Epidemic models with varying infectivity on a refining spatial grid. I. The SI model

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ABSTRACT. We consider a space-time SI epidemic model with infection age dependent infectivity and non-local infections constructed on a grid of the torus $\mathbb{T}^d = (0,1]^d$, where the individuals may migrate from node to another. The migration processes in either of the two states are assumed to be Markovian. We establish a functional law of large numbers by letting jointly N the initial approximate number of individuals on each node go to infinity and ε the mesh size of the grid go to zero. The limit is a system of parabolic PDE/integral equations. The constraint on the speed of convergence of the parameters N and ε is that $N\varepsilon^d \to \infty$ as $(N,\varepsilon) \to (+\infty, 0)$.

Key words. epidemic model, varying infectivity, non-local infections, law of large numbers, integral equations, space-time.

1. Introduction

We consider an epidemic model on a refining grid of the d dimensional torus $\mathbb{T}^d$. Like in the earlier work [8], the individuals move from one patch to its neighbors according to a random walk. The first novelty of this paper is that the infectivity of each individual is a random function, which evolves with the time elapsed since infection, as first considered in [6], and recently studied in [3] and [4]. The second novelty is that we allow infection of a susceptible individual by infectious individuals located in distinct patches, and we use a very general rate of infections.

There are two parameters in our model, N which is the order of the number of individuals in each patch, and ε, which is the distance between two neighboring sites. The total number of patches is ε^{-d}, and the total number of individuals in the model is $N\varepsilon^{-d}$. Our goal is to study the limit of the renormalized stochastic finite population model as both $N \to \infty$ and $\varepsilon \to 0$. In this paper we obtain a convergence result in L^∞ under the restriction that $N\varepsilon^d \to \infty$. In [8], the restriction was much weaker, thanks to clever martingale estimates due to Blount [2]. However, in contradiction with the model in [8], our model is non Markovian, and several of the fluctuating processes are not martingales. As a result, it does not seem possible to extend the techniques of [2] to the situation studied in the present paper.

There are three models in the present paper. The stochastic SDE model parametrized by the pair (N,ε), the deterministic model which is an ODE parametrized by ε on the patches (and is the LLN limit of the first model when $N \to \infty$ with ε fixed), and the PDE model on the torus $\mathbb{T}^d$, which is the limit of the ODE model as $\varepsilon \to 0$. The convergence of the ODE model to the PDE model exploits standard arguments on semigroup and their approximation, based on some result in [5]. The main new argument in the present paper consists in showing that the difference in L^∞ between the stochastic and the ODE models, which tends to zero as $N \to \infty$ while ε is fixed according to [4], tends also to zero when $(N,\varepsilon) \to (+\infty, 0)$, provided $N\varepsilon^d \to \infty$.

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In this paper, we consider the SI model, S as susceptible, I as infected. An infected individual has an age of infection dependent infectivity, which we suppose to vanish after some random time. It would be natural to decide that at that time the individual leaves the I compartment, and enters the R compartment, R as recovered. For the sake of simplifying our model, we decide that after being infected, an individual remains in the I compartment for ever. This does not affect the evolution of the epidemic, since when its infectivity remains zero, an individual does not contribute anymore to the propagation of the illness, exactly as an individual in the R compartment of an SIR model. However, there are two drawbacks of the present model. First, we do not follow the evolution of the number of infectious individuals, since we have so to speak merged the I and the R compartments. Second, while we distinguish the rate of movements of the S type and the I type individuals we do not distinguish that rate between the infectious and the recovered individuals. The reason for studying the SI model separately is that, in our "Varying Infectivity" model the techniques for proving the convergence as $\varepsilon \to 0$ of the ODE model to the PDE model which we are using in the SI case will not be available in the SIR case. One is forced to use different techniques. We will study the extension of the present results to the SIR model in a future publication. But our conviction is that it is worth to present the results in the SI case, due to the possibility in this case of using classical semigroup techniques.

