation of this result, which was first a surprise for the authors, is the following. In the case $\alpha,\ \beta > 0$, the limiting noise should be white both in time and space, i. e. the limiting PDE should be a "bilinear" SPDE driven a space–time white noise. But we know that the corresponding stochastic integral should be interpreted as a Stratonovich integral, i. e. an Itô integral plus a correction term. However, in the space–time white noise case, the correction term is infinite. Hence the correct choice of γ forces the Itô integral term to vanish, which is necessary for the "Itô–Stratonovich correction term" not to explode.

This result is consistent with those in Bal [1], where higher dimensional time independent situations are treated, with the severe restriction that the noise source c be a Gaussian random field, while our random field c is much

2

more general. One reason why we restrict ourselves to the one–dimensional case is that in higher spatial dimension (whether the problem is time dependent or time independent), the limit will always be a deterministic PDE, which restricts our motivation for studying that problem.

In fact, within the case α, $\beta > 0$, we have only been able to treat the case where $0 < \beta \leq \alpha/2$. The case $0 < \alpha < 2\beta$ remains open. Our methods do not seem to cover this last case.

Two variants of the same problem, but with coefficients not depending upon time t, have already been considered in [13] and in [8]. The case of random coefficients which are periodic in space was considered in [4].

The paper is organized as follows. In section 2 we state the Feynman–Kac formula for the solution u^ε of equation (1.1). Section 3 is devoted to a presentation of the various statements to be proved in the paper. In section 4 we give a criterion for convergence in law, which is used later. In section 5 we treat the case $\alpha = 0$, $\beta > 0$. In section 6 we treat the case $0 \leq 2\beta \leq \alpha$, starting with the case $\beta > 0$, and finally ending with the case $\beta = 0$, $\alpha > 0$.

2   The Feynman–Kac formula

Let $\{B_t;\ t \geq 0\}$ denote a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The pair

$$(\{c(t,x),\ t \geq 0, x \in \mathbb{R}\}, \{B_t;\ t \geq 0\})$$

is defined on the product probability space $(\Omega \times \mathcal{S}, \mathcal{F} \otimes \mathcal{A}, \mathbb{P} \times P)$, so that $\{c(t,x),\ t \geq 0, x \in \mathbb{R}\}$ and $\{B_t;\ t \geq 0\}$ are mutually independent.

The solution of equation (1.1) is given by the formula

$$\begin{aligned}
u^\varepsilon(t,x) &= \mathbb{E}\left[g(x + B_t)\exp\left(\varepsilon^{-\gamma}\int_0^t c\left(\frac{s}{\varepsilon^\alpha}, \frac{x + B_s}{\varepsilon^\beta}\right)ds\right)\right] \\
&= \mathbb{E}\left[g(x + B_t)\exp\left(\varepsilon^{-\gamma}\int_0^t \int_{\mathbb{R}} c\left(\frac{s}{\varepsilon^\alpha}, \frac{x + y}{\varepsilon^\beta}\right)L(ds, y)dy\right)\right],
\end{aligned} \tag{2.1}$$

where $L(t,x)$ denotes the local time at time t and at level x of the process B, and $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$. We shall use the notation $X_t^x = x + B_t$.

3   Statement of the results

We gather here the statements to be proved in the rest of the paper.

3.1   The case $\alpha = 0$, $\beta > 0$, $\gamma = \beta/2$

In this case we need the following additional assumptions.

For each $x \in \mathbb{R}$, $t \to c(t,x)$ is a. s. of class C^2, and the $\mathbb{R}^3$–valued random field

$$\{(c(t,x), c'(t,x), c''(t,x)); \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}, \tag{3.1}$$

is stationary, has zero mean, and is uniformly bounded; here and later on in this section we use the notation

$$c'(t,x) = \frac{\partial c}{\partial t}(t,x), \quad c''(t,x) = \frac{\partial^2 c}{\partial t^2}(t,x).$$

We assume that random field (3.1) is "ϕ–mixing in the x direction", in the sense that the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\phi(h) = \sup_{A \in \mathcal{G}_x, \ B \in \mathcal{G}^{x+h}, \ P(A) > 0} |P(B|A) - P(B)|,$$

where

$$\mathcal{G}_x = \sigma\{c(t,z), \ t \geq 0, z \leq x\} \quad \mathcal{G}^y = \sigma\{c(t,z), \ t \geq 0, z \geq y\},$$

satisfies

$$\phi \leq C(1+h)^{-(3+\delta)}, \tag{3.2}$$

for some C, $\delta > 0$.

We assume moreover that (by stationarity, the following quantities do not depend on t)

$$\int_{-\infty}^{\infty} |Ec(t,0)c(t,x)|dx < \infty, \quad \int_{-\infty}^{\infty} |Ec'(t,0)c'(t,x)|dx < \infty,$$

$$\int_{-\infty}^{\infty} |Ec''(t,0)c''(t,x)|dx < \infty.$$

In fact, these estimates follow from (3.2) and the boundedness of the functions in (3.1). Under those assumptions, we have the (see Theorem 5.10 below)

Theorem 3.1. *For any* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$u^\varepsilon(t, x) \to u(t, x) := \mathbb{E}\left[g(X_t^x) \exp\left(\int_0^t \int_{\mathbb{R}} L(ds, y - x) W(s, dy)\right)\right]$$

in P–law, as $\varepsilon \to 0$, *where* W *is a centered Gaussian noise which is "white in space and coloured in time" (see (5.3) for the accurate definition), defined on the probability space* $(\mathcal{S}, \mathcal{A}, P)$, *while* L *denotes the local time of a standard one–dimensional Brownian motion defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *In particular,* L *and* W *are independent.*

The limiting SPDE in this case is best written in the form (see equation (5.9) below)

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t, x) + \dfrac{\partial(uW)}{\partial x}(t, x) - \dfrac{\partial u}{\partial x}(t, x)W(t, x), & t \geq 0, \ x \in \mathbb{R}; \\ u(0, x) = g(x), & x \in \mathbb{R}. \end{cases}$$

Remark 3.2. *The argument used in the proof of Theorem 3.1 can be easily extended to prove weak convergence of all finite dimensional distributions of* u^ε *towards those of* u, *but we were not able to check tightness and prove convergence for the topology of locally uniform convergence (in* t *and* x).

3.2 The case $0 < 2\beta \leq \alpha$

We now introduce a new assumption, namely that

$$(H_{\text{um}}) \qquad\qquad \alpha_{\text{um}}(r) \leq C(1 + r)^{-(3+\delta)},$$

where α_{um} is the so–called the uniform mixing coefficient of the random field $c(t, x)$, defined as follows. For a set $A \subset \mathbb{R}^2$ denote by $\mathcal{F}_A$ the σ-algebra generated by $\{c(t, x) : (t, x) \in A\}$. We set

$$\alpha_{\text{um}}(r) = \sup |P(\mathcal{S}_1 | \mathcal{S}_2) - P(\mathcal{S}_1)|,$$

where the supremum is taken over all $\mathcal{S}_1 \in \sigma\{c(t, x), \ t \leq t_0, \ x \in \mathbb{R}\}$ and $\mathcal{S}_2 \in \sigma\{c(t, x), \ t \geq t_0 + r, \ x \in \mathbb{R}\}$, with $P(\mathcal{S}_2) > 0$. By stationarity, the supremum on the right hand side does not depend upon t_0.

Under the assumption (H_{um}), we have the (see Corollary 6.9, Corollary 6.11 and Proposition 6.14)

Theorem 3.3. *As $\varepsilon \to 0$, $u^\varepsilon(t,x)$ converges in probability, locally uniformly in t and x, to the deterministic function $u(t,x)$ given by*

$$u(t,x) = \mathbb{E}[g(x + B_t)] \exp\left(t\Sigma\right),$$

which is a solution of the deterministic parabolic PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + \Sigma u(t,x), & t \geq 0, \ x \in \mathbb{R}; \\ u(0,x) = g(x), \ x \in \mathbb{R}. \end{cases} \tag{3.3}$$

Here

$$\Sigma = \begin{cases} \int_0^\infty \mathbb{E}\Phi(r, B_r)dr, & \text{if } 2\beta = \alpha, \\ \int_0^\infty \Phi(r, 0)dr, & \text{if } 2\beta < \alpha, \end{cases}$$

and Φ has been defined by (1.3). Furthermore, for any compact set $K \subset \mathbb{R}$ we have

$$\lim_{\varepsilon \to 0} E\|u^\varepsilon - u\|^2_{L^2(((0,T)\times K))} = 0.$$

3.3 The case $\beta = 0$, $\alpha > 0$, $\gamma = \alpha/2$

In addition to (H_{um}), we assume here that the following assumption holds :

(Hö) For each $s \in \mathbb{R}$ the realizations $c(s,y)$ are a.s. Hölder continuous in $y \in \mathbb{R}$ with a deterministic exponent $\theta > 1/3$. Moreover,

$$|c(s,y_1) - c(s,y_2)| \leq c|y_1 - y_2|^\theta,$$

with a deterministic constant c.

Remark 3.4. *It is possible to weaken assumption (Hö), replacing the condition $\theta > 1/3$ by the condition $\theta > 0$, at the expense of replacing the exponent 3 in (H_{um}) with $k+1$, in case $1/(k+1) \leq \theta < 1/k$, $k \in \mathbb{N}$. Minor modifications in our proofs are necessary for each value of k.*

Under the three above assumptions, we have the (see Theorem 6.18 below)

Theorem 3.5. *Under assumptions (Hö) and (H_{um}), as $\varepsilon \to 0$,*

$$u^\varepsilon(t,x) \to u(t,x) := \mathbb{E}\left[g(X_t^x)\exp\left(\int_0^t \int_{\mathbb{R}} W(ds,y)L(s, y - x)dy\right)\right]$$

6

in P–law in $C(\mathbb{R}_+ \times \mathbb{R})$, equipped with the topology of local uniform convergence in t and x, where W is a centered Gaussian noise which is "white in time and coloured in space" (see Proposition 6.15) defined on the probability space $(\mathcal{S}, \mathcal{A}, P)$, while L denotes the local time of a standard one–dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In particular, L and W are independent.

In this case the limiting SPDE reads (in Stratonovich form)

$$\begin{cases} du(t, x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t, x)dt + u(t, x) \circ W(dt, x), \quad t \geq 0, \; x \in \mathbb{R}; \\ u(0, x) = g(x), \quad x \in \mathbb{R}. \end{cases}$$

4 A criterion for convergence in law

In the cases where the limit is deterministic, convergence in law is equivalent to convergence in probability. In fact in those cases we will establish convergence in $L^2(P)$. However, in the case where the limit is random, we are faced with true convergence in law. The quantity which should converge in law is a "partial expectation", by which we mean an expectation with respect to $\mathbb{P}$ alone (and not with respect to $\mathbb{P} \times P$), or in other words a conditional expectation. Taking the limit in law of such a quantity does not seem to be very common. In this section, we establish a criterion for convergence in law which is specially tailored for our needs.

Proposition 4.1. *Let* $\{Z^\varepsilon, \varepsilon > 0\}$ *be a collection of real-valued random variables, and suppose that there exist a random variable* Z *and, for each* $M > 0$, *random variables* Z_M^ε *and* Z_M *such that*

(i) *For any* M *the sequence* Z_M^ε *converges to* Z_M *in law, as* $\varepsilon \to 0$;

(ii) *It holds*
$$|Z^\varepsilon - Z_M^\varepsilon| \leq \frac{\chi^\varepsilon}{M}, \qquad |Z - Z_M| \leq \frac{\chi^0}{M},$$
where the family of r. v.'s $\{\chi^\varepsilon, \varepsilon \geq 0\}$ *is tight.*

Then Z^ε *converges to* Z *in law, as* $\varepsilon \to 0$.

Proof. Since $\{Z^\varepsilon, \ \varepsilon > 0\}$ is tight, it suffices to show that for all $\varphi \in C(\mathbb{R})$ with $|\varphi(x)| \leq 1$ for all $x \in \mathbb{R}$, and φ globally Lipschitz,

$$E\varphi(Z^\varepsilon) \to E\varphi(Z) \quad \text{as } \varepsilon \to 0.$$

Note that

$$E\varphi(Z^\varepsilon) - E\varphi(Z) = E[\varphi(Z) - \varphi(Z_M)] + E[\varphi(Z_M^\varepsilon) - \varphi(Z^\varepsilon)] + E\varphi(Z_M) - E\varphi(Z_M^\varepsilon).$$

If K stands for the Lipschitz constant of φ, then

$$|E\{\varphi(Z) - \varphi(Z_M) + \varphi(Z_M^\varepsilon) - \varphi(Z^\varepsilon)\}| \leq E \inf\left(4, \frac{K}{M}(\chi^\varepsilon + \chi^0)\right).$$

Consequently, as $M \to \infty$,

$$\sup_{\varepsilon > 0} |E\{\varphi(Z) - \varphi(Z_M) + \varphi(Z_M^\varepsilon) - \varphi(Z^\varepsilon)\}| \to 0.$$

The result follows since by (i) for each fixed M, $E\varphi(Z_M) - E\varphi(Z_M^\varepsilon) \to 0$, as $\varepsilon \to 0$. $\qquad\square$

Corollary 4.2. *Let $\mathbb{X}$ a Banach space, $\Psi : \Omega \times \mathbb{X} \to \mathbb{R}$ a mapping and $\{W^\varepsilon, \ \varepsilon > 0\}$ a family of $\mathbb{X}$–valued random variables defined on $(\mathcal{S}, \mathcal{A}, P)$ be such that*

(i) $x \to \Psi(\omega, x)$ is continuous, in $\mathbb{P}$–probability,

(ii) $\forall x \in \mathbb{X}$, $\omega \to \Psi(\omega, x)$ is $\mathcal{F}$–measurable,

(iii) for some $\delta > 0$, the family $\{\mathbb{E}|\Psi|^{1+\delta}(\cdot, W^\varepsilon), \ \varepsilon > 0\}$ is tight.

If moreover W^ε converges in law towards W, then as $\varepsilon \to 0$

$$\mathbb{E}\Psi(\cdot, W^\varepsilon) \text{ converges in law to } \mathbb{E}\Psi(\cdot, W).$$

Proof. For $M > 0$ and $z \in \mathbb{R}$ write $\psi_M(z) = (z \wedge M) \vee (-M)$. Note that

$$|\Psi(\cdot, W^\varepsilon) - \psi_M \circ \Psi(\cdot, W^\varepsilon)| \leq \frac{|\Psi|^{1+\delta}(\cdot, W^\varepsilon)}{M^\delta}.$$

Consequently, we can apply Proposition 4.1 with $Z^\varepsilon = \mathbb{E}\Psi(\cdot, W^\varepsilon)$, $Z = \mathbb{E}\Psi(\cdot, W)$, $Z_M^\varepsilon = \mathbb{E}\psi_M \circ \Psi(\cdot, W^\varepsilon)$, $Z_M = \mathbb{E}\psi_M \circ \Psi(\cdot, W)$, since by Lebesgue's dominated convergence theorem $x \to \mathbb{E}\psi_M \circ \Psi(\cdot, x)$ is continuous from $\mathbb{X}$ into $\mathbb{R}$. $\qquad\square$

Remark 4.3. *Writing*

$$u^\varepsilon(t, x) = \mathbb{E}\left[g(X_t^x) \exp(Y_{t,x}^\varepsilon) \right]$$

we will check the third condition of the Corollary with $\delta = 1/3$ and we shall use the following Hölder inequality

$$\mathbb{E}\left[|g|^{4/3}(X_t^x) \exp\left(\frac{4}{3} Y_{t,x}^\varepsilon \right) \right] \leq \left(\mathbb{E} g^2(X_t^x) \right)^{2/3} \left(\mathbb{E} \exp(4 Y_t^{x,\varepsilon}) \right)^{1/3}.$$

So we have to check that the family $\{\mathbb{E} \exp(4 Y_t^{x,\varepsilon}), \ \varepsilon > 0\}$ is tight.

