FILTERING OF A DIFFUSION PROCESS WITH POISSON-TYPE OBSERVATION

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We consider a filtering problem, where the signal X_t is a Markov diffusion process, and the observation is a marked point process (for instance a Poisson process), whose predictable projection (the stochastic intensity in the case of a point process) is a given function of the signal X_t.

We associate to this problem a backward stochastic PDE, whose solution is expressible in terms of the conditional law in the filtering problem. It then follows that the forward equation, adjoint to the backward one, governs the evolution of the "unnormalized conditional density".

Analogous results have been proved in the case of an observation corrupted by a Wiener noise in [5] and [6]. The proofs here are more direct.

§1. THE FILTERING PROBLEM

§.1.1. The signal Process.

We are given the following functions on $R_+ \times R^N$:

$\sigma_{i,j}(t,x)$ [resp. $b_i(t,x)$] continuous [resp. Borel measurable] and bounded on $[0,T] \times R^N$, $\forall T > 0$, $i,j = 1\ldots N$.

Let $a = \sigma\sigma^*$. We suppose :

(1.1) $\quad \exists \alpha > 0$ s.t. $(a(t,x)\xi,\xi) \geq \alpha|\xi|^2$, $\forall \xi \in R^N$, $\quad \forall (t,x) \in R_+ \times R^N$

(1.2) $\quad \dfrac{\partial a_{ij}}{\partial x_j} \in L^\infty(]0,T[\times R^N)$, $\forall T > 0$, $\quad i,j = 1\ldots N$

Let $\Omega_1 = C(R_+ ; R^N)$, $X_t(\omega_1) = \omega_1(t)$, $\mathcal{G}_t^s = \sigma\{X_\theta, s \leq \theta \leq t\}$, $\mathcal{G}^s = \bigvee_{t \geq s} \mathcal{G}_t^s$.

We consider the martingale problem, cf. STROOCK-VARADHAN [8], associated with the infinitesimal generator

$$L_t = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}(t,x) \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{N} b_i(t,x) \frac{\partial}{\partial x_i}$$

Let P^1_{sx} denote the unique probability measure on $(\Omega_1, \mathcal{G}^s)$, solution of the martingale problem, with initial condition $P^1_{sx}(X_s = x) = 1$.

Let $p_o \in L^2(R^N)$ satisfy :

$$p_o(x) \geq 0 \quad \text{a.e.}, \quad \int_{R^N} p_o(x)\, dx = 1 ,$$

and let P_1 be the unique probability measure on $(\Omega_1, \mathcal{G}^o)$ satisfying :

$$P_1(X_t \in B) = \int_{R^N} p_o(x)\, P^1_{ox}(X_t \in B) dx ,$$

$\forall$ B Borel subset of R^N.

Then there exists a $P_1 - \mathcal{G}^o_t$ standard Wiener process W_t with values in R^N, such that, P_1 a.s. :

$$(1.3) \qquad dX_t = b(t,X_t)dt + \sigma(t,X_t)\, dW_t$$

$(W_t - W_s,\ t \geq s)$ is also a $P^1_{sx} - \mathcal{G}^s_t$ Wiener process.

§.1.2. The observation process

Let $(\Omega_2, \mathcal{F}, P_2)$ be a probability space, on which we define a marked point process (cf. JACOD [3], BREMAUD-JACOD [2]). Let $(\mathcal{F}_t,\ t \geq 0)$ be an increasing and right continuous family of sub-σ-algebras of $\mathcal{F}$. Let T_n be a strictly increasing family of $\mathcal{F}_t$ stopping times. We suppose :

$$\lim_{n \to \infty} T_n = +\infty \quad \text{a.s.}$$

Let z_n be a sequence of $\mathcal{F}_{T_n}$ measurable random variables with values in a measurable space $(Z, \mathcal{Z})$ (the set of "marks").

We define a transition measure from $(\Omega, \mathcal{F})$ over $(R_+ \times Z, \mathcal{B} \otimes \mathcal{Z})$, where $\mathcal{B}$ is the Borel σ-field over R_+, by :

$$\mu(\omega\ ;\ A \times B) = \sum_{n>0} 1_{\{T_n \in A\}}\, 1_{\{z_n \in B\}}$$

where $A \in \mathcal{B}$, $B \in \mathcal{Z}$.

Let $\mathcal{F}^\mu_t = \sigma\{\mu(]o,s] \times B),\ s \leq t,\ B \in \mathcal{Z}\}$ $\mathcal{F}^\mu_t \subset \mathcal{F}_t$, and $\mathcal{F}^\mu_t$ is increasing and right continuous (see COURREGE - PRIOURET [11], and [3]).