Let us finally comment on the assumptions on the age of infection dependent infectivity. We assume that to each individual who gets infected is attached a random infectivity function, the functions attached to the various individuals being i.i.d., all having the law of a random function λ (the law is different for the initially infected individuals). In this paper, as in [4], we only assume that λ belongs a.s. to the Skorohod space of càlàg function $\mathbf{D}$, and satisfies $0 \le \lambda(t) \le \lambda^*$, for some $\lambda^* > 0$. This is weaker than the assumptions made in [3]. The proof in [4] is quite different from the proof in [3]. Here we use a proof similar to that in [3]. The limitation is that we obtain only the pointwise convergence of the renormalised total infectivity function, while we obtain uniform in t convergence of the proportions of susceptible and infected individuals. We believe that this proof is interesting, due to its simplicity.

Note that there is some literature on similar models, but mainly without movements of the various individuals, see in particular [1] for a SIS Markov model, and [9] for a SIR varying infectivity model. Our previous publication [8] treats a Markov SIR model with movements and only local infections.

The paper is organized as follows. We describe our model in detail in section 2, in particular the complex form of the rate of infection. In section 3, we state the law of large numbers limit as $N \to \infty$, with ε fixed, referring to [4] for the proof. In section 4, we take the limit as $\varepsilon \to 0$ in the ODE model. Finally, in section 5, we study the difference between the stochastic and the ODE model, as $(N, \varepsilon) \to (+\infty, 0)$, and conclude our main result.

2. Model description

We consider a total population size $N\varepsilon^{-d}$ initially distributed on the ε^{-d} nodes of a refining spatial grid $\mathrm{D}_\varepsilon := [0,1)^d \cap \varepsilon\mathbb{Z}^d$, in which an infection is introduced. Here ε is the mesh size of the grid (we assume that $\varepsilon^{-1} \in \mathbb{N}\backslash\{0\}$). We focus our attention to the periodic boundary conditions on the hypercube $[0,1]^d$, that is, our domain is the torus $\mathbb{T}^d := [0,1]^d$. Our results can be extended to a bounded domain of $\mathbb{R}^d$ with smooth boundary, and Neumann boundary conditions.

2.1. **Set-up and notations.**

We split the population in two subsets $S^{N,\varepsilon}$ and $I^{N,\varepsilon}$. $S^{N,\varepsilon}$ stands for the susceptible individuals, who do not have the disease and who can get infected, while $I^{N,\varepsilon}$ is referred to the subset of those individuals who are suffering from the illness or have recovered from the disease.

We shall denote by x_ε the nodes of the grid D_ε. $S^{N,\varepsilon}(t, x_\varepsilon)$ denotes the number of susceptible individuals at site x_ε at time t. Let $B^{N,\varepsilon}(t,x_\varepsilon)$ be the total number of individuals at site x_ε at time t, i.e. $B^{N,\varepsilon}(t,x_\varepsilon) := S^{N,\varepsilon}(t,x_\varepsilon) + I^{N,\varepsilon}(t,x_\varepsilon)$. We define $S^{N,\varepsilon}(t)$ (resp. $I^{N,\varepsilon}(t)$) as the total number of susceptible individuals (resp. infected individuals) at time t in the whole population, that is:

$$S^{N,\varepsilon}(t) := \sum_{x_\varepsilon} S^{N,\varepsilon}(t,x_\varepsilon), \quad \text{and} \quad I^{N,\varepsilon}(t) := \sum_{x_\varepsilon} I^{N,\varepsilon}(t,x_\varepsilon)\,, \quad \forall t \geq 0.$$

We have $B^{N,\varepsilon}(t) := \sum_{x_\varepsilon} B^{N,\varepsilon}(t) = N\varepsilon^{-d}\,, \quad \forall t \geq 0.$

To each individual j is attached a random infection-age dependent infectivity process $\{\lambda_{-j}(t)\,:t \geq 0\}$ or $\{\lambda_j(t)\,:t\geq 0\}$. $\lambda_{-j}(t)$ is the infectivity at time t of the j-th initially infected individual. The initially susceptible individual j who is infected at a random time $\tau_j^{N,\varepsilon}$, has at time t the infectivity $\lambda_j(t-\tau_j^{N,\varepsilon})$, i.e. $\lambda_j(t)$ is the infectivity at time t after its time of infection of the j-th initially susceptible individual. We assume that $\lambda_j = 0$ on $\mathbb{R}_-$ and that $\{\lambda_{-j}\,:j\geq 1\}$ and $\{\lambda_j\,:j\geq 1\}$ are two mutually independent sequences of i.i.d $\mathbb{R}_+$-valued random functions.