5 The case $\alpha = 0$, $\beta > 0$.

In this case, $\gamma = \beta/2$. Without loss of generality, we restrict ourselves to the case $\beta = 1$. For each $\varepsilon > 0$, $x \in \mathbb{R}$, we define the process

$$Y_{t,x}^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^t c\left(s, \frac{X_s^x}{\varepsilon} \right) ds, \ t \geq 0.$$

It will be convenient in this section to assume that for each $x \in \mathbb{R}$, $t \to c(t, x)$ is a. s. of class C^2, and that the $\mathbb{R}^3$–valued random field

$$\{(c(t,x), c'(t,x), c''(t,x)); \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}, \tag{5.1}$$

is stationary, has zero mean, and is uniformly bounded; here and later on in this section we use the notation

$$c'(t,x) = \frac{\partial c}{\partial t}(t,x), \quad c''(t,x) = \frac{\partial^2 c}{\partial t^2}(t,x).$$

We assume that random field (5.1) is "ϕ–mixing in the x direction", in the sense that the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\phi(h) = \sup_{A \in \mathcal{G}_x, \ B \in \mathcal{G}^{x+h}, \ P(A)>0} |P(B|A) - P(B)|,$$

where

$$\mathcal{G}_x = \sigma\{c(t,z), \ t \geq 0, z \leq x\} \quad \mathcal{G}^y = \sigma\{c(t,z), \ t \geq 0, z \geq y\},$$

satisfies
$$\phi \leq C(1+h)^{-(3+\delta)},$$
for some C, $\delta > 0$.

We assume moreover that (by stationarity, the following quantities do not depend on t)

$$\int_{-\infty}^{\infty} |Ec(t,0)c(t,x)|dx < \infty, \quad \int_{-\infty}^{\infty} |Ec'(t,0)c'(t,x)|dx < \infty,$$

$$\int_{-\infty}^{\infty} |Ec''(t,0)c''(t,x)|dx < \infty.$$

Remark 5.1. *We suspect that the assumption of C^2 regularity is much stronger than what is necessary for the result that follows to hold. However, in the case of weaker regularity assumptions, there are technical difficulties which we were not able to overcome.*

5.1 Weak convergence

The aim of this subsection is to prove the

Theorem 5.2. *For each $t > 0$, $x \in \mathbb{R}$,*

$$Y_{t,x}^{\varepsilon} \to Y_{t,x} := \int_0^t \int_{\mathbb{R}} L(ds, y - x)W(s, dy), \tag{5.2}$$

in P–law, as $\varepsilon \to 0$, where, as above, $L(t,y)$ is the local time at level y and time t of the Brownian motion $\{X_t^0, \ t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{W(t,y), \ y \in \mathbb{R}\}$ is a centered Gaussian random field defined on $(\mathcal{S}, \mathcal{A}, P)$, with the covariance function

$$E(W(t,x)W(t',x')) = \begin{cases} \Psi(t-t')|x| \wedge |x'|, & \text{if } x\,x' > 0; \\ 0, & \text{if } x\,x' < 0, \end{cases} \tag{5.3}$$

where for each $r \in \mathbb{R}$,

$$\Psi(r) = \int_{\mathbb{R}} \Phi(r, y)dy,$$

and the double integral in (5.2) is defined below. In particular (X, L) and W are independent.

10

We define

$$W_\varepsilon(t,x) = \frac{1}{\sqrt{\varepsilon}} \int_0^x c\left(t, \frac{y}{\varepsilon}\right) dy, \quad W_\varepsilon'(t,x) = \frac{1}{\sqrt{\varepsilon}} \int_0^x c'\left(t, \frac{y}{\varepsilon}\right) dy.$$

Note that $\{W_\varepsilon(t,x), W_\varepsilon'(t,x)\}$ is a random field defined on the probability space $(\mathcal{S}, \mathcal{A}, P)$.

We first prove

Proposition 5.3. *The sequence of random fields $\{(W_\varepsilon, W_\varepsilon')\}$ converges weakly as random fields defined on the probability space $(\mathcal{S}, \mathcal{A}, P)$, as $\varepsilon \to 0$, in the space $C(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}^2)$ equipped with the topology of uniform convergence on compact sets, to a centered Gaussian random field*

$$\{(W(t,x), W'(t,x)), \ t \geq 0, x \in \mathbb{R}\},$$

where the covariance function of $\{W(t,x)\}$ is given by (5.3), and

$$W'(t,x) = \frac{\partial W}{\partial t}(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \ a. \ s.$$

PROOF: For the sake of clarity of the exposition, we prove the convergence result for $\{W(t,x)\}$, while the proof for the pair $\{(W(t,x), W'(t,x))\}$ is essentially identical. The last statement of Proposition 5.3 can be obtained by taking the weak limit in the identity

$$W_\varepsilon(t,x) = W_\varepsilon(s,x) + \int_s^t W_\varepsilon'(r,x)dr.$$

Tightness of the sequence of random fields $\{W_\varepsilon, \ \varepsilon > 0\}$ follows from the proof of Proposition 6.15 below, upon interchanging the variables t and x. Note that (Hö) and (H_{um}) are satisfied here, with t and x interchanged. We postpone the proof to section 6 since the result needed there is more general.

Now it remains to identify the limit law of the vector of random processes

$$(W_\varepsilon(t_1, \cdot), \ldots, W_\varepsilon(t_n, \cdot)),$$

for any $n \geq 1$, any $0 \leq t_1 < t_2 < \cdots < t_n$. It follows from Theorem 20.1 in [3], together with the comments on pages 177 and 178 of that book that the above converges as $\varepsilon \to 0$ towards an n–dimensional Wiener process

$$(W(t_1, \cdot), \ldots, W(t_n, \cdot)),$$

11

which is such that the (i, j) entry of the covariance matrix of the random vector $(W(t_1, x), \ldots, W(t_n, x))$ is $\Psi(t_i - t_j)|x|$. $\qquad\square$

We can now proceed with the

PROOF OF THEOREM 5.2 : We deduce from Itô's formula that, if

$$\mathcal{W}_\varepsilon(t, x) := \int_0^x W_\varepsilon(t, y) dy,$$

$$
\begin{aligned}
\mathcal{W}_\varepsilon(t, X_t^x) = \mathcal{W}_\varepsilon(0, x) &+ \int_0^t \frac{\partial \mathcal{W}_\varepsilon}{\partial s}(s, X_s^x) ds \\
&+ \int_0^t W_\varepsilon(s, X_s^x) dX_s^x + \frac{1}{2} \int_0^t \frac{\partial W_\varepsilon}{\partial x}(s, X_s^x) ds,
\end{aligned}
\tag{5.4}
$$

consequently

$$
\begin{aligned}
Y_{t,x}^\varepsilon &= \int_0^t \frac{\partial W_\varepsilon}{\partial x}(s, X_s^x) ds \\
&= 2[\mathcal{W}_\varepsilon(t, X_t^x) - \mathcal{W}_\varepsilon(0, x) - \int_0^t \frac{\partial \mathcal{W}_\varepsilon}{\partial s}(s, X_s^x) ds - \int_0^t W_\varepsilon(s, X_s^x) dX_s^x].
\end{aligned}
\tag{5.5}
$$

The mapping which to $f \in C(\mathbb{R}_+ \times \mathbb{R})$ associates $g(t, x) = \int_0^x f(t, y) dy$ is continuous from $C(\mathbb{R}_+ \times \mathbb{R})$ into itself. Hence it follows from Proposition 5.3 that $(W_\varepsilon', W_\varepsilon, \mathcal{W}_\varepsilon) \Rightarrow (W', W, \mathcal{W})$ in $C(\mathbb{R}_+ \times \mathbb{R})^3$ as $\varepsilon \to 0$, where $\mathcal{W}(t, x) = \int_0^x W(t, y) dy$, $t \geq 0$, $x \in \mathbb{R}$.

Moreover the mappings

$$
f \to \int_0^t f(s, X_s^x) dX_s^x, \quad f \to \int_0^t f(s, X_s^x) ds
$$

are continuous from $C(\mathbb{R}_+ \times \mathbb{R})$ into $L^1(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the topology of convergence in probability. Consequently

$$
Y_{t,x}^\varepsilon \to 2\Big[\mathcal{W}(t, X_t^x) - \mathcal{W}(0, x) - \int_0^t \frac{\partial \mathcal{W}}{\partial s}(s, X_s^x) ds - \int_0^t W(s, X_s^x) dX_s^x\Big]
$$

in P law and $\mathbb{P}$ probability, hence also in $P \times \mathbb{P}$ law.

The result now follows from the

Lemma 5.4. *The following relation holds a. s.*

$$\mathcal{W}(t, X_t^x) = \mathcal{W}(0, x) + \int_0^t \frac{\partial \mathcal{W}}{\partial s}(s, X_s^x)ds + \int_0^t W(s, X_s^x)dX_s^x$$

$$+ \frac{1}{2}\int_0^t \int_{\mathbb{R}} L(ds, y - x)W(s, dy).$$

PROOF: Let

$$W_n(t, x) = (W(t, \cdot) * \rho_n)(x), \qquad (5.6)$$

where $\rho_n(x) = n\rho(nx)$ and ρ is a smooth map from $\mathbb{R}$ into $\mathbb{R}_+$ with compact support, whose integral over $\mathbb{R}$ equals one, and $\mathcal{W}_n(t, x) = \int_0^x W_n(t, y)dy$. Then from Itô's formula

$$\mathcal{W}_n(t, X_t^x) = \mathcal{W}_n(0, x) + \int_0^t \frac{\partial \mathcal{W}_n}{\partial s}(s, X_s^x)ds + \int_0^t W_n(s, X_s^x)dX_s^x$$

$$+ \frac{1}{2}\int_0^t \frac{\partial W_n}{\partial x}(s, X_s^x)ds$$

$$= \mathcal{W}_n(0, x) + \int_0^t \frac{\partial \mathcal{W}_n}{\partial s}(s, X_s^x)ds + \int_0^t W_n(X_s^x)dX_s^x$$

$$+ \frac{1}{2}\int_0^t \int_{\mathbb{R}} L(ds, y - x)\frac{\partial W_n}{\partial x}(s, y)dy.$$

The Lemma now follows by taking the limit as $n \to \infty$, provided we take the limit in the last term, which is done in the

Proposition 5.5. *There exists a unique linear mapping*

$$L \to \{\Lambda(L); \; t \geq 0, \; x \in \mathbb{R}\}$$

from the set of jointly continuous L's which are increasing with respect to the t variable and have compact support in the x variable for all t, into the set of centered Gaussian random fields, with the coraviance function given by

$$\mathbb{E}(\Lambda_{t,x}\Lambda_{t',x'}) = \int_{\mathbb{R}} dy \int_0^t \int_0^{t'} \Psi(s - r)L(ds, y - x)L(dr, y - x'),$$

where

$$\Lambda_{t,x}(L) = L^2(\mathbb{P}) - \lim_{n \to \infty} \int_0^t \int_{\mathbb{R}} L(ds, y - x)\frac{\partial W_n}{\partial x}(s, y)dy.$$

13

PROOF: We first need to show that the right hand side of the formula for the covariance function of the process $\{\Lambda_{t,x}(L),\ t \geq 0,\ x \in \mathbb{R}\}$ is well defined. This follows from the fact that

$$
0 \leq \int_{\mathbb{R}} dy \int_0^t \int_0^{t'} |\Psi(s-r)| L(ds, y-x) L(dr, y-x')
$$
$$
\leq \Psi(0) \int_{\mathbb{R}} L(t, y-x) L(t', y-x') dy
$$
$$
< \infty.
$$

The last inequality follows from the fact that both $L(t, \cdot)$ and $L(t', \cdot)$ are continuous and have compact support.

Now define

$$
\Lambda_{t,x}^{(n)}(L) = \int_{\mathbb{R}} dy \int_0^t \frac{\partial W_n}{\partial y}(s, y) L(ds, y-x)
$$
$$
= \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_n'(y-z) \left(\int_0^t W(s, z) L(ds, y-x) \right) dy dz.
$$

In order to complete the proof of the Proposition, it suffices to show that

$$
E\left[\Lambda_{t,x}^{(n)}(L) \Lambda_{t',x'}^{(m)}(L) \right] \to \int_{\mathbb{R}} dy \int_0^t \int_0^{t'} \Psi(s-r) L(ds, y-x) L(dr, y-x'),\ \ (5.7)
$$

as $n, m \to \infty$. Let

$$
\Sigma_{n,m}(y, y') = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{\{zz'>0\}} |z| \wedge |z'| \rho_n'(y-z) \rho_m'(y'-z') dz dz'.
$$

We have

$$
E\left[\Lambda_{t,x}^{(n)}(L) \Lambda_{t',x'}^{(m)}(L) \right] = \int_{\mathbb{R}} \int_{\mathbb{R}} \Sigma_{n,m}(y, y') dy dy' \int_0^t \int_0^{t'} \Psi(s-s') L(ds, y-x) L(ds', y'-x').
$$

Now an elementary computation based on integration by parts yields

$$
\Sigma_{n,m}(y, y') = \int_{\mathbb{R}} \rho_n(y-z) \rho_m(y'-z) dz,
$$

and (5.7) follows from this and the last identity. $\qquad \square$

14

We now turn to the case where $L(t, x)$ is the local time of the standard Brownian motion $\{X_t, t \geq 0\}$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus we now define the stochastic process $\{\Lambda_{t,x}(L), t \geq 0, x \in \mathbb{R}\}$ on the product probability space $(\mathcal{S} \times \Omega, \mathcal{A} \otimes \mathcal{F}, P \times \mathbb{P})$, and denote $\bar{\mathbb{P}} := P \times \mathbb{P}$. We have the

Proposition 5.6. *For each fixed $x \in \mathbb{R}$, the process $\{\Lambda_{t,x}(L), t \geq 0\}$ has a $\bar{\mathbb{P}}$ a. s. continuous modification.*

PROOF: We have, for $0 \leq s < t$,

$$\bar{\mathbb{E}}\left(|\Lambda_{t,x}(L) - \Lambda_{s,x}(L)|^p\right) = \mathbb{E}\left(\left|\int_{\mathbb{R}} dy \int_s^t \int_s^t \Psi(r - r') L(dr, y) L(dr', y)\right|^{p/2}\right)$$

$$\leq \Psi(0)^{p/2} \mathbb{E}\left(\left|\int_{\mathbb{R}} (L(t, y) - L(s, y))^2 dy\right|^{p/2}\right)$$

$$\leq \Psi(0)^{p/2} \mathbb{E}\left(\left|\sup_y (L(t, y) - L(s, y))(t - s)\right|^{p/2}\right),$$

where we have used the following well known formula

$$\int_{\mathbb{R}} L(t, x) dx = t.$$

Now from (III) page 200 of Barlow, Yor [2], there exists a universal constant c_p such that

$$\mathbb{E}\left(\sup_x (L(t, x) - L(s, x))^{p/2}\right) \leq c_p \mathbb{E}\left(\sup_{0 \leq s \leq t} |X_s|^{p/2}\right).$$

The above right hand side is finite, and

$$\bar{\mathbb{E}}\left(|\Lambda_t(L) - \Lambda_s(L)|^p\right) \leq C_p(t - s)^{p/2},$$

from which the result follows, if we choose $p > 2$.