We suppose that there exists a positive finite measure ν on $(Z, \mathcal{Z})$ such that :

$$\mu(]o, t] \times B) - t\,\nu(B) \quad \text{is a } P_2 - \mathcal{F}_t^{\mu} \text{ martingale}, \ \forall B \in \mathcal{Z} .$$

It follows from WATANABE's result (see [9], [10]) that $\mu(]o, t] \times B)$ is a $P_2 - \mathcal{F}_t^{\mu}$ Poisson process with intensity $\nu(B)$. In other words (JACOD [3]), $dt \times \nu(dz)$ is the predictable projection of $\mu(dt \times dz)$.

Let now $(\Omega, \mathcal{F}^S) = (\Omega_1 \times \Omega_2, \mathcal{G}^S \otimes \mathcal{F})$. We will write $\mathcal{G}_t^S$ for $\mathcal{G}_t^S \otimes \{\Omega_2, \emptyset\}$ and $\mathcal{F}_t^{\mu}$ for $\{\Omega_1, \emptyset\} \otimes \mathcal{F}_t^{\mu}$. Define $\mathcal{F}_t = \mathcal{G}_t^o \vee \mathcal{F}_t^{\mu}$.

Let $\tilde{P}_{sx} = P_{sx}^1 \times P_2$

$\tilde{P} = P_1 \times P_2$

Let $\Psi(t,x,z)$ be a measurable, non negative and bounded function defined on $R_+ \times R^N \times Z$. Consider the process :

$$\rho(t,z) = \Psi(t, X_t, z)$$

If $\mathcal{P}$ denotes the σ-field of $\mathcal{F}_t$-previsible subsets of $\Omega \times R_+$, ρ is $\mathcal{P} \otimes \mathcal{Z}$ measurable and bounded. We can define :

$$M_t^{\theta} = \prod_{\{n/\theta < T_n \leq t\}} \rho(T_n, z_n) \times \exp\Big\{ - \int_{]\theta,t] \times Z} (\rho(s,z)-1)\nu(dz)ds \Big\}$$

We define the measure P on $(\Omega, \mathcal{F}_t)$ by :

$$\frac{dP}{d\tilde{P}} = M_t^o$$

can show (see BREMAUD [1]).

<u>Lemma 1.1.</u>

(i) P and $\tilde{P}$ coincide on $\mathcal{G}^o$

(ii) $\forall B \in \mathcal{Z}$, $\mu(]o,t] \times B) - \int_{]o,t] \times B} \rho(s,z)\,\nu(dz)ds$ is a $\mathcal{F}_t - P$ martingale. ∎

We will denote by E [resp. $\tilde{E}$, $\tilde{E}_{sx}$] the expectation with respect to P [resp. $\tilde{P}$, $\tilde{P}_{sx}$].

§.1.3. <u>The filtering problem</u>

Our aim is to caracterize the law of X_t, conditionned by $\mathcal{F}_t^{\mu}$, or equivalently quantities of the form : $E(f(X_t) / \mathcal{F}_t^{\mu})$.

We will see that this conditionnal law has a density with respect to Lebesgue measure, which we shall caracterize through the solution of a stochastic partial differential equation.

The following result is easy to prove :

Lemma 1.2.

Let f be a bounded Borel measurable function from R^N into R. Then $\forall t > 0$:

$$(1.4) \qquad E(f(X_t) \,/\, \mathcal{F}_t^\mu) = \frac{\tilde{E}(f(X_t)\, M_t^o \,/\, \mathcal{F}_t^\mu)}{\tilde{E}(M_t^o \,/\, \mathcal{F}_t^\mu)} \quad \text{a.s.}$$

2. STUDY OF A BACKWARD STOCHASTIC PDE

Define $H^1(R^N) = \{u \in L^2(R^N) \;;\; \frac{\partial u}{\partial x_i} \in L^2(R^N),\ i = 1 \ldots N\}$,

and $H^{-1}(R^N)$ its dual space.

We will denote by $|.|$ and $||.||$ the norms in $L^2(R^N)$ and $H^1(R^N)$, $(.,.)$ the scalar product in $L^2(R^N)$, and $\langle .,. \rangle$ the duality between $H^1(R^N)$ and $H^{-1}(R^N)$.

L_t can be considered as a family of elements of $\mathcal{L}(H^1(R^N) \;;\; H^{-1}(R^N))$, defined by :

$$\langle L_t u, v\rangle = -\frac{1}{2} \sum_{i,j=1}^{N} \int_{R^N} a_{ij}(t,x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx \; - \; \sum_{i=1}^{N} \int_{R^N} a_i(t,x) \frac{\partial u}{\partial x_i}\, v\, dx$$

where $a_i = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial a_{ij}}{\partial x_j} - b_i$.