We define the infected periods of newly and initially infected individual $j>0$ and $j<0$ respectively, by the random variables

$$\eta_j := \sup\{t>0:\ \lambda_j(t)>0\},\ j\in\mathbb{Z}\backslash\{0\}.$$

We define $F(t) := \mathbb{P}\left(\eta_1 \leq t\right),\,,\, F_0(t) := \mathbb{P}\left(\eta_{-1}\leq t\right)$, the distributions functions of λ_j for $j\geq 1$ and for $j\leq -1$ respectivelly. Let $F^c(t) := 1-F(t)$ and $F_0^c(t) := 1-F_0(t)$. We moreover define

$$\overline{\lambda}(t) := \mathbb{E}\left[\lambda_1(t)\right] \ \text{ and } \overline{\lambda}_0(t) := \mathbb{E}\left[\lambda_{-1}(t)\right].$$

Note that, under the i.i.d. assumption of the random variables $\{\lambda_j(.)\}_{j\geq 1}$, the sequence of random variables $\{\eta_j\}_{j\geq 1}$ is i.i.d. Also, the sequence of random variables $\{\eta_j\}_{j\leq -1}$ is i.i.d.

We assume that susceptible individuals move from patch to patch according to a time-homogenous Markov process $X(t)$ with jump rates ν_S/ε^2 and transition function

$$p_\varepsilon^{x_\varepsilon,y_\varepsilon}(s,t) = \mathbb{P}\left(X(t)=y_\varepsilon | X(s)=x_\varepsilon\right),$$

and while infectious individuals move from patch to patch according to a time-homogeneous Markov process $Y(t)$ with jump rates ν_I/ε^2 and transition function

$$q_\varepsilon^{x_\varepsilon,y_\varepsilon}(s,t) = \mathbb{P}\left(Y(t)=y_\varepsilon | Y(s)=x_\varepsilon\right).$$

ν_S and ν_I are positive diffusion coefficients for the susceptible and infected subpopulations, respectively. We assume that those movements of the various individuals are mutually independent.
In addition, we use $X_j^{s,x_\varepsilon}(t)$ (resp. $Y_j^{s,x_\varepsilon}(t)$) to denote the position at time t of the individual j if it is susceptible (resp. infected) during the time interval (s,t), and was in location/node x_ε at time s.
For all $x_\varepsilon \in \mathrm{D}_\varepsilon$, let $V_\varepsilon(x_\varepsilon)$ be the cube centered at the site x_ε with volume ε^d. Let $\mathrm{H}^\varepsilon \subset L^2(\mathbb{T}^d)$ denote the space of real valued step functions that are constant on each cell $V_\varepsilon(x_\varepsilon)$.
Δ_ε is the discrete Laplace operator defined as follows

$$\Delta_\varepsilon f(x_\varepsilon) = \sum_{i=1}^d \varepsilon^{-d}\big[f(x_\varepsilon+\varepsilon e_i) - 2f(x_\varepsilon) + f(x_\varepsilon - \varepsilon e_i)\big], \quad f\in\mathrm{H}^\varepsilon$$

and we define the operators $\Delta_\varepsilon^S f := \nu_S\Delta_\varepsilon f$ and $\Delta_\varepsilon^I f := \nu_I \Delta_\varepsilon f$, $f\in\mathrm{H}^\varepsilon$.
Δ denotes the d-dimensional Laplace operator. Let $T_{S,\varepsilon}$ (resp. $T_{I,\varepsilon}$) be the semigroup acting on H^ε generated by $\nu_S\Delta_\varepsilon$ (resp. $\nu_I\Delta_\varepsilon$). Similary, we denote by T_S (resp. T_I) the semigroup acting on $L^2(\mathbb{T}^d)$ generated by $\nu_S\Delta$ (resp. $\nu_I\Delta$).