5.2 Convergence of the sequence u^ε

In order to deduce the convergence of u^ε from that of $Y_{t,x}^\varepsilon$ and Corollary 4.2, we need some uniform integrability under $\mathbb{P}$ of the collection of random variables

$$\left\{\exp\left[\frac{1}{\sqrt{\varepsilon}} \int_0^t c\left(s, \frac{X_s^x}{\varepsilon}\right) ds\right], \ \varepsilon > 0\right\}.$$

15

For each $0 < \gamma < 1/2$, $t > 0$, $\varepsilon > 0$, we define the $\mathbb{R}_+$–valued random variables

$$\xi_{t,\gamma}^{\varepsilon} = \sup_{0 \le s \le t,\ x \in \mathbb{R}} \frac{|W_\varepsilon(s,x)|}{(1+|x|)^{1-\gamma}}, \quad \eta_{t,\gamma}^{\varepsilon} = \sup_{0 \le s \le t,\ x \in \mathbb{R}} \frac{\left|\frac{\partial W_\varepsilon}{\partial s}(s,x)\right|}{(1+|x|)^{1-\gamma}}.$$

We now prove the

Lemma 5.7. *For each $t > 0$, $0 < \gamma < 1/2$ and $\varepsilon_0 > 0$, the two collections of random variables $\{\xi_{t,\gamma}^{\varepsilon},\ 0 < \varepsilon \le \varepsilon_0\}$ and $\{\eta_{t,\gamma}^{\varepsilon},\ 0 < \varepsilon \le \varepsilon_0\}$ are tight.*

PROOF: We have

$$\eta_{t,\gamma}^{\varepsilon} \le \sup_{x \in \mathbb{R}} \frac{\left|\frac{\partial W_\varepsilon}{\partial s}(0,x)\right|}{(1+|x|)^{1-\gamma}} + \sup_{x \in \mathbb{R}} \int_0^t \frac{\left|\frac{\partial^2 W_\varepsilon}{\partial s^2}(s,x)\right|}{(1+|x|)^{1-\gamma}} ds,$$

and similarly

$$\xi_{t,\gamma}^{\varepsilon} \le \sup_{x \in \mathbb{R}} \frac{|W_\varepsilon(0,x)|}{(1+|x|)^{1-\gamma}} + \sup_{x \in \mathbb{R}} \int_0^t \frac{\left|\frac{\partial W_\varepsilon}{\partial s}(s,x)\right|}{(1+|x|)^{1-\gamma}} ds.$$

It remains to show that each of the four collections of r. v. appearing in the two above right hand sides is tight. Each of the four terms can be treated by the exact same argument as used in the proof of Lemma 5 page 295–296 of [13], which we now reproduce for the convenience of the reader, in the case of the first term of the second right–hand side. $\qquad\square$

We drop the index t for simplicity, and define

$$\zeta_\gamma^{\varepsilon} = \sup_{x \in \mathbb{R}} \frac{|W_\varepsilon(x)|}{(1+|x|)^{1-\gamma}}.$$

We have the

Lemma 5.8. *For any $0 < \gamma < 1/2$ and $\varepsilon_0 > 0$, the collection of random variables $\{\zeta_\gamma^{\varepsilon},\ 0 < \varepsilon \le \varepsilon_0\}$ is tight.*

PROOF: Due to the symmetry it is sufficient to estimate $|W_\varepsilon(x)|$ for $x > 0$.

16

We have

$$E(|W_\varepsilon(r)|^2) = \varepsilon \int_0^{r/\varepsilon} \int_0^{r/\varepsilon} E(c(x)c(y))dxdy$$

$$\leq 2\varepsilon \int_0^{r/\varepsilon} \int_0^{\infty} |E(c(0)c(x))|dsxdy$$

$$\leq 2rc_0.$$

Denote by $\mathcal{G}_x = \sigma\{c(y), \ y \leq x\}$ and

$$\eta_x = \int_0^\infty E(c(y+x)|\mathcal{G}_x)dy.$$

Combining the estimate (2.23) in the case $p = \infty$ in Proposition 7.2.6. from [5] with our condition that the correlation function Φ is both bounded and integrable, we deduce that the stationary process $\{\eta_x, \ x \geq 0\}$ satisfies $|\eta_x| \leq c_1$ a.s. for all $x > 0$, with a non-random constant c_1. Moreover,

$$\int_0^x c(r)dr - \eta_x$$

is a square integrable $\mathcal{G}_x$ martingale. Denote it by $\mathcal{N}_x$. Clearly

$$W_\varepsilon(x) = \frac{\sqrt{\varepsilon}}{\bar{c}} \int_0^{x/\varepsilon} c(y)dy$$

$$= \frac{\sqrt{\varepsilon}}{\bar{c}} \mathcal{N}_{x/\varepsilon} + \frac{\sqrt{\varepsilon}}{\bar{c}} \eta_{x/\varepsilon},$$

and thus we deduce from Doob's inequality

$$E\left(\sup_{0 \leq x \leq r} |W_\varepsilon(x)|^2\right) \leq \frac{2}{\bar{c}^2} E\left(\sup_{0 \leq x \leq r/\varepsilon} \left(\sqrt{\varepsilon}\mathcal{N}_x\right)^2\right) + 2\frac{c_1^2\varepsilon}{\bar{c}^2}$$

$$\leq \frac{4}{\bar{c}^2} E((\sqrt{\varepsilon}\mathcal{N}_{r/\varepsilon})^2) + 2\frac{c_1^2\varepsilon}{\bar{c}^2}$$

$$\leq 8E(|W_\varepsilon(r)|^2) + 10\frac{c_1^2\varepsilon}{\bar{c}^2}$$

$$\leq C(\varepsilon + r),$$

17

provided $C = (16c_0) \vee (10c_1^2/\bar{c}^2)$. Now for $j \geq 1$, $M > 0$,

$$P\left(\sup_{2^{j-1} < r \leq 2^j} \frac{|W_\varepsilon(r)|}{(1+r)^{1-\gamma}} \geq M\right) \leq P\left(\sup_{0 \leq r \leq 2^j} |W_\varepsilon(r)| \geq (1 + 2^{j-1})^{1-\gamma} M\right)$$

$$\leq \frac{C(\varepsilon + 2^j)}{M^2(1 + 2^{j-1})^{2-2\gamma}}$$

$$\leq (\varepsilon \vee 1)\frac{2C}{M^2}(1 + 2^{j-1})^{2\gamma-1}.$$

Summing up over $j \geq 1$, we deduce that

$$P\left(\zeta_\gamma^\varepsilon \geq M\right) \leq 2P\left(\sup_{r > 0} \frac{|W_\varepsilon(r)|}{(1+r)^{1-\gamma}} \geq M\right)$$

$$\leq (\varepsilon \vee 1)\frac{4C}{M^2} \sum_{j=0}^{\infty}(1 + 2^j)^{2\gamma-1}$$

$$\leq (\varepsilon \vee 1)\frac{C'}{M^2}.$$

This completes the proof of Lemma. $\square$

We can now establish the required uniform integrability

Proposition 5.9. *For each $t > 0$, $x \in \mathbb{R}$, the collection of random variables*

$$\left\{\mathbb{E}\left(\exp\left[\frac{4}{\sqrt{\varepsilon}}\int_0^t c\left(s, \frac{X_s^x}{\varepsilon}\right)ds\right]\right), \; \varepsilon > 0\right\}$$

is P–tight.

PROOF: We make use of the following easy estimate : if Z is an $N(0,1)$ random variable, $c > 0$ and $0 < p < 2$,

$$\mathbb{E}\exp(c|Z|^p) \leq \sqrt{2}\exp\left[\frac{2-p}{2}(4c)^{\frac{2}{2-p}}\right]. \tag{5.8}$$

From the identity (5.4) in the proof of Theorem 5.2, we deduce that

$$\frac{4}{\sqrt{\varepsilon}}\int_0^t c\left(s, \frac{X_s^x}{\varepsilon}\right)ds = 8\mathcal{W}_\varepsilon(t, X_t^x) - 8\mathcal{W}_\varepsilon(0, x) - 8\int_0^t \frac{\partial \mathcal{W}_\varepsilon}{\partial s}(s, X_s^x)ds$$

$$- 8\int_0^t W_\varepsilon(s, X_s^x)dB_s.$$

18

Hence

$$\mathbb{E}\left(\exp\left[\frac{4}{\sqrt{\varepsilon}}\int_0^t c\left(s,\frac{X_s^x}{\varepsilon}\right)ds\right]\right) \le e^{-8\mathcal{W}_\varepsilon(0,x)}\left[\mathbb{E}\left(e^{24\mathcal{W}_\varepsilon(t,X_t^x)}\right)\right]^{1/3}$$

$$\times\left[\mathbb{E}\left(e^{-24\int_0^t \frac{\partial\mathcal{W}_\varepsilon}{\partial s}(s,X_s^x)ds}\right)\right]^{1/3}\left[\mathbb{E}\left(e^{-24\int_0^t \mathcal{W}_\varepsilon(s,X_s^x)dB_s}\right)\right]^{1/3}.$$

It remains to dominate each of the 4 factors of the right–hand side of the last identity by a tight sequence, which we now do, with the help of Lemma 5.7. Below γ is an arbitrarily fixed number in the interval $(0,1/2)$. Clearly,

$$-8\mathcal{W}_\varepsilon(0,x) \le 8|x|(1+|x|)^{1-\gamma}\xi_{0,\gamma}^\varepsilon,$$

and the sequence on the right-hand side is tight as well as the sequence of the exponentials $\exp\left(8|x|(1+|x|)^{1-\gamma}\xi_{0,\gamma}^\varepsilon\right)$. Next

$$24\mathcal{W}_\varepsilon(t,x+B_t) = 24\int_0^{x+B_t} W_\varepsilon(t,y)dy$$

$$\le 24|x+B_t|(1+|x+B_t|)^{1-\gamma}\xi_{t,\gamma}^\varepsilon$$

$$\le 48[(1+|x|)^{2-\gamma}+|B_t|^{2-\gamma}]\xi_{t,\gamma}^\varepsilon.$$

Hence from (5.8),

$$\mathbb{E}\left(e^{24\mathcal{W}_\varepsilon(t,X_t^x)}\right) \le \sqrt{2}\exp\left[48(1+|x|)^{2-\gamma}\xi_{t,\gamma}^\varepsilon\right]\exp\left[\frac{\gamma}{2}\left(192\xi_{t,\gamma}^\varepsilon t^{1-\gamma/2}\right)^{2/\gamma}\right].$$

Similarly,

$$-24\int_0^t \frac{\partial\mathcal{W}_\varepsilon}{\partial s}(s,X_s^x)ds \le 24\eta_{t,\gamma}^\varepsilon\int_0^t\left(|x+B_s|+\frac{1}{1-\gamma}|x+B_s|^{2-\gamma}\right)ds,$$

so using Jensen's inequality, we get

$$\exp\left(-24\int_0^t \frac{\partial\mathcal{W}_\varepsilon}{\partial s}(s,X_s^x)ds\right) \le \frac{1}{t}\int_0^t\exp\left[24t\eta_{t,\gamma}^\varepsilon\left(|x+B_s|+\frac{1}{1-\gamma}|x+B_s|^{2-\gamma}\right)\right]ds,$$

from which the result follows as above. Next from Cauchy–Schwarz,

$$\mathbb{E}\exp\left(-24\int_0^t \mathcal{W}_\varepsilon(s,X_s^x)dB_s\right) \le \left[\mathbb{E}\exp\left(-48\int_0^t \mathcal{W}_\varepsilon(s,X_s^x)dB_s - 1152\int_0^t W_\varepsilon^2(s,X_s^x)ds\right)\right]^{1/2}$$

$$\times\left[\mathbb{E}\exp\left(1152\int_0^t W_\varepsilon^2(s,X_s^x)ds\right)\right]^{1/2}$$

$$\le \left[\mathbb{E}\exp\left(1152\int_0^t W_\varepsilon^2(s,X_s^x)ds\right)\right]^{1/2},$$

19

but

$$\int_0^t W_\varepsilon^2(s, X_s^x) ds \le \left[\xi_{t,\gamma}^\varepsilon\right]^2 \int_0^t (1 + |x + B_s|)^{2-2\gamma} ds,$$

and we estimate this term again using Jensen's inequality and the inequality (5.8). $\qquad\square$

It now follows from Theorem 5.2, Propositions 5.3 and 5.9, and the fact that by formula (5.5) the exponent in the Feynman–Kac formula is a continuous function of (W, W_t'), that we can apply Corollary 4.2, yielding

Theorem 5.10. *For any* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$u^\varepsilon(t, x) \to u(t, x) := \mathbb{E}\left[g(X_t^x) \exp\left(\int_0^t \int_{\mathbb{R}} L(ds, y - x) W(s, dy)\right)\right]$$

in P–law, as $\varepsilon \to 0$.

Remark 5.11. *Note that it is not clear how the limiting exponent in the Feynman–Kac formula could be written in terms of* W *and* B.

The corresponding limiting SPDE reads

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t, x) dt + u(t, x) \circ W(t, dx), & t \ge 0,\ x \in \mathbb{R}; \\ u(0, x) = g(x), & x \in \mathbb{R}, \end{cases}$$

where the stochastic integral should be interpreted as an anticipative Stratonovich integral, see [11], [12]. Since anticipating stochastic integrals are not very easy to handle, we prefer to rewrite the above SPDE as follows, using the same trick as in [13]. We note that $u(t, x) \circ W(t, dx)$ is a convenient notation for the product

$$u(t, x)\frac{\partial W}{\partial x}(t, x) = \frac{\partial (uW)}{\partial x}(t, x) - \frac{\partial u}{\partial x}(t, x) W(t, x).$$

Hence we rewrite the above SPDE in the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t, x) + \dfrac{\partial (uW)}{\partial x}(t, x) - \dfrac{\partial u}{\partial x}(t, x) W(t, x), & t \ge 0,\ x \in \mathbb{R}; \\ u(0, x) = g(x), & x \in \mathbb{R}. \end{cases}$$

$$(5.9)$$

6   The case $0 \leq 2\beta \leq \alpha$, $\alpha > 0$

We first prove two Propositions which will be useful in two of the three following subcases.

We first recall the definition of the uniform mixing coefficient $\alpha_{\mathrm{um}}(r)$ of the random field $c(t, x)$. We set

$$\alpha_{\mathrm{um}}(r) = \sup |P(\mathcal{S}_1 | \mathcal{S}_2) - P(\mathcal{S}_1)|,$$

where the supremum is taken over all $\mathcal{S}_1 \in \sigma\{c(t, x),\ t \leq t_0,\ x \in \mathbb{R}\}$ and $\mathcal{S}_2 \in \sigma\{c(t, x),\ t \geq t_0 + r,\ x \in \mathbb{R}\}$, with $P(\mathcal{S}_2) > 0$. By stationarity, the supremum on the right hand side does not depend upon t_0.