We consider the following backward stochastic P.D.E. :

$$(2.1) \qquad \begin{cases} dv(t) + L_t\, v(t)dt + v(t) \int_Z h(t,z)(\mu(dt,dz) - \nu(dz)dt) = 0 \\ v(T) = f \end{cases}$$

where $h(t,x,z) = \Psi(t,x,z) - 1$ (we omit the variable x). We are looking for a solution $v(t,x)$, which we will consider as a process with values in $L^2(R^N)$.

Write $\tilde{\Omega}$ for $(\Omega, \mathcal{F}_T, \tilde{P})$ and define :

$$D^2(0,T;\, H^1(R^N), L^2(R^N)) = \{v \in L^2(\tilde{\Omega} \times\,]0,T[;\, H^1(R^N)) \;,$$
$$\text{s.t. } v \text{ belongs a.s. to } \mathcal{D}(0,T \;;\; L^2(R^N))\}$$

where $D(0,T\ ;\ L^2(R^N))$ denotes the space of right-continuous functions having left limits, in $L^2(R^N)$.

<u>Theorem 2.1.</u>
Suppose $f \in L^2(R^N)$, then equation (2.1) has a unique solution :

$$v \in D^2(0,T\ ;\ H^1(R^N),\ L^2(R^N))$$

<u>Proof</u> :
We solve equation (2.1) backward for each ω.
Suppose for instance that $T_n(\omega) = T$. Then :

$$f - v(T_n^-) + f.\ h(T_n,z_n) = 0 \tag{2.2}$$

(2.2) defines $v(T_n^-)$ as an element of $L^2(R^N)$.

We then solve equation (2.1) backward from T_n to T_{n-1}, i.e. :

$$\begin{cases} v'(t) + L_t\, v(t) = \left(\int_Z h(t,z)d\nu(z)\right) v(t)\ ,\ T_{n-1} \le t < T_n \\ v(T_n^-) = f(1+h(T_n,z_n)) \end{cases} \tag{2.3}$$

(2.3) defines a unique element of $L^2(T_{n-1}, T_n\ ;\ H^1(R^N)) \cap C([T_{n-1},T_n[;\ L^2(R^N))$.
Repeating this procedure, we define for each ω a unique element $v(\omega) \in L^2(0,T;H^1(R^N)) \cap D(0,T\ ;\ L^2(R^N))$. It is easy to check that $\omega \to v(\omega)$ is measurable.
Moreover,

$$|v(t)|^2 + 2\int_t^T \langle -Lv,v\rangle\, ds = |f|^2 + \int_t^T\int_Z |hv|^2\, \mu(ds\times dz) \ +$$
$$+\ 2\int_t^T\int_Z \left(\int_{R^N} v^2\, h\, dx\right)(\mu(ds \times dz) - ds\ \nu(dz))$$

Let $t_n = \sup\{0 \le t \le T,\ |v(t)| > n\}$. Using backward martingale properties with respect to $\mathcal{F}_\mu^t = \sigma\{\mu([s,T]\times B), t\le s\le T, B\in \mathcal{Z}\}$, we get:

$$\tilde{E}(|v(t\vee t_n)|^2) + 2\tilde{E}\int_{t\vee t_n}^T \langle -Lv,v\rangle\, ds \le |f|^2 + c\,\tilde{E}\int_{t\vee t_n}^T |v|^2\, ds$$

But from (1.1), it is easy to check that $\exists\ \lambda$, s.t. :

$$\langle -Lv,v\rangle + \lambda|v|^2 \ge \frac{\alpha}{2}\, ||v||^2$$

$$(2.4)\qquad \tilde{E}|v(t\vee t_n)|^2 + \alpha\,\tilde{E}\int_{t\vee t_n}^{T} ||v||^2 ds \le |f|^2 + c_1\,\tilde{E}\int_{t\vee t_n}^{T} |v|^2\, ds$$

One can take the limit in (2.4) when $n \to \infty$, yielding :

$$\tilde{E}\int_0^T ||v||^2\, dt \;<\; +\infty$$

We now prove our main result, which is a sort of generalization of the well-known FEYMAN-KAC formula.

Theorem 2.2.