2.2. **Model formulation.**
All random variables and processes are defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider a SI epidemic model where each infectious individual has an infectivity that is randomly varying with the time elapsed since infection. We assume that a susceptible individual in patch x_ε has contacts with infectious individuals of patch y_ε at rate $\beta_\varepsilon^{x_\varepsilon,y_\varepsilon}(t)$ at time t.
Given a site x_ε, the total force of infection at each time t at site x_ε is the aggregate infectivity of all the individuals that are currently infectious in this site:

$$\mathfrak{F}^{N,\varepsilon}(t,x_\varepsilon) = \sum_{j=1}^{I^{N,\varepsilon}(0)} \lambda_{-j}(t)\mathbb{1}_{Y_j(t)=x_\varepsilon} + \sum_{y_\varepsilon}\int_0^t\int_0^\infty\int_{\mathbf{D}}\int_{\mathbf{D}} \lambda(t-s)\mathbb{1}_{u\le S^{N,\varepsilon}(s^-,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s^-,y_\varepsilon)}\mathbb{1}_{Y^{s,y_\varepsilon}(t)=x_\varepsilon} Q^{y_\varepsilon}(ds,du,d\lambda,dY),$$

where

$$\overline{\Gamma}^{N,\varepsilon}(t,y_\varepsilon) := \frac{1}{N^{1-\gamma}[B^{N,\varepsilon}(t,y_\varepsilon)]^\gamma}\sum_{x_\varepsilon}\beta_\varepsilon^{y_\varepsilon,x_\varepsilon}(t)\mathfrak{F}^{N,\varepsilon}(t,x_\varepsilon)$$

is the force of infection exerted on each susceptible individual in patch x_ε, and $\{Q^{y_\varepsilon}, y_\varepsilon \in D_\varepsilon\}$ are i.i.d. standard Poisson random measures (PRM) on $\mathbb{R}_+^2 \times \mathbf{D}^2$ with intensity $ds\otimes du\otimes d\mathbb{P}_\lambda \otimes d\mathbb{P}_Y$. $\mathbf{D}$ denotes the space of càdlàg paths from $\mathbb{R}_+$ into $\mathbb{R}_+$, which we equip with the Skorohod topology. We assume that $\gamma\in[0,1]$. By an abuse of notation, we denote by $Q^{x_\varepsilon}(ds,du)$ the projection of $Q^{x_\varepsilon}(ds,du,d\lambda,dY)$ on the first two coordinates. Let, with $\Upsilon^{N,\varepsilon}(t,x_\varepsilon) := S^{N,\varepsilon}(t,x_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(t,x_\varepsilon)$,

$$A^{N,\varepsilon}(t,x_\varepsilon) := \int_0^t\int_0^\infty \mathbb{1}_{u\le\Upsilon^{N,\varepsilon}(s^-,x_\varepsilon)}Q^{x_\varepsilon}(ds,du).$$

In what follows, $x_\varepsilon \sim y_\varepsilon$ means that the nodes x_ε and y_ε are neighbors (each point of D_ε has $2d$ neighbors).

The epidemic dynamic of the model can be described by the following equations

$$\begin{aligned} S^{N,\varepsilon}(t,x_\varepsilon) &= S^{N,\varepsilon}(0,x_\varepsilon) - A^{N,\varepsilon}(t,x_\varepsilon) - \sum_{y_\varepsilon\sim x_\varepsilon} P_S^{x_\varepsilon,y_\varepsilon}\left(\int_0^t \frac{\nu_S}{\varepsilon^2}S^{N,\varepsilon}(s,x_\varepsilon)ds\right) + \sum_{y_\varepsilon\sim x_\varepsilon} P_S^{y_\varepsilon,x_\varepsilon}\left(\int_0^t \frac{\nu_S}{\varepsilon^2}S^{N,\varepsilon}(s,y_\varepsilon)ds\right)\\ I^{N,\varepsilon}(t,x_\varepsilon) &= I^{N,\varepsilon}(0,x_\varepsilon) + A^{N,\varepsilon}(t,x_\varepsilon) - \sum_{y_\varepsilon\sim x_\varepsilon} P_I^{x_\varepsilon,y_\varepsilon}\left(\int_0^t \frac{\nu_I}{\varepsilon^2}I^{N,\varepsilon}(s,x_\varepsilon)ds\right) + \sum_{y_\varepsilon\sim x_\varepsilon} P_I^{y_\varepsilon,x_\varepsilon}\left(\int_0^t \frac{\nu_I}{\varepsilon^2}I^{N,\varepsilon}(s,y_\varepsilon)ds\right), \end{aligned} \tag{2.1}$$

where $P_S^{x_\varepsilon,y_\varepsilon}$, $P_I^{x_\varepsilon,y_\varepsilon}$, $x_\varepsilon, y_\varepsilon \in \mathrm{D}_\varepsilon$ are mutually independent standard Poisson processes.
In the above equations $P_S^{x_\varepsilon,y_\varepsilon}$ (resp. $P_I^{x_\varepsilon,y_\varepsilon}$) is the counting process of susceptible (resp. infected) individuals that migrate from the patch x_ε to y_ε.