Next we recall the definition of the maximum correlation coefficient $\rho(r)$:

$$\rho_{\mathrm{mc}}(r) = \sup\ |E(\xi\eta)|,$$

where the supremum is taken over all r. v.'s ξ (resp. η), which are assumed to be measurable with respect to $\sigma\{c(t, x),\ t \leq t_0,\ x \in \mathbb{R}\}$ (resp. $\sigma\{c(t, x),\ t \geq t_0 + r,\ x \in \mathbb{R}\}$), and such that $E\xi\ E\eta = 0$, $|\xi| \leq 1$, $|\eta| \leq 1$.

We shall assume in this section that there exists $C,\ \delta > 0$ such that

$$(H_{\mathrm{um}}) \qquad\qquad \alpha_{\mathrm{um}}(r) \leq C(1 + r)^{-(3+\delta)}.$$

Proposition 7.2.2, page 346 of [5], with, using the notations there, $s = \infty$, $r = 1$, $p = \infty$ and $q = 1$, yields the

Lemma 6.1. *It follows from (H_{um}) that for some constant C',*

$$\rho_{\mathrm{mc}}(r) \leq C'(1 + r)^{-(3+\delta)},$$

and in particular $\rho_{\mathrm{mc}} \in L^1(\mathbb{R}_+)$.

An immediate consequence of the Lemma is the

Corollary 6.2. *There exists a constant C such that for all $t \geq 0$, $x \in \mathbb{R}$,*

$$|\Phi(t, x)| \leq C(1 + t + |x|)^{-(3+\delta)}.$$

21

Recall the function Φ defined in (1.3). It will be convenient in the sequel to use the fact that there exists a bounded function $\Psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ such that
$$|\Phi(s,x)| \leq \Psi(s,x),$$
$x \to \Psi(s,x)$ is decreasing on $\mathbb{R}_+$ for all $s \in \mathbb{R}_+$, $\Psi(s,-x) = \Psi(s,x)$, and
$$\int_0^\infty \Psi(t,0)dt < \infty; \quad \int_0^\infty \Psi(t,x)dt \to 0, \quad \text{as } |x| \to \infty. \qquad (6.1)$$
For example, we might set (for $x > 0$)
$$\Psi(t,x) = \sup_{\substack{s \geq t \\ |y| \geq x}} |\Phi(s,y)|.$$

In this case, (6.1) follows from our standing assumption (H_{um}), see Corollary 6.2.

Whithout loss of generality, we assume now that $\alpha = 1$. Hence we want to treat the case $0 \leq \beta \leq 1/2$. The exponent in the Feynman–Kac formula reads
$$Y_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^t c\left(\frac{s}{\varepsilon}, \frac{x + B_s}{\varepsilon^\beta}\right) ds.$$
Let us first prove the

Proposition 6.3. *Assume that the condition (H_{um}) holds. Then for all $0 \leq \beta \leq 1/2$, the limit relation holds in $\mathbb{P}$–probability*
$$\lim_{\varepsilon \to 0} E \exp(Y_t^{\varepsilon,x}) = \exp(t\Sigma) \qquad (6.2)$$

with
$$\Sigma(\beta) = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \Phi(u,0)du, & \text{if } 0 \leq \beta < 1/2, \\[2mm] \displaystyle\int_{-\infty}^{+\infty} \mathbb{E}\Phi(u,B_u)du, & \text{if } \beta = 1/2. \end{cases} \qquad (6.3)$$

PROOF: We only consider the case $\beta = 1/2$, for $\beta \in (0, 1/2)$ the desired statement can be justified in the same way with some simplifications.

We introduce a partition of the interval $(0, t/\varepsilon)$ into alternating subintervals of the form
$$\mathcal{I}_j^\varepsilon = \left((\varepsilon^{-1/3} + \varepsilon^{-\nu})j, \ (\varepsilon^{-1/3} + \varepsilon^{-\nu})j + \varepsilon^{-1/3}\right), \quad j = 1, 2, \ldots, K^\varepsilon,$$

22

$$\mathcal{J}_j^\varepsilon = \big((\varepsilon^{-1/3} + \varepsilon^{-\nu}) j + \varepsilon^{-1/3}, \ (\varepsilon^{-1/3} + \varepsilon^{-\nu})(j+1) \big), \quad j = 1, 2, \dots, K^\varepsilon;$$

here $K^\varepsilon = [(\varepsilon^{-1} t)/(\varepsilon^{-1/3} + \varepsilon^{-\nu})]$, $[\cdot]$ stands for the integer part, and $0 < \nu < 1/3$. This implies that $K^\varepsilon = t\varepsilon^{-2/3}(1 + o(1))$. Denote

$$\eta_j^\varepsilon = \sqrt{\varepsilon} \int_{\mathcal{I}_j^\varepsilon} c\big(s, \tfrac{x}{\sqrt{\varepsilon}} + \tilde{B}_s\big) ds, \qquad \zeta_j^\varepsilon = \sqrt{\varepsilon} \int_{\mathcal{J}_j^\varepsilon} c\big(s, \tfrac{x}{\sqrt{\varepsilon}} + \tilde{B}_s\big) ds$$

where the new Wiener process $\tilde{B}_s$ has been obtained from the original one by the scaling $\sqrt{\varepsilon}\tilde{B}_{s/\varepsilon} = B_s$. We may assume without loss of generality that the process $\tilde{B}_s$ is fixed. Then

$$Y_t^{\varepsilon,x} = \sum_{j=0}^{K^\varepsilon} (\eta_j^\varepsilon + \zeta_j^\varepsilon) + V_\varepsilon,$$

where $|V_\varepsilon| \leq C\varepsilon^{1/3}$ $P \times \mathbb{P}$-a.s.

Notice that, due to the standing assumptions on $c(s,x)$, there exists a constant C such that

$$|\eta_j^\varepsilon| \leq C\varepsilon^{1/6}, \qquad |\zeta_j^\varepsilon| \leq C\varepsilon^{1/2-\nu}. \tag{6.4}$$

To use efficiently the mixing properties of the coefficients it is convenient to represent $Y_t^{\varepsilon,x}$ as follows

$$Y_t^{\varepsilon,x} = \sum_{j \text{ is even}} \eta_j^\varepsilon + \sum_{j \text{ is odd}} \eta_j^\varepsilon + \sum_{j=0}^{K^\varepsilon} \zeta_j^\varepsilon \ := Y_e^\varepsilon + Y_o^\varepsilon + \mathcal{Y}^\varepsilon.$$

First, let us compute the limit of $E\exp(Y_e^\varepsilon)$. For the sake of definiteness we may assume that K^ε is odd. The case of even K^ε can be treated in exactly the same way. Using the notation $\mathcal{A}_j^\varepsilon = \sigma\{c(s,x) : s \leq (\varepsilon^{-1/3} + \varepsilon^{-\nu})j, \ x \in \mathbb{R}\}$, we have

$$E\exp(Y_e^\varepsilon) = E\exp\Big(\sum_{j=0}^{(K^\varepsilon-1)/2} \eta_{2j}^\varepsilon \Big) = E\Big(E\Big\{ \exp\Big(\sum_{j=0}^{(K^\varepsilon-1)/2} \eta_{2j}^\varepsilon \Big) \Big| \mathcal{A}_{(K^\varepsilon-2)}^\varepsilon \Big\} \Big)$$

$$= E\Big(\exp\Big(\sum_{j=0}^{(K^\varepsilon-3)/2} \eta_{2j}^\varepsilon \Big) E\big\{ \exp(\eta_{K^\varepsilon-1}^\varepsilon) \big| \mathcal{A}_{(K^\varepsilon-2)}^\varepsilon \big\} \Big)$$

23

$$= E\left(\exp\left(\sum_{j=0}^{(K^\varepsilon-3)/2} \eta_{2j}^\varepsilon \right) \left[E \exp(\eta_{K^\varepsilon-1}^\varepsilon) + E\{(\exp(\eta_{K^\varepsilon-1}^\varepsilon) - E\exp(\eta_{K^\varepsilon-1}^\varepsilon)) | \mathcal{A}_{(K^\varepsilon-2)}^\varepsilon\} \right] \right).$$

Since, according to (6.4), $|\exp(\eta_{K^\varepsilon-1}^\varepsilon) - E\exp(\eta_{K^\varepsilon-1}^\varepsilon)| \leq C\varepsilon^{1/6}$, then, by Proposition 2.6 page 349 in [5], we have

$$\left| E\{(\exp(\eta_{K^\varepsilon-1}^\varepsilon) - E\exp(\eta_{K^\varepsilon-1}^\varepsilon)) | \mathcal{A}_{(K^\varepsilon-2)}^\varepsilon\} \right| \leq C\alpha_{\text{um}}(\varepsilon^{-1/3})\varepsilon^{1/6} \leq C\varepsilon^{(3+\delta)/3+1/6}$$

Combining this estimate with the evident bound $1/2 \leq E\exp(\eta_{K^\varepsilon-1}^\varepsilon) \leq 2$, we obtain

$$E\exp(Y_e^\varepsilon) = E\left(\exp\left(\sum_{j=0}^{(K^\varepsilon-3)/2} \eta_{2j}^\varepsilon \right) E\exp(\eta_{K^\varepsilon-1}^\varepsilon) \left(1 + O(\varepsilon^{(3+\delta)/3+1/6})\right) \right)$$

with $|O(\varepsilon^{(3+\delta)/3+1/6})| \leq C\varepsilon^{(3+\delta)/3+1/6}$. Iterating this process, after $K^\varepsilon/2$ steps we arrive at the equality

$$E\exp(Y_e^\varepsilon) = \prod_{j=0}^{(K^\varepsilon-1)/2} E\exp(\eta_{2j}^\varepsilon) \left(1 + O(\varepsilon^{(3+\delta)/3+1/6})\right).$$

Since $\displaystyle\prod_{j=0}^{(K^\varepsilon-1)/2} \left(1 + O(\varepsilon^{(3+\delta)/3+1/6})\right)$ converges to 1 as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} E\exp(Y_e^\varepsilon) = \lim_{\varepsilon\to 0} \prod_{j=0}^{(K^\varepsilon-1)/2} E\exp(\eta_{2j}^\varepsilon) \tag{6.5}$$

We proceed with estimating the term $E\exp(\eta_j^\varepsilon)$. Using Taylor expansion of the exponent about zero results in the following relation

$$E\exp(\eta_j^\varepsilon) = 1 + E\eta_j^\varepsilon + \frac{1}{2}E\big((\eta_j^\varepsilon)^2\big) + \frac{1}{6}E\big((\eta_j^\varepsilon)^3\big) + \frac{1}{24}E\big((\eta_j^\varepsilon)^4\big) + O(\varepsilon^{5/6}), \tag{6.6}$$

here we have also used the bound $|\eta_j^\varepsilon| \leq C\varepsilon^{1/6}$. By the centering condition on $c(\cdot)$, $E\eta_j^\varepsilon = 0$. Considering λ_j^ε defined in the proof of Lemma 6.6 below in the particular case $\gamma = 1/3$, $\nu = 0$, we have that η_j^ε and $\lambda_j^{\sqrt{\varepsilon}}$ have the same law. It then follows from (6.22) that

$$\frac{1}{6}\big|E\big((\eta_j^\varepsilon)^3\big)\big| \leq C\varepsilon^{5/6}, \qquad \frac{1}{24}E\big((\eta_j^\varepsilon)^4\big) \leq C\varepsilon. \tag{6.7}$$

24

The contribution of the term $\frac{1}{2}E\big((\eta_j^\varepsilon)^2\big)$ can be computed as follows

$$\frac{1}{2}E\big((\eta_j^\varepsilon)^2\big) = \frac{\varepsilon}{2}\int\limits_{\mathcal{I}_j^\varepsilon}\int\limits_{\mathcal{I}_j^\varepsilon} E\Big\{c\big(r,\frac{x}{\sqrt{\varepsilon}}+\tilde{B}_r\big)c\big(s,\frac{x}{\sqrt{\varepsilon}}+\tilde{B}_s\big)\Big\}\,ds\,dr \qquad (6.8)$$

$$= \frac{\varepsilon}{2}\int\limits_{\mathcal{I}_j^\varepsilon}\int\limits_{\mathcal{I}_j^\varepsilon} \Phi(r-s,\tilde{B}_r-\tilde{B}_s)\,dr\,ds := \Xi_j^\varepsilon.$$

By definition and due to the properties of the Wiener process, the random variables $\Xi_j^\varepsilon = \Xi_j^\varepsilon(\omega)$, $j = 1,2\ldots,K^\varepsilon$, are independent, identically distributed and satisfy the following bounds

$$C_0\varepsilon^{2/3} \leq \Xi_j^\varepsilon \leq C_1\varepsilon^{2/3}, \qquad \mathbb{E}\Xi_j^\varepsilon = \Sigma(1/2)\varepsilon^{2/3} + O(\varepsilon) \qquad (6.9)$$

with $0 < C_0 < C_1 < \infty$ and $|O(\varepsilon)| \leq C_2\varepsilon$; the quantity $\Sigma(1/2)$ has been defined in (6.3). Combining (6.5)–(6.9) yields

$$\lim_{\varepsilon\to 0} E\exp(Y_e^\varepsilon) = \lim_{\varepsilon\to 0}\prod_{j=0}^{(K^\varepsilon-1)/2}\big(1+\Xi_j^\varepsilon+O(\varepsilon^{5/6})\big) \qquad (6.10)$$

$$= \exp\Big(\lim_{\varepsilon\to 0}\sum_{j=0}^{(K^\varepsilon-1)/2}\Xi_j^\varepsilon\Big) = \exp\Big(\lim_{\varepsilon\to 0}[(K^\varepsilon/2)\,\mathbb{E}\Xi_j^\varepsilon]\Big) = \exp\Big(\frac{t\Sigma(1/2)}{2}\Big)$$

in $\mathbb{P}$ probability, from the weak law of large numbers. Similarly

$$\lim_{\varepsilon\to 0} E\exp(Y_o^\varepsilon) = \exp\Big(\frac{t\Sigma(1/2)}{2}\Big). \qquad (6.11)$$

Exploiting exactly the same arguments one can show that

$$\lim_{\varepsilon\to 0} E\exp(\mathcal{Y}^\varepsilon) = 1$$

in $\mathbb{P}$ probability. In view of the strict convexity and the strict positivity for $x \neq 0$ of the function $\varphi(x) = e^x - 1 - x$, this implies that, as $\varepsilon \to 0$,

$$\mathcal{Y}^\varepsilon \to 0 \quad \text{in } \mathbb{P}\times P \text{ probability}. \qquad (6.12)$$

Following the line of the proof of estimate (6.18) in Lemma 6.6 below, one can show that

$$E\exp(4Y_{e,o}^\varepsilon) \leq C, \qquad E\exp(4\mathcal{Y}^\varepsilon) \leq C$$

with a deterministic constant C. Thanks to these bounds we deduce from (6.12) that

$$\lim_{\varepsilon \to 0} E \exp(Y_e^\varepsilon + Y_o^\varepsilon + \mathcal{Y}^\varepsilon) = \lim_{\varepsilon \to 0} E \exp(Y_e^\varepsilon + Y_o^\varepsilon) \qquad (6.13)$$

in $\mathbb{P}$ probability.