Suppose f is Borel measurable and both bounded and square integrable, then, $\forall\theta \in [0,T]$,

$$(2.5)\qquad v(\theta,x) = \tilde{E}_{\theta x}\,(f(X_T)\, M_T^{\theta} \,/\, \mathcal{F}_T^{\mu}) \quad \text{a.e. and a.s.}$$

Proof :

It suffices to prove (2.5) in the case of regular (in x) coefficients a, b, h and f. We can then take the limit of both sides of (2.5) when $a_n \to a$ uniformly on compact sets, b_n, h_n and f_n converge in measure on compact sets, as in [7].

But if a, b, h and f are regular-say C^∞ in x with compact support- then any partial derivative in x of v belongs to $D^2(0,T\,;\,H^1(R^N),L^2(R^N))$, because it is a solution of an equation similar to (2.1).

It then follows, by the properties of Sobolev spaces, that a.s. $v(\omega) \in C_b^{1,2}([T_k,T_{k+1}[\times R^N)$, $\forall k$ s.t. $[T_k(\omega), T_{k+1}(\omega)] \subset [0,T]$.

Define $V_t = v(t, X_t)$. It follows from Ito formula that between the jump times T_k :

$$d\,V_t = v'_t(t,X_t)dt + L_t\, v(t,X_t)dt + \nabla v(t,X_t).\,\sigma(t,X_t)d\,W_t$$

Then at any time t,

$$d\,V_t = dv(t,X_t) + L_t\, v(t,X_t)\,dt + \nabla v(t,X_t).\,\sigma(t,X_t)\,d\,W_t$$

On the other hand :

$$d\,M_t^\theta = M_{t^-}^\theta \times \int_Z h(t,X_t,z)(\mu(dt\times dz) - dt\times\nu(dz))$$

Define $\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{F}_T^\mu$. V_t is a $\mathcal{H}_t$ semi-martingale, and M_t^θ is an $\mathcal{H}_t$-adapted process with finite variation. It then follows (see MEYER [4]) :

$$d\, V_t\, M_t^\theta = M_{t-}^\theta \cdot d\, V_t + V_t \cdot d\, M_t^\theta$$

$$d\, V_t\, M_t^\theta = M_{t-}^\theta \cdot \nabla v(t,X_t).\ \sigma(t,X_t) d\, W_t \tag{2.6}$$

because v satisfies equation (2.1) at any point (t,x).

Integrating (2.6) from θ to T, and taking $E_{\theta x}(\,.\,/\,\mathcal{F}_T^\mu)$ of both sides yield (2.5).

3. THE EQUATION FOR THE UNNORMALIZED CONDITIONAL DENSITY

Consider now equation :

$$\begin{cases} du(t) - L_t^* u(t)dt = u(t^-) \int_Z h(t,z)(\mu(dt \times dz) - dt \times \nu(dz)) \\ u(o) = p_o \end{cases} \tag{3.1}$$

where p_o is the density of the law of X_o.

Again equation (3.1) has a unique solution in $D^2(0,T\ ;\ H^1(R^N),\ L^2(R^N))$.

The interesting fact is that (3.1) is the adjoint of (2.1) :

<u>Theorem 3.1.</u>

The following holds a.s. :

$$(u(t),v(t)) = (u(s),v(s)),\quad \forall s,\ t \in [0,T] \tag{3.2}$$

<u>Proof</u> :

Between jump times,

$$\frac{d}{dt}(u(t),v(t)) = \langle L^* u,v\rangle - \langle u,\ Lv\rangle + (u \int h d\nu, v) - (v \int h d\nu, u) = 0$$

On the other hand,

$$u(T_n) = u(T_n^-)(1 + h(T_n,z_n))$$
$$v(T_n^-) = v(T_n)(1 + h(T_n,z_n))$$
$$(u(T_n),v(T_n)) = (u(T_n^-),\ v(T_n^-))$$

We now deduce from theorems 2.2 and 3.1 :

Corollary

(i) $\tilde{E}\,(f(X_T) M_T^o \,/\, \mathcal{F}_T^\mu) = (u(T),f)$

(ii) $E\,(f(X_T) \,/\, \mathcal{F}_T^\mu) = \dfrac{(u(T),f)}{(u(T),1)}$

Proof :

(2.5) and (3.2) yield :

$$\int_{R^N} p_o(x)\, \tilde{E}_{ox}(f(X_T) M_T^o \,/\, \mathcal{F}_T^\mu) = (u(T),f)$$

which proves (i), (ii) then follows from (1.4).

T is arbitrary. We have proved that equation (3.1) describes the evolution of the unnormalized conditional density.

CONCLUSION

Our method can be applied to more general filtering problems. Combining this result with that in [6], we can treat observation processes with both Wiener and Poisson noise, as in BREMAUD [1]. On the other hand, X_t could be in principle any Markov process - for instance a diffusion with jumps.

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