In the sequel of this paper we may use the same notation for different constants (we use the generic notations c, C for positive constants). These constants can depend upon some parameters of the model, as long as these are independent of ε and N, and we will not necessarily mention this dependence explicitly. The exact value may change from line to line.

3. Law of large numbers as $N\to\infty$, ε being fixed

We consider the renormalized model by dividing the number of individuals in each compartment and at each patch by N. Hence, we define

$$\overline{S}^{N,\varepsilon}(t,x_\varepsilon) := \frac{1}{N}S^{N,\varepsilon}(t,x_\varepsilon),\quad \overline{I}^{N,\varepsilon}(t,x_\varepsilon) := \frac{1}{N}I^{N,\varepsilon}(t,x_\varepsilon), \text{ and } \overline{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon) := \frac{1}{N}\mathfrak{F}^{N,\varepsilon}(t,x_\varepsilon).$$

Assumption 3.1 *We make the following assumptions on the initial conditions. We assume that:*

(i) *there exists a collection of positive numbers* $\{\,\overline{S}^{\varepsilon}(0,x_\varepsilon),\ \overline{I}^{\varepsilon}(0,x_\varepsilon),\ x_\varepsilon\in \mathrm{D}_\varepsilon\,\}$ *such that*

$$\sum_{x_\varepsilon}\Big[\overline{S}^{\varepsilon}(0,x_\varepsilon)+\overline{I}^{\varepsilon}(0,x_\varepsilon)\Big]=\varepsilon^{-d}\,,$$

and $\Big|S^{N,\varepsilon}(0)-N\overline{S}^{\varepsilon}(0)\Big|\le 1\,,\qquad \Big|I^{N,\varepsilon}(0)-N\overline{I}^{\varepsilon}(0)\Big|\le 1$*;*

(ii) *there exists two continuous functions* $\overline{\mathbf{S}},\,\overline{\mathbf{I}}:\mathbb{T}^d\longrightarrow\mathbb{R}_+$ *such that* $c\le\overline{\mathbf{S}}(x)\le C,\quad \overline{\mathbf{I}}(x)\le C$ *for all* $x\in\mathbb{T}^d$, $\int_{\mathbb{T}^d}\Big[\overline{\mathbf{S}}(x)+\overline{\mathbf{I}}(x)\Big]dx=1$ *and*

$$\overline{S}^{\,\varepsilon}(0,x_\varepsilon)=\varepsilon^{-d}\int_{V_\varepsilon(x_\varepsilon)}\overline{\mathbf{S}}(x)dx,\quad \overline{I}^{\,\varepsilon}(0,x_\varepsilon)=\varepsilon^{-d}\int_{V_\varepsilon(x_\varepsilon)}\overline{\mathbf{I}}(x)dx\,.$$

(iii) $\{X_j(0)\,,1\le j\le S^{N,\varepsilon}(0)\}$ *and* $\{Y_j(0)\,,1\le j\le I^{N,\varepsilon}(0)\}$ *are two mutually independent collections of i.i.d. random variables satisfying* $\mathbb{P}\left(X_j(0)=x_\varepsilon\right)=\dfrac{\overline{S}^{\,\varepsilon}(0,x_\varepsilon)}{\overline{S}^{\,\varepsilon}(0)}$, *and* $\mathbb{P}\left(Y_j(0)=x_\varepsilon\right)=\dfrac{\overline{I}^{\,\varepsilon}(0,x_\varepsilon)}{\overline{I}^{\,\varepsilon}(0)}$ *for all* $x_\varepsilon\in\mathrm{D}_\varepsilon$, *where* $\overline{S}^{\,\varepsilon}(0):=\sum_{x_\varepsilon}\overline{S}^{\,\varepsilon}(0,x_\varepsilon)$ *and* $\overline{I}^{\,\varepsilon}(0):=\sum_{x_\varepsilon}\overline{I}^{\,\varepsilon}(0,x_\varepsilon)$. *Moreover* $S^{N,\varepsilon}(0,x_\varepsilon)=\sum_{j=1}^{S^{N,\varepsilon}(0)}\mathbb{1}_{X_j(0)=x_\varepsilon}$ *and* $I^{N,\varepsilon}(0,x_\varepsilon)=\sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{1}_{Y_j(0)=x_\varepsilon}$.