Denote

$$\mathcal{A}_e^\varepsilon = \sigma\Big\{ c(s,x) \, : \, s \in \bigcup_{j=0}^{(K^\varepsilon-1)/2} \mathcal{I}_{2j}^\varepsilon, \ x \in \mathbb{R} \Big\},$$

By construction,

$$\mathrm{dist}\Big(\bigcup_{j=0}^{(K^\varepsilon-1)/2} \mathcal{I}_{2j}^\varepsilon, \ \bigcup_{j=0}^{(K^\varepsilon-1)/2} \mathcal{I}_{2j+1}^\varepsilon \Big) = \varepsilon^{-\nu}.$$

Therefore,

$$E \exp(Y_e^\varepsilon + Y_o^\varepsilon) = E\big\{ \exp(Y_e^\varepsilon) E\big(\exp(Y_o^\varepsilon) | \mathcal{A}_e^\varepsilon \big) \big\} =$$

$$E\{\exp(Y_e^\varepsilon)\} \, E\{\exp(Y_e^\varepsilon)\} + o(\varepsilon^{2\nu})$$

and

$$\lim_{\varepsilon \to 0} E \exp(Y_t^{\varepsilon,x}) = \lim_{\varepsilon \to 0} E\{\exp(Y_e^\varepsilon)\} \lim_{\varepsilon \to 0} E\{\exp(Y_e^\varepsilon)\} = \exp(t\Sigma),$$

as required. $\qquad\square$

Now, consider the process $\exp\big(Y_t^{\varepsilon,x}(\omega)\big) \exp\big(Y_t^{\varepsilon,x}(\omega_1)\big)$ defined on the product space $\Omega \times \Omega$ with the product measure $\mathbb{P} \times \mathbb{P}$.

Proposition 6.4. *Assume that the condition (H_{um}) holds. Then for all $0 \leq \beta \leq 1/2$, the limit relation holds in $\mathbb{P} \times \mathbb{P}$–probability*

$$\lim_{\varepsilon \to 0} E\big\{ \exp\big(Y_t^{\varepsilon,x}(\omega)\big) \exp\big(Y_t^{\varepsilon,x}(\omega_1)\big) \big\} = \exp(2t\Sigma) \qquad (6.14)$$

PROOF: It is easy to check that for the standard Brownian motion B_s and for any $t > 0$ the limit relation holds

$$\lim_{\delta \to 0} \mathrm{meas}\big\{ s \in [0,t] \, : \, |B_s(\omega) - B_s(\omega_1)| < \delta \big\} = 0 \qquad (6.15)$$

26

$\mathbb{P} \times \mathbb{P}$-a.s. Due to the condition (H_{um}), for any pair (ω, ω_1) such that (6.15) is fulfilled, we have

$$\lim_{\varepsilon \to 0} E\big\{ \exp\big(Y_t^{\varepsilon,x}(\omega)\big) \exp\big(Y_t^{\varepsilon,x}(\omega_1)\big)\big\}$$

$$= \lim_{\varepsilon \to 0} E\big\{ \exp\big(Y_t^{\varepsilon,x}(\omega)\big)\big\} \lim_{\varepsilon \to 0} E\big\{ \exp\big(Y_t^{\varepsilon,x}(\omega_1)\big)\big\},$$

and the desired statement follows from Proposition 6.3. $\qquad\square$

The next result will be needed below

Lemma 6.5. *Under our standing assumptions, for any $\nu \geq 0$, there exists a constant C such that for all $0 \leq s < t$, $\mathbb{P}$ a. s.,*

$$E\left[\left(\varepsilon \int_{s_1/\varepsilon^2}^{s_2/\varepsilon^2} c\left(t, \frac{x}{\varepsilon} + \varepsilon^\nu B_t\right) dt\right)^6\right] \leq C|s_2 - s_1|^3.$$

PROOF: In all this proof, y stands for $x/\varepsilon + \varepsilon^\nu B_s$. Its dependence upon s, ε and ω is harmless.

$$E\left(\varepsilon \int_{s_1/\varepsilon^2}^{s_2/\varepsilon^2} c\big(t, y\big) dt\right)^6 = \varepsilon^6 \int_{s_1/\varepsilon^2}^{s_2/\varepsilon^2} \dots \int_{s_1/\varepsilon^2}^{s_2/\varepsilon^2} E\{c(t_1, y) \dots c(t_6, y)\} dt_1 \dots dt_6.$$

Let us now introduce the set

$$S(r) = \left\{ (t_1, \dots, t_6) \in \left[\frac{s_1}{\varepsilon^2}, \frac{s_2}{\varepsilon^2}\right]^6 : \max_{1 \leq i \leq 6} \min_{j \neq i} |t_i - t_j| \leq r\right\}.$$

It is an easy exercise to check that

$$V(r) = \text{Vol}(S(r)) \leq Cr^3 \frac{(s_2 - s_1)^3}{\varepsilon^6}.$$

If for some $i \in \{1, \dots, 6\}$ it holds $|t_i - t_j| \geq r$ for all $j \neq i$ (without loss of generality $i = 1$), then, taking into account (H_{um}), we have

$$|E\{c(t_1, y)c(t_2, y) \dots c(t_6, y)\}| = |E(c(t_2, y) \dots c(t_6, y)E\{c(t_1, y)|\mathcal{F}_{\{t_2, \dots, t_6\}}\})|$$
$$\leq CE(|c(t_2, y) \dots c(t_6, y)|)(1 + r)^{-(3+\delta)}\|c(t_1, y)\|_{L^\infty(\mathcal{A})}$$
$$\leq (1 + r)^{-(3+\delta)}\|c\|_{L^\infty(\mathcal{A})}^6$$
$$\leq C(1 + r)^{-(3+\delta)}.$$

Therefore

$$\varepsilon^6 \int\limits_{s_1/\varepsilon^2}^{s_2/\varepsilon^2} \cdots \int\limits_{s_1/\varepsilon^2}^{s_2/\varepsilon^2} E\{c(t_1, y) \ldots c(t_6, y)\} dt_1 \ldots dt_6$$

$$\leq C\varepsilon^6 \int\limits_0^{\sqrt{6}(s_2-s_1)/\varepsilon^2} \frac{dV(r)}{(1+r)^{3+\delta}}$$

$$\leq C\varepsilon^6 \int\limits_0^\infty \frac{dV(r)}{(1+r)^{3+\delta}}$$

$$= C\varepsilon^6 \left(V(r)(1+r)^{-(3+\delta)}\right)\big|_0^\infty + (3+\delta)C\varepsilon^6 \int\limits_0^\infty \frac{V(r)dr}{(1+r)^{(4+\delta)}}$$

$$\leq C(s_2-s_1)^3 \int\limits_0^\infty \frac{r^3\,dr}{(1+r)^{(4+\delta)}} \leq C(s_2-s_1)^3.$$

$\square$

6.1   The case $\alpha = 2\beta > 0$

This is the "central case", where $\alpha/4 + \beta/2 = \alpha/2$. In this case, $\gamma = \beta = \alpha/2$, and we consider w. l. o. g. the case where $\gamma = \beta = 1$, $\alpha = 2$. This means that we consider the PDE

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial x^2}(t,x) + \dfrac{1}{\varepsilon}c\left(\dfrac{t}{\varepsilon^2}, \dfrac{x}{\varepsilon}\right)u^\varepsilon(t,x), & t \geq 0,\ x \in \mathbb{R}; \\ u^\varepsilon(0,x) = g(x), & x \in \mathbb{R}, \end{cases} \tag{6.16}$$

whose solution is given by the Feynman–Kac formula

$$u^\varepsilon(t,x) = \mathbb{E}\left[g(x + B_t)\exp\left(\varepsilon^{-1}\int_0^t c\left(\frac{s}{\varepsilon^2}, \frac{x+B_s}{\varepsilon}\right)ds\right)\right].$$

We will show that the limit of $u^\varepsilon(t,x)$, as $\varepsilon \to 0$, is a deterministic function.

Let us define

$$Y_{t,x}^\varepsilon = \varepsilon^{-1} \int_0^t c\left(\frac{s}{\varepsilon^2}, \frac{x+B_s}{\varepsilon}\right) ds,$$

Then

$$u^\varepsilon(t,x) = \mathbb{E}\left[g(x+B_t)\exp(Y_{t,x}^\varepsilon)\right].$$

The random variable $Y_{t,x}^\varepsilon$ is defined on the product probability space $(\mathcal{S} \times \Omega, \mathcal{A} \otimes \mathcal{F}, P \times \mathbb{P})$.

The limit of $u^\varepsilon(t,x)$ will be obtained by a combination of Proposition 6.3 (in the case $\beta = 1/2$) and some uniform integrability property, which we now establish. Let us prove the uniform in $\varepsilon > 0$ and $\omega \in \Omega$ integrability with respect to the measure P of the random variable

$$\exp(Y_t^\varepsilon) = \exp\left(\varepsilon \int_0^{t/\varepsilon^2} c\left(s, \frac{x}{\varepsilon} + B_s\right) ds\right), \qquad t > 0.$$

Because we need slightly different versions of the same result in other sections of this paper, we prove a more general result, which will be used in this section with $\nu = 0$.

Lemma 6.6. *If the assumption (H_{um}) is satisfied, then for any $\theta > 0$, there exists a constant $C(\theta)$ such that for all $\varepsilon > 0$ and $\nu \in \mathbb{R}$,*

$$E \exp\left(\theta\varepsilon \int_0^{t/\varepsilon^2} c(s, x\varepsilon^{\frac{\nu}{2}-1} + B_{\varepsilon^\nu s}) ds\right) \le C(\theta). \qquad (6.17)$$

Remark 6.7. *The condition $\alpha(r) \le C(1+r)^{-(1+\delta/2)}$, which is weaker than (H_{um}), does imply that $\rho \in L^1(\mathbb{R}_+)$. However, the proof would be slightly more delicate. In particular, the parameter γ which appears in the proof below should be choosen as a function of δ.*

PROOF: Let γ be an arbitrary positive number such that $0 < \gamma < 1/2$, and consider an equidistant partition of the interval $[0, \frac{t}{\varepsilon^2}]$, the length of all subintervals being equal to $\varepsilon^{\gamma-1}$ (without loss of generality we assume that $t\varepsilon^{-(\gamma+1)}$ is an integer and, moreover, an even number). We estimate separately the contribution of all the subintervals with even numbers and of

29

those with odd numbers. It suffices to show that, with $\delta = \nu/2 - 1$,

$$E \exp\left(2\theta \sum_{j=1}^{t\varepsilon^{-(\gamma+1)/2}}{}' \varepsilon \int_{2(j-1)\varepsilon^{(\gamma-1)}}^{(2j-1)\varepsilon^{(\gamma-1)}} c(s, x\varepsilon^\delta + B_{\varepsilon^\nu s})ds\right) \le C, \qquad (6.18)$$

$$E \exp\left(2\theta \sum_{j=1}^{t\varepsilon^{-(\gamma+1)/2}} \varepsilon \int_{(2j-1)\varepsilon^{(\gamma-1)}}^{2j\varepsilon^{(\gamma-1)}} c(s, x\varepsilon^\delta + B_{\varepsilon^\nu s})ds\right) \le C.$$

We introduce the notation

$$\lambda_j^\varepsilon = 2\theta\varepsilon \int_{2(j-1)\varepsilon^{(\gamma-1)}}^{(2j-1)\varepsilon^{(\gamma-1)}} c(t, x\varepsilon^\delta + B_{\varepsilon^\nu t})dt; \qquad \mathcal{F}_j^\varepsilon = \sigma\{c(t,x) : t \le 2(j-1)\varepsilon^{(\gamma-1)}, \ x \in \mathbb{R}\}.$$

Since $|c(s,x)| \le C$, we have the bound

$$|\lambda_j^\varepsilon| \le c\varepsilon^\gamma \qquad (6.19)$$

and, moreover,

$$E \exp(\lambda_j^\varepsilon) = E\left(1 + \lambda_j^\varepsilon + \frac{(\lambda_j^\varepsilon)^2}{2!} + \cdots + \frac{(\lambda_j^\varepsilon)^k}{k!}\right) + \circ(\varepsilon^{\gamma+1}), \qquad (6.20)$$

provided $k \ge (\frac{1}{\gamma} + 1)$. The last term on the right hand side admits the bound $|\circ(\varepsilon^{\gamma+1})| \le \kappa(\varepsilon)\varepsilon^{\gamma+1}$, where κ is a deterministic function defined on $\mathbb{R}_+$, which is such that $\kappa(\varepsilon) \to 0$, as $\varepsilon \to 0$. Since the random field $\{c(t,x), \ t \ge 0, x \in \mathbb{R}\}$ is centered, $E\lambda_j^\varepsilon = 0$. Then

$$E((\lambda_j^\varepsilon)^2) = 4\theta^2\varepsilon^2 \int_{2(j-1)\varepsilon^{(\gamma-1)}}^{(2j-1)\varepsilon^{(\gamma-1)}} \int_{2(j-1)\varepsilon^{(\gamma-1)}}^{(2j-1)\varepsilon^{(\gamma-1)}} E\big(c(t, x\varepsilon^\delta + B_{\varepsilon^\nu t})c(s, x\varepsilon^\delta + B_{\varepsilon^\nu s})\big)dsdt$$

$$\le c\varepsilon^2 \int_0^{\varepsilon^{(\gamma-1)}} \int_0^{\varepsilon^{(\gamma-1)}} \rho(t-s)dtds$$

$$\le C\varepsilon^{1+\gamma};$$

$$(6.21)$$

30

For $m \geq 2$ we obtain

$$|E(\lambda_j^\varepsilon)^m| \leq c_m \varepsilon^m \int\limits_{2(j-1)\varepsilon^{(\gamma-1)}}^{(2j-1)\varepsilon^{(\gamma-1)}} \cdots \int\limits_{2(j-1)\varepsilon^{(\gamma-1)}}^{(2j-1)\varepsilon^{(\gamma-1)}} |E\big(c(t_1, x\varepsilon^\delta + B_{\varepsilon^\nu t_1}) \ldots c(t_m, x\varepsilon^\delta + B_{\varepsilon^\nu t_m})\big)| dt_1 \ldots dt_m$$

$$= c_m \varepsilon^m \int\limits_{2(j-1)\varepsilon^{(\gamma-1)}}^{(2j-1)\varepsilon^{(\gamma-1)}} \cdots \int\limits_{2(j-1)\varepsilon^{(\gamma-1)}}^{(2j-1)\varepsilon^{(\gamma-1)}} \Big|E\Big(c(t_1, x\varepsilon^\delta + B_{\varepsilon^\nu t_1}) \prod_{i=2}^{m} c(t_i, x\varepsilon^\delta + B_{\varepsilon^\nu t_i})\Big)\Big| dt_1 \ldots dt_m$$

$$\leq c_m \varepsilon^m \|c(\cdot,\cdot)\|_\infty^m \int\limits_0^{\varepsilon^{(\gamma-1)}} \cdots \int\limits_0^{\varepsilon^{(\gamma-1)}} \rho\big(\min_{2\leq i \leq m} |t_i - t_1|\big) dt_1 \ldots dt_m$$

$$\leq \sum_{i=2}^{m} c_m \varepsilon^m \int\limits_0^{\varepsilon^{(\gamma-1)}} \cdots \int\limits_0^{\varepsilon^{(\gamma-1)}} \rho(|t_i - t_1|) dt_1 \ldots dt_m$$

$$\leq c_m \varepsilon^m \varepsilon^{(\gamma-1)(m-1)} = c_m \varepsilon^{(1+(m-1)\gamma)}.$$

$$(6.22)$$

Combining (6.20)–(6.22) together gives

$$E \exp(\lambda_j^\varepsilon) \leq 1 + c\varepsilon^{(\gamma+1)}. \tag{6.23}$$

Now, letting $L = t/(2\varepsilon^{\gamma+1})$, we can estimate the left hand side of (6.18) as follows

31

$$E \exp \Big(\sum_{j=1}^{L} \lambda_j^\varepsilon \Big) = E \Big(E \{ \exp \Big(\sum_{j=0}^{L} \lambda_j^\varepsilon \Big) \, | \, \mathcal{F}_{L-1}^\varepsilon \} \Big)$$

$$= E \Big[\exp \Big(\sum_{j=0}^{L-1} \lambda_j^\varepsilon \Big) E \{ \exp \big(\lambda_L^\varepsilon \big) \, | \, \mathcal{F}_{L-1}^\varepsilon \} \Big]$$

$$= E \exp \Big(\sum_{j=0}^{L-1} \lambda_j^\varepsilon \Big) E \big(\exp(\lambda_L^\varepsilon) \big)$$

$$+ E \Big[\exp \Big(\sum_{j=0}^{L-1} \lambda_j^\varepsilon \Big) E \{ \big[\exp \big(\lambda_L^\varepsilon \big) - E \exp \big(\lambda_L^\varepsilon \big) \big] \, | \, \mathcal{F}_{L-1}^\varepsilon \} \Big]$$

$$\leq (1 + c\varepsilon^{(1+\gamma)}) E \exp \Big(\sum_{j=0}^{L-1} \lambda_j^\varepsilon \Big)$$

$$+ E \Big[\exp \Big(\sum_{j=0}^{L-1} \lambda_j^\varepsilon \Big) E \{ \big[\exp \big(\lambda_L^\varepsilon \big) - E \exp \big(\lambda_L^\varepsilon \big) \big] \, | \, \mathcal{F}_{L-1}^\varepsilon \} \Big]$$

Using successfully Proposition 7.2.6 from [5], the obvious inequality

$$\| \exp(\xi) - E \exp(\xi) \|_\infty \leq \| \exp(\xi) \|_\infty \| \xi \|_\infty,$$

the bound (6.19), and the fact that $\gamma < 1/2$, we obtain the inequality

$$\begin{aligned} \big| E \{ \big[\exp \big(\lambda_L^\varepsilon \big) - E \exp \big(\lambda_L^\varepsilon \big) \big] \, | \, \mathcal{F}_{L-1}^\varepsilon \} \big| &\leq c\alpha(\varepsilon^{(\gamma-1)}) \| \exp \big(\lambda_L^\varepsilon \big) - E \exp \big(\lambda_L^\varepsilon \big) \|_{L^\infty} \\ &\leq c \| \exp \big(\lambda_L^\varepsilon \big) \|_{L^\infty} \| \lambda_L^\varepsilon \|_{L^\infty} \alpha(\varepsilon^{(\gamma-1)}) \\ &\leq c\varepsilon^\gamma (\varepsilon^{(\gamma-1)})^{-(3+\delta)} \\ &= c\varepsilon^{(2-\gamma+\delta-\gamma\delta)} \\ &\leq c\varepsilon^{(\gamma+1)}. \end{aligned}$$

Finally, we conclude that

$$E \exp \Big(\sum_{j=0}^{L} \lambda_j^\varepsilon \Big) \leq E \exp \Big(\sum_{j=0}^{L-1} \lambda_j^\varepsilon \Big) (1 + c\varepsilon^{(\gamma+1)}).$$

Iterating this inequality, we get after L steps:

$$E \exp \Big(\sum_{j=0}^{L} \lambda_j^{\varepsilon} \Big) \leq \big(1 + c\varepsilon^{(\gamma+1)} \big)^L \leq \big(1 + c\varepsilon^{(\gamma+1)} \big)^{(t/\varepsilon^{(\gamma+1)})} \leq \exp(2ct).$$

The contribution of the odd terms can be estimated exactly in the same way, and the proof is complete. $\square$

Proposition 6.8. *We have that*

$$E\Big(\big(\mathbb{E}(g(x + B_t)[e^{Y_t^{\varepsilon,x}} - e^{t\Sigma}]) \big)^2 \Big) \to 0$$

as $\varepsilon \to 0$, where

$$\Sigma = \int_0^{\infty} \mathbb{E}\Phi(r, B_r) dr.$$

PROOF: We have to compute

$$E\Big(\big(\mathbb{E}(g(x + B_t)[e^{Y_t^{\varepsilon,x}} - e^{t\Sigma}]) \big)^2 \Big) \tag{6.24}$$

$$= \int_{\Omega} \int_{\Omega} g(x + B_t(\omega))g(x + B_t(\omega')) \, E\big(e^{Y_t^{\varepsilon,x}(\omega) + Y_t^{\varepsilon,x}(\omega')} \big) \mathbb{P}(d\omega)\mathbb{P}(d\omega')$$

$$- 2e^{t\Sigma}\mathbb{E}g(x + B_t) \int_{\Omega} g(x + B_t(\omega)) E e^{Y_t^{\varepsilon,x}(\omega)} \mathbb{P}(d\omega) + (\mathbb{E}g(x + B_t))^2 e^{2t\Sigma}.$$

It follows from Proposition 6.3, Proposition 6.4 and Lemma 6.6 that

$$E e^{Y_t^{\varepsilon,x}} \to e^{t\Sigma}, \tag{6.25}$$

in $\mathbb{P}$–probability as $\varepsilon \to 0$, and

$$E e^{Y_t^{\varepsilon,x}(\omega) + Y_t^{\varepsilon,x}(\omega')} \to e^{2t\Sigma} \tag{6.26}$$

in $\mathbb{P}(d\omega) \times \mathbb{P}(d\omega')$–probability as $\varepsilon \to 0$. Passing to the limit, as $\varepsilon \to 0$, on the right-hand side of (6.24) we arrive at the required assertion. $\square$

An immediate consequence of the last Proposition is the

Corollary 6.9. *For any $t > 0$ and $x \in \mathbb{R}$, the limit in probability $u(t,x)$ of $u^\varepsilon(t,x)$ is given by*

$$u(t,x) = \mathbb{E}[g(x + B_t)] \exp\left(t\Sigma\right),$$

which is a solution of the deterministic parabolic PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + \Sigma u(t,x), & t \geq 0, \ x \in \mathbb{R}; \\ u(0,x) = g(x), \ x \in \mathbb{R}. \end{cases} \tag{6.27}$$

We will now state an additional result.

Proposition 6.10. *Under the assumptions of the above results, the collection $\{u^\varepsilon(t,x), \ t \geq 0, \ x \in \mathbb{R}\}_{\varepsilon > 0}$ is tight in $C([0,\infty) \times \mathbb{R})$.*

The proof of this proposition is the object of the next subsection. We first state a clear consequence of the last two statements

Corollary 6.11. *As $\varepsilon \to 0$, $u^\varepsilon(t,x) \to u(t,x)$ in probability, locally uniformly in t and x.*

6.2 Proof of Proposition 6.10

We begin by proving tightness of the family

$$\left\{\mathbb{E}\left[\exp(Y_{t,x}^\varepsilon)\right], \ t \geq 0, \ x \in \mathbb{R}\right\}$$

in the topology of locally uniform convergence. This tightness is an immediate consequence of the following two inequalities (see Theorem 20 from Appendix I in [7]):

$$E\left|\mathbb{E}\left(\exp(Y_{t,x}^\varepsilon)\right) - \mathbb{E}\left(\exp(Y_{s,x}^\varepsilon)\right)\right|^5 \leq C_N |t - s|^{5/2} \tag{6.28}$$

and

$$E\left|\mathbb{E}\left(\exp(Y_{t,x}^\varepsilon)\right) - \mathbb{E}\left(\exp(Y_{t,y}^\varepsilon)\right)\right|^5 \leq C_N |x - y|^{9/4} \tag{6.29}$$

for all $s, t \in [0, N]$ and $x, y \in [-N, N]$, and the estimate

$$E\,\mathbb{E}\left(\exp(Y_{t,x}^\varepsilon)\right) \leq C. \tag{6.30}$$

The latter estimate follows from Lemma 6.6.

We first prove inequality (6.28) which is easier.

Without loss of generality we assume that $t > s$. Then, considering the evident inequality

$$e^a - e^b \le (e^a + e^b)(a - b), \tag{6.31}$$

we obtain

$$E\left[\left|\mathbb{E}\big(\exp(Y_{t,x}^\varepsilon)\big) - \mathbb{E}\big(\exp(Y_{s,x}^\varepsilon)\big)\right|^5\right] \le E\left[\left|\mathbb{E}\{\big(\exp(Y_{t,x}^\varepsilon) + \exp(Y_{s,x}^\varepsilon)\big)|Y_{t,x}^\varepsilon - Y_{s,x}^\varepsilon|\}\right|^5\right]$$

$$\le E\left[\big(\mathbb{E}|Y_{t,x}^\varepsilon - Y_{s,x}^\varepsilon|^6\big)^{5/6}\big(\mathbb{E}\big(\exp(Y_{t,x}^\varepsilon) + \exp(Y_{s,x}^\varepsilon)\big)^{30}\big)^{1/6}\right]$$

$$\le \big(E\,\mathbb{E}|Y_{t,x}^\varepsilon - Y_{s,x}^\varepsilon|^6\big)^{5/6}\big(E\,\mathbb{E}\big(\exp(Y_{t,x}^\varepsilon) + \exp(Y_{s,x}^\varepsilon)\big)^{30}\big)^{1/6}$$

Lemma 6.6 implies that

$$E\left[\big(\exp(Y_{t,x}^\varepsilon) + \exp(Y_{s,x}^\varepsilon)\big)^{30}\right] \le C \tag{6.32}$$

with a constant C that does not depend on ε. It follows from Lemma 6.5 that

$$E\left[|Y_{t,x}^\varepsilon - Y_{s,x}^\varepsilon|^6\right] \le C|t - s|^3.$$

Combining the last three estimates we obtain (6.28).

We proceed with the proof of (6.29). In this case the proof is more involved. We have

$$\mathbb{E}\big(\exp(Y_{t,x}^\varepsilon)\big) - \mathbb{E}\big(\exp(Y_{t,y}^\varepsilon)\big)$$

$$= \mathbb{E}\left(\exp\left[\varepsilon\int_0^{t/\varepsilon^2} c\Big(s, \frac{x}{\varepsilon} + B_s\Big)ds\right] - \exp\left[\varepsilon\int_0^{t/\varepsilon^2} c\Big(s, \frac{y}{\varepsilon} + \tilde{B}_s\Big)ds\right]\right), \tag{6.33}$$

where B_s and $\tilde{B}_s$ are two standard Wiener processes.

Let B_s^1, B_s^2 and B_s^3 be three standard independent Wiener processes, and denote

$$\tau_\varepsilon^{x-y} = \inf\{t > 0 : \frac{x}{\varepsilon} + B_t^1 = \frac{y}{\varepsilon} + B_t^2\}$$

$$= \inf\{t > 0 : \frac{x-y}{\varepsilon} + B_t^1 = B_t^2\}. \tag{6.34}$$

35

We set

$$
\begin{cases}
B_s = B_s^1 & \text{for } s \le \tau_\varepsilon^{x-y}, \\[2mm]
\tilde{B}_s = B_s^2 & \text{for } s \le \tau_\varepsilon^{x-y}, \\[2mm]
B_s = \tilde{B}_s = B_s^3 - B_{\tau_\varepsilon^{x-y}}^3 + B_{\tau_\varepsilon^{x-y}}^1 + \frac{x}{\varepsilon} & \text{for } s \ge \tau_\varepsilon^{x-y}.
\end{cases}
$$

Then B_s and $\tilde{B}_s$ are two standard Wiener processes such that $\frac{x}{\varepsilon} + B_s$ and $\frac{y}{\varepsilon} + \tilde{B}_s$ coincide for $s \ge \tau_\varepsilon^{x-y}$.

The next statement is easy to prove using the well–known reflection principle

Lemma 6.12. *Let $\{B_t, \ t \ge 0\}$ be a standard Brownian motion, $x \in \mathbb{R}$ and $\tau_x := \inf\{t > 0, \ B_t = x\}$. Then*

$$
\mathbb{P}(\tau_x > t) = \mathbb{P}(-|x| < B_t < |x|) \le \sqrt{\frac{2}{\pi}} \frac{|x|}{\sqrt{t}}.
$$

It follows that

$$
\mathbb{P}\left(\tau_\varepsilon^{x-y} > \frac{|x-y|}{\varepsilon^2}\right) \le c \, |x-y|^{1/2}. \tag{6.35}
$$

Assuming without loss of generality that $x-y > 0$, we introduce the following two events and the corresponding indicator functions:

$$
\mathbb{I}_\varepsilon^+ = \{\omega \in \Omega \ : \ \tau_\varepsilon^{x-y} > \frac{x-y}{\varepsilon^2}\}, \qquad \mathbb{I}_\varepsilon^- = \{\omega \in \Omega \ : \ \tau_\varepsilon^{x-y} \le \frac{x-y}{\varepsilon^2}\},
$$

$$
I_\varepsilon^+ = \mathbf{1}_{\mathbb{I}_\varepsilon^+}, \qquad I_\varepsilon^- = \mathbf{1}_{\mathbb{I}_\varepsilon^-}; \tag{6.36}
$$

here $\mathbf{1}$ stands for the indicator function of a set. Letting

$$
Z_t^\varepsilon = \exp(Y_{t,x}^\varepsilon) - \exp(\tilde{Y}_{t,y}^\varepsilon)
$$

with

$$
Y_{t,x}^\varepsilon = \varepsilon \int_0^{t/\varepsilon^2} c\left(s, \frac{x}{\varepsilon} + B_s\right) ds, \qquad \tilde{Y}_{t,y}^\varepsilon = \varepsilon \int_0^{t/\varepsilon^2} c\left(s, \frac{y}{\varepsilon} + \tilde{B}_s\right) ds,
$$

we have

$$E\big|\mathbb{E}(Z_t^\varepsilon)\big|^5 = E\big|\mathbb{E}(I_\varepsilon^+ Z_t^\varepsilon) + \mathbb{E}(I_\varepsilon^- Z_t^\varepsilon)\big|^5$$
$$\leq 2^4 E\big|\mathbb{E}(I_\varepsilon^+ Z_t^\varepsilon)\big|^5 + 2^4 E\big|\mathbb{E}(I_\varepsilon^- Z_t^\varepsilon)\big|^5$$
$$= J_1 + J_2.$$