Assumption 3.2 **(i)** *We assume that* $\beta_\varepsilon^{x_\varepsilon,y_\varepsilon}(t)=\beta_t(x_\varepsilon,V_\varepsilon(x_\varepsilon))$, *where* $\beta_t(x,A)$ *is a transition kernel and there exists a constant* β^* *such that* $\beta_t(x,\mathbb{T}^d)\le\beta^*$, *for all* $x\in\mathbb{T}^d$ *and* $t\ge 0$.

(ii) *there exists a positive constant* $\lambda^*>0$ *such that* $0\le\lambda_j(t)\le\lambda^*$, *for all* $j\in\mathbb{Z}\backslash\{0\}$ *and* $t\ge 0$.

Under 3.1 and 3.2, we have the

Theorem 3.1 (**Law of Large Numbers: $\mathbf{N}\to\infty$, ε being fixed**)
As $N\to\infty$, $\Big(\overline{S}^{N,\varepsilon}(t,x_\varepsilon),\,\overline{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon),\,\overline{I}^{N,\varepsilon}(t,x_\varepsilon),\ \ t\ge 0,\ x_\varepsilon\in\mathrm{D}_\varepsilon\Big)$ *converges in* $\mathbf{D}^{3\varepsilon^{-d}}$ *in probability, to the unique solution* $\Big(\overline{S}^{\varepsilon}(t,x_\varepsilon),\,\overline{\mathfrak{F}}^{\varepsilon}(t,x_\varepsilon),\,\overline{I}^{\varepsilon}(t,x_\varepsilon),\ \ t\ge 0,\ x_\varepsilon\in\mathrm{D}_\varepsilon\Big)$ *of the following system of integral equations*

$$\begin{cases}\overline{S}^{\,\varepsilon}(t,x_\varepsilon)=\overline{S}^{\,\varepsilon}(0,x_\varepsilon)-\displaystyle\int_0^t\overline{S}^{\,\varepsilon}(s,x_\varepsilon)\overline{\Gamma}^{\,\varepsilon}(s,x_\varepsilon)ds+\int_0^t\big[\Delta_\varepsilon^S\overline{S}^{\,\varepsilon}\big](s,x_\varepsilon)ds\\ \overline{\mathfrak{F}}^{\,\varepsilon}(t,x_\varepsilon)=\overline{\lambda}_0(t)\displaystyle\sum_{y_\varepsilon}\overline{I}^{\,\varepsilon}(0,y_\varepsilon)q^{y_\varepsilon,x_\varepsilon}(0,t)+\sum_{y_\varepsilon}\int_0^t\overline{\lambda}(t-s)\overline{S}^{\,\varepsilon}(s,y_\varepsilon)\overline{\Gamma}^{\,\varepsilon}(s,y_\varepsilon)q^{y_\varepsilon,x_\varepsilon}(s,t)ds\\ \overline{I}^{\,\varepsilon}(t,x_\varepsilon)=\overline{I}^{\,\varepsilon}(0,x_\varepsilon)+\displaystyle\int_0^t\overline{S}^{\,\varepsilon}(s,x_\varepsilon)\overline{\Gamma}^{\varepsilon}(s,x_\varepsilon)ds+\int_0^t\big[\Delta_\varepsilon^I\overline{I}^{\,\varepsilon}\big](s,x_\varepsilon)ds,\\ \quad t\ge 0,\ \ x_\varepsilon\in\mathrm{D}_\varepsilon,\end{cases}\tag{3.1}$$

where

$$\overline{\Gamma}^{\,\varepsilon}(t,x_\varepsilon)=\frac{1}{\big[\,\overline{B}^{\,\varepsilon}(t,x_\varepsilon)\big]^\gamma}\sum_{y_\varepsilon}\beta_\varepsilon^{x_\varepsilon,y_\varepsilon}(t)\overline{\mathfrak{F}}^{\,\varepsilon}(t,y_\varepsilon)\ \ \text{and}\ \ \overline{B}^{\,\varepsilon}(t,x_\varepsilon)=\overline{S}^{\,\varepsilon}(t,x_\varepsilon)+\overline{I}^{\,\varepsilon}(t,x_\varepsilon).$$

This Theorem is a special case of Theorem 3.1 in [4], whose proof written for a multi-patch multi-group SIR model is easily adapted to our case.