In order to estimate J_1 we use the inequality

$$\big|\mathbb{E}(I_\varepsilon^+ Z_t^\varepsilon)\big|^5 \leq \big(\mathbb{E}[(I_\varepsilon^+)^{1+\delta}]\big)^{\frac{5}{1+\delta}} \left[\mathbb{E}\big(|Z_t^\varepsilon|^{\frac{1+\delta}{\delta}}\big)\right]^{\frac{5\delta}{1+\delta}}$$
$$\leq c(x-y)^{\frac{5}{2(1+\delta)}} \left[\mathbb{E}\big(|Z_t^\varepsilon|^{\frac{1+\delta}{\delta}}\big)\right]^{\frac{5\delta}{1+\delta}};$$

here we have also used (6.35). This yields, with $\delta = 1/9$,

$$J_1 \leq c\,(x-y)^{9/4}\, E\left(\left[\mathbb{E}\big(|Z_t^\varepsilon|^{10}\big)\right]^{1/2}\right)$$
$$\leq c\,(x-y)^{9/4}\left[E\mathbb{E}\big(|Z_t^\varepsilon|^{10}\big)\right]^{1/2}.$$

But from Lemma 6.6, there exists C such that for all $\varepsilon > 0$,

$$E\big(|Z_t^\varepsilon|^{10}\big) \leq C.$$

Consequently

$$J_1 \leq \bar{c}\,|x-y|^{9/4}. \tag{6.37}$$

We proceed with estimating J_2. With the help of (6.31), we obtain

$$2^{-4} J_2 \leq E\,\mathbb{E}\big(I_\varepsilon^- |Z_t^\varepsilon|^5\big)$$
$$\leq \mathbb{E}\,E\left[I_\varepsilon^-\big(\exp(Y_{t,x}^\varepsilon) + \exp(\tilde{Y}_{t,y}^\varepsilon)\big)^5 |Y_{t,x}^\varepsilon - \tilde{Y}_{t,y}^\varepsilon|^5\right]$$
$$\leq \left(\mathbb{E}\,E\Big(\big[\big(\exp(Y_{t,x}^\varepsilon) + \exp(\tilde{Y}_{t,y}^\varepsilon)\big)\big]^{30}\Big)\right)^{1/6} \times \left(\mathbb{E}\,E\Big[I_\varepsilon^- |Y_{t,x}^\varepsilon - \tilde{Y}_{t,y}^\varepsilon|^6\Big]\right)^{5/6}.$$

Again from Lemma 6.6, we conclude that the first factor on the right-hand side is bounded, uniformly in ε :

$$\mathbb{E}\,E\Big(\big[\big(\exp(Y_{t,x}^\varepsilon) + \exp(\tilde{Y}_{t,y}^\varepsilon)\big)\big]^{30}\Big) \leq C. \tag{6.38}$$

Denoting $R^\varepsilon(s) = c(s, \frac{x}{\varepsilon} + B_s) - c(s, \frac{y}{\varepsilon} + \tilde{B}_s)$, for the second factor we get

$$E\Big[I_\varepsilon^- |Y_{t,x}^\varepsilon - \tilde{Y}_{t,y}^\varepsilon|^6\Big] = I_\varepsilon^- E\Big[\Big(\varepsilon \int_0^{\tau_\varepsilon^{x-y}} R^\varepsilon(s)ds\Big)^6\Big];$$

here we have also used the definition of I_ε^-. It follows from Lemma 6.5 that

$$E\Big[\Big(\varepsilon \int_0^{\tau_\varepsilon^{x-y}} R^\varepsilon(s)ds\Big)^6\Big] \le c(\varepsilon^2 \tau_\varepsilon^{x-y})^3.$$

This yields, since $\varepsilon^2 \tau_\varepsilon^{x-y} \le |x-y|$ on the set $\mathbb{I}_\varepsilon^-$,

$$\begin{aligned}
\Big(\mathbb{E}\, E\Big[I_\varepsilon^- |Y_t^{x,\varepsilon} - \tilde{Y}_t^{y,\varepsilon}|^6\Big]\Big)^{5/6} &\le c\Big(\mathbb{E}\, I_\varepsilon^- (\varepsilon^2 \tau_\varepsilon^{x-y})^3\Big)^{5/6} \\
&\le C|x-y|^{5/2};
\end{aligned} \tag{6.39}$$

Combining this bound with (6.38) and then with (6.37), we arrive at (6.29).

From estimates (6.28)–(6.30) we deduce that for any $N > 0$ there is $C_N > 0$ such that

$$E \sup_{\substack{|x| \le N, \\ 0 \le t \le N}} \mathbb{E}\, \exp(Y_{t,x}^\varepsilon) \le C_N. \tag{6.40}$$

In exactly the same way one can prove the estimate

$$E \sup_{\substack{|x| \le N, \\ 0 \le t \le N}} \mathbb{E}\, \exp(2Y_{t,x}^\varepsilon) \le C_N. \tag{6.41}$$

We can clearly deduce from (6.40)

$$E \sup_{\substack{|x| \le N, \\ 0 \le t \le N}} \mathbb{E}\big\{|g(x + \varepsilon B_{t/\varepsilon^2})| \exp(Y_{t,x}^\varepsilon)\big\} \le C_N \|g\|_{L^\infty(\mathbb{R})}. \tag{6.42}$$

Furthermore, (6.28)–(6.29) imply (see the proof of Theorem 20 in Appendix I from [7])

$$\sup_{\varepsilon > 0} P\Big\{ \sup_{\mathcal{Q}_\delta^N} \big|\mathbb{E}\big\{ \exp(Y_{t,x}^\varepsilon) - \exp(Y_{s,y}^\varepsilon)\big\}\big| \ge \eta \Big\} \longrightarrow 0, \tag{6.43}$$

as $\delta \to 0$, for any $\eta > 0$ and $N > 0$; here $\mathcal{Q}_\delta^N$ stands for the set

$$\mathcal{Q}_\delta^N = \{(t, s, x, y) \in [0, N]^2 \times [-N, N]^2 \,:\, |t - s| \leq \delta, \, |x - y| \leq \delta\}.$$

From (6.43) and (6.41) follows

$$\limsup_{\delta \to 0} \, E \sup_{\mathcal{Q}_\delta^N} \left| \mathbb{E}\big\{ \exp(Y_{t,x}^\varepsilon) - \exp(Y_{s,y}^\varepsilon)\big\} \right| = 0. \tag{6.44}$$

Let us first prove the tightness of $\{u^\varepsilon\}_{\varepsilon > 0}$ for $g \in C_0^\infty(\mathbb{R})$. It suffices to prove that

$$E \left| \mathbb{E}\big(g(x + B_t)\exp(Y_{t,x}^\varepsilon)\big) - \mathbb{E}\big(g(x + B_s)\exp(Y_{s,x}^\varepsilon)\big) \right|^5 \leq C_N |t - s|^{5/2} \tag{6.45}$$

and

$$E \left| \mathbb{E}\big(g(x + B_t)\exp(Y_{t,x}^\varepsilon)\big) - \mathbb{E}\big(g(y + B_t)\exp(Y_{t,y}^\varepsilon)\big) \right|^5 \leq C_N |x - y|^{9/4} \tag{6.46}$$

for all s, $t \in [0, N]$ and x, $y \in [-N, N]$.

For $g \in C_0^\infty$, the inequality (6.45) can be justified in exactly the same way as we proved (6.28), taking into account the fact that g is uniformly Lipschitz, and we leave its proof to the reader.

In order to derive the second estimate we recall the definition of τ_{x-y}^ε, I_ε^+ and I_ε^- (see (6.34) and (6.36)), let

$$\mathcal{Z}_t^\varepsilon = \big(g(x + \varepsilon B_{t/\varepsilon^2})Y_{t,x}^\varepsilon - g(y + \varepsilon \tilde{B}_{t/\varepsilon^2})\tilde{Y}_{t,y}^\varepsilon\big),$$

and rewrite the left-hand side of this estimate as follows

$$\begin{aligned}
E\left|\mathbb{E}(\mathcal{Z}_t^\varepsilon)\right|^5 &= E\left|\mathbb{E}(I_\varepsilon^+ \mathcal{Z}_t^\varepsilon) + \mathbb{E}(I_\varepsilon^- \mathcal{Z}_t^\varepsilon)\right|^5 \\
&\leq 2^4 E\left|\mathbb{E}(I_\varepsilon^+ \mathcal{Z}_t^\varepsilon)\right|^5 + 2^4 E\left|\mathbb{E}(I_\varepsilon^- \mathcal{Z}_t^\varepsilon)\right|^5 \\
&= \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned}$$

Using the arguments of the proof of inequality (6.37) we readily obtain

$$\mathcal{J}_1 \leq c\|g\|_{L^\infty(\mathbb{R})}^5 |x - y|^{9/4}. \tag{6.47}$$

We turn to $\mathcal{J}_2$ and consider two different cases. Namely, $t \geq |x - y|$ and $t < |x - y|$. In the former case we have $I_\varepsilon^- g(x + \varepsilon B_{t/\varepsilon^2}) = I_\varepsilon^- g(x + \varepsilon \tilde{B}_{t/\varepsilon^2})$.

With the help of (6.31) this yields

$$\mathcal{J}_2 \le \mathbb{E}\, E\left[I_\varepsilon^- \|g\|_{L^\infty}^5 \left(\exp(Y_{t,x}^\varepsilon) + \exp(\tilde{Y}_{t,y}^\varepsilon) \right)^5 |Y_{t,x}^\varepsilon - \tilde{Y}_{t,y}^\varepsilon|^5 \right]$$

$$\le \left(\mathbb{E}\, E\left(\left[\|g\|_{L^\infty} \left(\exp(Y_{t,x}^\varepsilon) + \exp(\tilde{Y}_{t,y}^\varepsilon) \right) \right]^{30} \right) \right)^{1/6}$$

$$\times \left(\mathbb{E}\, E\left[I_\varepsilon^- |Y_{t,x}^\varepsilon - \tilde{Y}_{t,y}^\varepsilon|^6 \right] \right)^{5/6},$$

and, by (6.32) and (6.39), we obtain

$$\mathcal{J}_2 \le c\|g\|^5 |x - y|^{5/2}. \tag{6.48}$$

In the case $t < |x - y|$, in view of (6.45) we have

$$E|u^\varepsilon(t,x) - u^\varepsilon(t,y)|^5 \le 3^4 \big(E|u^\varepsilon(t,x) - u^\varepsilon(0,x)|^5 + E|u^\varepsilon(0,x) - u^\varepsilon(0,y)|^5$$

$$+ E|u^\varepsilon(0,y) - u^\varepsilon(t,y)|^5 \big)$$

$$\le C(t^{5/2} + |g(x) - g(y)|^5 + t^{5/2}).$$

Since $t < |x - y|$ and g is smooth, we finally conclude that

$$E|u^\varepsilon(t,x) - u^\varepsilon(t,y)|^5 \le C|x - y|^{5/2}.$$

This completes the proof of the tightness of $\{u^\varepsilon\}$ in the case of a $C_0^\infty(\mathbb{R})$ initial function.

As a consequence we deduce the following property of $\{u^\varepsilon\}$, in the case $g \in C_0^\infty(\mathbb{R})$, by the exact same argument which led us to (6.44). For any $N > 0$,

$$\limsup_{\delta \to 0} E \sup_{\varepsilon > 0} \sup_{\mathcal{Q}_\delta^N} \left| \mathbb{E}\{ g(x + \varepsilon B_{t/\varepsilon^2}) \exp(Y_{t,x}^\varepsilon) - g(y + \varepsilon B_{s/\varepsilon^2}) \exp(Y_{s,y}^\varepsilon) \} \right| = 0. \tag{6.49}$$

It remains to prove that (6.49) holds true for any $g \in C_b(\mathbb{R})$.

We next assume that $g \in C_b(\mathbb{R})$ and has a compact support. For an arbitrary $\nu > 0$ denote by g_ν a C_0^∞ function such that $|g - g_\nu|_{L^\infty} < \nu$. Then

by (6.49) and (6.42) we get

$$\limsup_{\delta \to 0} \sup_{\varepsilon > 0} E \sup_{\mathcal{Q}_\delta^N} \left| \mathbb{E}\{ g(x + \varepsilon B_{t/\varepsilon^2}) \exp(Y_{t,x}^\varepsilon) - g(y + \varepsilon B_{s/\varepsilon^2}) \exp(Y_{s,y}^\varepsilon) \} \right|$$

$$\leq \limsup_{\delta \to 0} \sup_{\varepsilon > 0} E \sup_{\mathcal{Q}_\delta^N} \left| \mathbb{E}\{ \left(g(x + \varepsilon B_{t/\varepsilon^2}) - g_\nu(x + \varepsilon B_{t/\varepsilon^2}) \right) \exp(Y_{t,x}^\varepsilon) \} \right|$$

$$+ \limsup_{\delta \to 0} \sup_{\varepsilon > 0} E \sup_{\mathcal{Q}_\delta^N} \left| \mathbb{E}\{ g_\nu(x + \varepsilon B_{t/\varepsilon^2}) \exp(Y_{t,x}^\varepsilon) - g_\nu(y + \varepsilon B_{s/\varepsilon^2}) \exp(Y_{s,y}^\varepsilon) \} \right|$$

$$+ \limsup_{\delta \to 0} \sup_{\varepsilon > 0} E \sup_{\mathcal{Q}_\delta^N} \left| \mathbb{E}\{ \left(g_\nu(y + \varepsilon B_{s/\varepsilon^2}) - g(y + \varepsilon B_{s/\varepsilon^2}) \right) \exp(Y_{s,y}^\varepsilon) \} \right|$$

$$\leq C_N \nu + 0 + C_N \nu = 2 C_N \nu.$$

Since $\nu > 0$ is arbitrary, the last relation implies (6.49) for $g \in C_b(\mathbb{R})$ with compact support.

It remains to justify (6.49) for a generic $g \in C_b(\mathbb{R})$. We will use the localization arguments based on the following statement.

Lemma 6.13. *For any $\delta > 0$ and $N > 0$ there is $M = M(\delta, N)$ such that for each $g \in C_b(\mathbb{R})$ with $\operatorname{supp}(g) \cap [-M, M] = \emptyset$ the inequality holds*

$$E \sup_{\substack{|x| \leq N, \\ 0 \leq t \leq N}} \mathbb{E}\{ |g(x + \varepsilon B_{t/\varepsilon^2})| \exp(Y_{t,x}^\varepsilon) \} \leq \delta \|g\|_{L^\infty(\mathbb{R})}.$$

Proof. From (6.41), all we need to do is to estimate

$$\sup_{|x| \leq N, 0 \leq t \leq N} \sqrt{\mathbb{E}\left[g^2(x + B_t) \right]}.$$

The result follows from the fact that for any $\eta > 0$, we can choose M large enough such that

$$\mathbb{P}\left(\sup_{|x| \leq N, 0 \leq t \leq N} |x + B_t| > M \right) \leq \eta.$$

$\square$

Next, for an arbitrary ν, representing the function g as $g = g_{\nu,N}^1 + g_{\nu,N}^2$ with $g_{\nu,N}^1$ having a compact support and $g_{\nu,N}^2$ such that $\operatorname{supp}(g_{\nu,N}^2) \cap [-M(\nu, N), M(\nu, N)] = \emptyset$, one gets in the same way as above

$$\limsup_{\delta \to 0} \sup_{\varepsilon > 0} E \sup_{\mathcal{Q}_\delta^N} \left| \mathbb{E}\{ g(x + \varepsilon B_{t/\varepsilon^2}) \exp(Y_{t,x}^\varepsilon) - g(y + \varepsilon B_{s/\varepsilon^2}) \exp(Y_{s,y}^\varepsilon) \} \right| \leq C_N \nu.$$

Since $\nu > 0$ is arbitrary, this yields (6.49) for a generic $g \in C_b(\mathbb{R})$ and completes the proof of Proposition 6.10.

6.3 The case $0 < 2\beta < \alpha$

Without loss of generality we choose $\alpha = 1$ and $0 < \beta < 1/2$. Hence $\gamma = 1/2$. We know from Proposition 6.3 that

$$Y_t^{\varepsilon,x} = \frac{1}{\sqrt{\varepsilon}} \int_0^t c\left(\frac{s}{\varepsilon}, \frac{x + B_s}{\varepsilon^\beta}\right) ds,$$

converges, as $\varepsilon \to 0$, in $\mathbb{P}$–probability weakly under P to the Gaussian law $N(0, t\int_{\mathbb{R}} \Phi(u,0)du)$.

We now note that the r. v. $Y_{t,x}^\varepsilon$ can be rewritten as

$$Y_{t,x}^\varepsilon = \sqrt{\varepsilon} \int_0^{t/\varepsilon} c\left(s, x\varepsilon^{-\beta} + B_{s\varepsilon(1-\beta)}\right) ds.$$

Hence it follows from Lemma 6.6 with $\nu = (1 - \beta)$ that

$$\sup_{\varepsilon > 0} E\left(\exp[4Y_{t,x}^\varepsilon]\right) \leq C.$$

Consequently, by the same arguments as those in the previous section, we can show the

Proposition 6.14. *As $\varepsilon \to 0$, $u^\varepsilon(t,x)$ converges in probability, locally uniformly in t and x, to $u(t,x)$, which is given by*

$$u(t,x) = \mathbb{E}[g(x + B_t)] \exp\left(t\Sigma'\right),$$

and solves the deterministic parabolic PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + \Sigma' u(t,x), & t \geq 0, \ x \in \mathbb{R}; \\ u(0,x) = g(x), \ x \in \mathbb{R}, \end{cases} \tag{6.50}$$

where $\Sigma' = \int_0^\infty \Phi(u,0)du$.

6.4 The case $\beta = 0$

In this case, $\gamma = \alpha/2$. Without loss of generality, we restrict ourselves to the case $\alpha = 1$.

We will study the limiting behaviour of u^ε as $\varepsilon \to 0$ under the following additional assumption:

(*Hö*) For each $s \in \mathbb{R}$ the realizations $c(s, y)$ are a.s. Hölder continuous in $y \in \mathbb{R}$ with a deterministic exponent $\theta > 1/3$. Moreover,

$$|c(s, y_1) - c(s, y_2)| \le c|y_1 - y_2|^\theta,$$

with a deterministic constant c.

Of course, $(H_{\mathrm{um}})]$ is still assumed to be in force.

Proposition 6.3 still applies here. However, it is not sufficiently precise to be useful in this case. The reason is that the limit of u^ε will not be deterministic in this case. Convergence will be only in law, not in probability or in mean square. Going back to the proof of Proposition 6.8, which is not valid in the present case, we note that while the limiting law of $Y^\varepsilon_{\cdot,x}(\omega)$ is the same as above, that of $(Y^\varepsilon_{\cdot,x}(\omega), Y^\varepsilon_{\cdot,x}(\omega'))$ will be dramatically different.

Consider the exponent in the above Feynman–Kac formula, written in its first form. It reads

$$Y^\varepsilon_{t,x} = \frac{1}{\sqrt{\varepsilon}} \int_0^t c\left(\frac{s}{\varepsilon}, x + B_s\right) ds = \int_0^t W^\varepsilon(ds, x + B_s),$$

where

$$W^\varepsilon(t, x) := \frac{1}{\sqrt{\varepsilon}} \int_0^t c\left(\frac{s}{\varepsilon}, x\right) ds.$$

We have the

Proposition 6.15. *Under assumptions (Hö) and (H_{um}), as $\varepsilon \to 0$,*

$$W^\varepsilon \to W$$

in P–law, as random elements of $C(\mathbb{R}_+ \times \mathbb{R})$ equipped with the topology of locally uniform convergence in t and x, where $\{W(t, x), t \ge 0, x \in \mathbb{R}\}$ is a centered Gaussian process with covariance function given by

$$E(W(t, x)W(t', x')) = t \wedge t' \times R(x - x'),$$

with

$$R(x) = \int_{\mathbb{R}} \Phi(r, x) dr.$$

PROOF: The convergence of finite dimensional distributions is a direct consequence of the functional Central Limit Theorem for stationary processes having good enough mixing properties. Namely, according to the statements in [3], Chapter 4, §20, under the assumption (H_{um}), for any finite set $x^1, x^2, \ldots, x^m$ the family

$$\{W^\varepsilon(\cdot, x^1), \ldots, W^\varepsilon(\cdot, x^m)\}$$

converges in law, as $\varepsilon \to 0$, in the space $(C([0, T]))^m$, towards a m-dimensional Wiener process with covariance matrix

$$\sigma_{ij} = \int\limits_0^\infty E\big(c(s, x^i)c(0, x^j) + c(s, x^j)c(0, x^i)\big)ds = \int\limits_{-\infty}^\infty \Phi(s, x^i - x^j)ds = R(x^i - x^j).$$

The desired result will follow if we prove the tightness of $\{W^\varepsilon, \varepsilon > 0\}$ in $C(\mathbb{R}_+ \times \mathbb{R})$. In order to prove that this family is tight it suffices to show that there are two numbers $\nu_1 > 0$ and $\nu_2 > 2$ such that

$$E|W^\varepsilon(s_1, y_1) - W^\varepsilon(s_2, y_2)|^{\nu_1} \le C(|s_1 - s_2|^{\nu_2} + |y_1 - y_2|^{\nu_2})$$

with a constant C which does not depend on ε.

It follows from Lemma 6.5 that

$$E\left(\varepsilon^{-1/2} \int\limits_{s_1}^{s_2} c\left(\frac{t}{\varepsilon}, y\right)dt\right)^6 \le C(s_2 - s_1)^3.$$

Similarly, by $(H\ddot{o})$ and (H_{um}) one has

$$E\left(\varepsilon^{-1/2} \int\limits_0^s \left(c\left(\frac{t}{\varepsilon}, y_1\right) - c\left(\frac{t}{\varepsilon}, y_2\right)dt\right)\right)^6$$

$$= \varepsilon^3 \int\limits_0^{s/\varepsilon} \ldots \int\limits_0^{s/\varepsilon} E\{(c(t_1, y_1) - c(t_1, y_2)) \ldots (c(t_6, y_1) - c(t_6, y_2))\}dt_1 \ldots dt_6$$

$$\le C\varepsilon^3 |y_1 - y_2|^{6\theta} \int\limits_0^{\sqrt{6}T/\varepsilon} \frac{dV_T(r)}{(1 + r)^{3+\delta}},$$

44

where $V_T(r)$ stands for the volume of the set

$$S_T(r) = \left\{ (t_1, \ldots, t_6) \in \left[0, \frac{T}{\varepsilon}\right]^6 : \max_{1 \leq i \leq 6} \min_{j \neq i} |t_i - t_j| \leq r \right\}.$$

Straightforward computations show that

$$V_T(r) \leq Cr^3 \frac{T^3}{\varepsilon^3},$$

This yields

$$E\left(\varepsilon^{-1/2} \int_0^s \left(c\left(\frac{t}{\varepsilon}, y_1\right) - c\left(\frac{t}{\varepsilon}, y_2\right) dt \right)^6 \leq CT^3 |y_1 - y_2|^{6\theta} \int_0^\infty \frac{r^3 dr}{(1+r)^{4+\delta}} \right.$$

Since $\theta > 1/3$, this implies the desired estimate. $\qquad\square$

As we shall see below, the exponent in the Feynman–Kac formula converges towards

$$\int_0^t W(ds, x + B_s) = \int_0^t \int_{\mathbb{R}} W(ds, y) L(s, y - x) dy,$$

where again $L(t, z)$ stands for the local time of the process B at time t and location z.

Let us note that the left hand–side of the last identity can be defined without any reference to local time. Recall that W and B are independent, hence it suffices to define the stochastic integral

$$\int_0^t W(ds, f(s)), \quad t \geq 0,$$

with $f \in C(\mathbb{R}_+)$.

Proposition 6.16. *To any $f \in C(\mathbb{R}_+)$, we associate the continuous centered Gaussian process*

$$\{Y_t := \int_0^t W(ds, f(s)), \ t \geq 0\}$$

with the covariance function $(t \wedge t') R(0)$, which is, for each $t > 0$, the limit in probability as $n \to \infty$ of the sequence

$$Y_t^n := \sum_{k=1}^{[t2^n]} \left[W(k2^{-n}, f(k2^{-n})) - W((k-1)2^{-n}, f(k2^{-n})) \right].$$

45

PROOF: The fact that $\{Y_t^n, n \geq 1\}$ is a Cauchy sequence in $L^2(P)$ will follow from the fact that $E(Y_t^n Y_t^m)$ converges to a finite limit as n and m tend to infinity. This is indeed the case, since for $n > m$,

$$E(Y_t^n Y_t^m) = [t2^{-n}]2^n \sum_{\ell=1}^{[t2^m]} \sum_{k=(\ell-1)2^{n-m}}^{\ell 2^{n-m}} R(f(k2^{-n}) - f(\ell 2^{-m}))$$
$$\rightarrow tR(0),$$

as n and m tend to infinity, with $n > m$. The fact that Y_t is Gaussian and centered follows easily, as well as the formula for the covariance. $\square$

Note that the conditional law of $Y_t = \int_0^t W(ds, x + B_s)$, given $\{B_s, 0 \leq s \leq t\}$ is the law $N(0, tR(0))$. It does not depend on the realization of $\{B_s, 0 \leq s \leq t\}$, in agreement with Proposition 6.3. However, Y_t does depend on $\{B_s, 0 \leq s \leq t\}$. This follows in particular from the fact that if B and B' are two trajectories of the Brownian motion,

$$E\left[\int_0^t W(ds, B_s) \int_0^{t'} W(ds, B'_s)\right] = \int_0^{t \wedge t'} R(B_s - B'_s)ds.$$

The uniform integrability here is easy to establish. Indeed, we saw in the previous section that it is sufficient to prove that the collection of r. v.

$$\left\{\mathbb{E}\exp(2Y_{t,x}^\varepsilon), \; \varepsilon > 0\right\}$$

is P–tight. Since those are non–negative random variables, a sufficient condition is that

$$\sup_{\varepsilon > 0} E\mathbb{E}\exp(2Y_{t,x}^\varepsilon) < \infty,$$

and we can very well interchange the order of expectation. Now Lemma 6.6 above, in the case $\nu = 2$, implies that

$$\sup_{\varepsilon > 0} E\left(\exp\left[\frac{1}{\sqrt{\varepsilon}} \int_0^t c\left(\frac{s}{\varepsilon}, x + B_s\right) ds\right]\right) \leq C,$$

where C is a finite constant. This is easily seen by making the following change of variable :

$$\frac{1}{\sqrt{\varepsilon}} \int_0^t c\left(\frac{s}{\varepsilon}, x + B_s\right) ds = \eta \int_0^{t/\eta^2} c(r, x + B_{\varepsilon^2 r})dr,$$

46

with $\eta = \sqrt{\varepsilon}$.

We can now establish the

Theorem 6.17. *Under assumptions (Hö) and (H$_{\text{um}}$) for each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,*

$$u^\varepsilon(t, x) \to u(t, x) := \mathbb{E}\left[g(X_t^x) \exp\left(\int_0^t \int_{\mathbb{R}} W(ds, y)L(s, y - x)dy\right)\right]$$

in P–law, as $\varepsilon \to 0$.

PROOF: Note that

$$Y_{t,x}^\varepsilon = \int_0^t \int_{\mathbb{R}} W^\varepsilon(ds, y)L(s, y - x)dy$$

$$= \int_{\mathbb{R}} W^\varepsilon(t, y)L(t, y - x)dy - \int_0^t \int_{\mathbb{R}} W^\varepsilon(s, y)L(ds, y - x)dy$$

Define the functional $\Psi_{t,x} : [0, t] \times \mathbb{R} \to \mathbb{R}$ as

$$\Psi_{t,x}(\varphi) := \mathbb{E}\left[g(X_t^x) \exp\left(\int_{\mathbb{R}} \varphi(t, y)L(t, y - x)dy - \int_0^t \int_{\mathbb{R}} \varphi(s, y)L(ds, y - x)dy\right)\right].$$

All we have to show is that

$$u^\varepsilon(t, x) = \Psi_{t,x}(W^\varepsilon) \to \Psi_{t,x}(W)$$

in P–law, which follows from Proposition 6.15 and uniform integrability, since $\Psi_{t,x}$ is continuous. $\square$

The corresponding limiting SPDE reads (in Stratonovich form)

$$\begin{cases} du(t, x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t, x)dt + u(t, x) \circ W(dt, x), & t \geq 0, \ x \in \mathbb{R}; \\ u(0, x) = g(x), & x \in \mathbb{R}. \end{cases}$$

We can rewrite this SPDE in Itô form as follows

$$\begin{cases} du(t, x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t, x)dt + \dfrac{1}{2}u(t, x)R(0)dt + u(t, x)W(dt, x), & t \geq 0, \ x \in \mathbb{R}; \\ u(0, x) = g(x), & x \in \mathbb{R}. \end{cases}$$

47

Let us finally improve the convergence result stated in Theorem 6.17. First of all, it is not hard to show that this result can be extended to the proof of the convergence of the finite dimensional distributions of u^ε towards those of u. In other words, for all $n \geq 1$, all $(t_1, x_1), \ldots (t_n, x_n) \in \mathbb{R}_+ \times \mathbb{R}$, $(u^\varepsilon(t_1, x_1), \ldots, u^\varepsilon(t_n, x_n))$ converges weakly to $(u(t_1, x_1), \ldots, u(t_n, x_n))$ as $\varepsilon \to 0$. Finally tightness in $C(\mathbb{R}_+ \times \mathbb{R})$ of the collection of random fields$\{u^\varepsilon, \ \varepsilon > 0\}$ in easier to prove than in the above sub–sections. Indeed, an immediate adaptation of the proof of Proposition 6.15 yields that the collection $\{Y_\varepsilon^{t,x}, \ t \geq 0, x \in \mathbb{R}\}_{\varepsilon>0}$, defined on the product probability space $(\Omega \times \mathcal{S}, \mathcal{F} \otimes \mathcal{A}, \mathbb{P} \times P)$, is tight in $C(\mathbb{R}_+ \times \mathbb{R})$. This, combined with the uniform integrability of $\exp(Y_\varepsilon^{t,x})$ and the continuity of g, implies the tightness of u^ε. We have proved

Theorem 6.18. *Under assumptions (Hö) and (H_{um}), as $\varepsilon \to 0$,*

$$u^\varepsilon(t, x) \to u(t, x) := \mathbb{E}\left[g(X_t^x) \exp\left(\int_0^t \int_{\mathbb{R}} W(ds, y) L(s, y - x) dy\right)\right]$$

in P–law in $C(\mathbb{R}_+ \times \mathbb{R})$ equipped with the topology of locally uniform convergence in t and x.

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