4. Limit as $\varepsilon \to 0$ in the deterministic model

Before letting ε go to zero in the limit system (3.1) extended on the whole space $\mathbb{T}^d$, we prove some technical lemmas.

Lemma 4.1 *Let $T > 0$. There exists a positive constant C such that $\left\|\overline{S}^{\varepsilon}(t)\right\|_\infty \leq C$ and $\left\|\overline{I}^{\varepsilon}(t)\right\|_\infty \leq C$, for all $\varepsilon > 0$ and $t \in [0\,,\,T]$.*

Proof. Using the Duhamel formula, we have $\|\overline{S}^{\varepsilon}(t)\|_\infty \leq \sup_{x_\varepsilon} \overline{S}^{\varepsilon}(0, x_\varepsilon) \leq C$.

We now consider the term $\overline{I}^{\varepsilon}$. First using the previous estimate, we obtain

$$\frac{\overline{S}^{\varepsilon}(s, x_\varepsilon)}{\left(\overline{B}^{\varepsilon}(s, x_\varepsilon)\right)^\gamma} = \left(\frac{\overline{S}^{\varepsilon}(s, x_\varepsilon)}{\overline{B}^{\varepsilon}(s, x_\varepsilon)}\right)^\gamma \left[\overline{S}^{\varepsilon}(s, x_\varepsilon)\right]^{1-\gamma} \leq C(T, \gamma).$$

Next we have $\sum_{y_\varepsilon} \beta_\varepsilon^{x_\varepsilon, y_\varepsilon} \overline{\mathfrak{F}}^{\varepsilon}(s, y_\varepsilon) \leq \lambda^* \left\| \overline{I}^{\varepsilon}(s) \right\|_\infty \sum_{y_\varepsilon} \beta_\varepsilon^{x_\varepsilon, y_\varepsilon}(s) \leq \lambda^* \beta^* \left\|\overline{I}^{\varepsilon}(s)\right\|_\infty$. Thus

$$\begin{aligned}
\left\| \overline{I}^{\varepsilon}(t) \right\|_\infty &\leq \left\| \left(T_{I,\varepsilon}(t)\overline{I}^{\varepsilon}(0)\right) \right\|_\infty + \int_0^t T_{I,\varepsilon}(t-s) C \left\| \overline{I}^{\varepsilon}(s) \right\|_\infty ds \\
&\leq C + C \int_0^t \left\| \overline{I}^{\varepsilon}(s) \right\|_\infty ds.
\end{aligned}$$

The second statement then follows from Gronwall's Lemma. □

Lemma 4.2 *For any $T > 0$, there exists ε_0 and $c > 0$ such that $\overline{B}^{\varepsilon}(t, x_\varepsilon) \geq c$, for all $0 < \varepsilon \leq \varepsilon_0$, $x_\varepsilon \in \mathrm{D}_\varepsilon$ and $0 \leq t \leq T$.*

Proof. Let c and C be two positive constants such that $0 < c \leq \dfrac{\inf_{x_\varepsilon} \overline{S}^{\varepsilon}(0, x_\varepsilon)}{2} \leq \dfrac{C}{2}$, and let $T_c^\varepsilon := \inf\{t > 0\,,\, \inf_{x_\varepsilon} \overline{S}^{\varepsilon}(t, x_\varepsilon) < c\}$. On the interval $[0\,,\,T_c^\varepsilon]$, $\overline{S}^{\varepsilon}(t, x_\varepsilon) \geq c$, $\forall x_\varepsilon \in \mathrm{D}_\varepsilon$. For $t \leq T_c^\varepsilon$, we have

$$\overline{\Gamma}^{\varepsilon}(t, x_\varepsilon) = \frac{1}{\left[\overline{B}^{\varepsilon}(t, x_\varepsilon)\right]^\gamma} \sum_{y_\varepsilon} \beta_\varepsilon^{x_\varepsilon, y_\varepsilon}(t) \overline{\mathfrak{F}}^{\varepsilon}(t, y_\varepsilon) \leq \frac{\lambda^* \beta^*}{c^\gamma} \left\|\overline{I}^{\varepsilon}(t)\right\|_\infty := \overline{c},$$

$$\overline{S}^{\varepsilon}(t, x_\varepsilon) \leq \overline{S}^{\varepsilon}(0, x_\varepsilon) - \overline{c} \int_0^t \overline{S}^{\varepsilon}(s, x_\varepsilon) + \int_0^t \left[\Delta_\varepsilon^S \overline{S}^{\varepsilon}\right](s, x_\varepsilon) ds.$$

Hence $e^{\overline{c}t}\overline{S}^{\varepsilon}(t, x_\varepsilon) \geq \inf_{y_\varepsilon} \overline{S}^{\varepsilon}(0, y_\varepsilon) = 2c$, and consequently $T_c^\varepsilon \geq \log 2/\overline{c}$. Then for all $0 \leq t \leq T_c^\varepsilon$, we have $e^{\overline{c}t}\overline{S}^{\varepsilon}(t, x_\varepsilon) \geq 2c$. So $\overline{S}^{\varepsilon}(t, x_\varepsilon) \geq 2e^{-\overline{c}t}c \geq c$ iff $e^{-\overline{c}t} \geq \frac{1}{2}$

From 3.1 **(ii)** and the fact that $\bar{\mathbf{I}}(0) \neq 0$, there exists a ball $B(x_0, \rho)$ and $a > 0$ such that $\bar{\mathbf{I}}(y) \geq a$, for all $y \in B(x_0, \rho)$. Let us consider the following ODE

$$\frac{d\, u_\varepsilon}{dt} = \nu_I \Delta_\varepsilon u_\varepsilon, \quad u_\varepsilon(0) = a \mathbb{1}_{B(x_0, \rho)}.$$

We have that $u_\varepsilon \longrightarrow u$ in $L^\infty\left([0, T] \times \mathbb{T}^d\right)$ as $\varepsilon \to 0$, where u is the solution of

$$\frac{d\, u}{dt} = \nu_I \Delta u, \quad u(0) = a \mathbb{1}_{B(x_0, \rho)}.$$

For all $\dfrac{\log 2}{\overline{c}} < t \leq T$, there exists a positive constant $\underline{c}$, such that $u(t, x) \geq 2\underline{c}$, $\forall x \in \mathbb{T}^d$. Then, there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \leq \varepsilon_0$, $\overline{I}^{\varepsilon}(t, x_\varepsilon) \geq u_\varepsilon(t, x_\varepsilon) \geq \underline{c}$, for all $\dfrac{\log 2}{\overline{c}} < t \leq T$.

We have shown that $\overline{B}^{\varepsilon}(t, x_\varepsilon) \geq c \wedge \underline{c}$, for all $0 \leq t \leq T$, $x \in \mathrm{D}_\varepsilon$, $\varepsilon_0 \leq \varepsilon$.

□

We now extend the solution of the system (3.1) to the whole space $\mathbb{T}^d$. So, we define
$\overline{\mathbf{S}}^{\varepsilon}(t,x) := \sum_{x_\varepsilon} \overline{S}^{\varepsilon}(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x)$, $\overline{\mathbf{I}}^{\varepsilon}(t,x) := \sum_{x_\varepsilon} \overline{I}^{\varepsilon}(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x)$, $\overline{\mathbf{F}}^{\varepsilon}(t,x) := \sum_{x_\varepsilon} \overline{\mathfrak{F}}^{\varepsilon}(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x)$,

$\overline{\mathbf{X}}^{\varepsilon} := (\overline{\mathbf{S}}^{\varepsilon}, \overline{\mathbf{F}}^{\varepsilon}, \overline{\mathbf{I}}^{\varepsilon})$.

Theorem 4.1 *For all $T \geq 0$, $\sup_{0\leq t\leq T} \left\|\overline{\mathbf{X}}^{\varepsilon}(t) - \overline{\mathbf{X}}(t)\right\|_\infty \longrightarrow 0$ as $\varepsilon \to 0$, where $\overline{\mathbf{X}} := (\overline{\mathbf{S}}, \overline{\mathbf{F}}, \overline{\mathbf{I}})$ is the unique solution of the following system of parabolic PDE/integral equations.*

$$\begin{cases} \overline{\mathbf{S}}(t,x) = \overline{\mathbf{S}}(0,x) - \displaystyle\int_0^t \overline{\mathbf{S}}(s,x)\overline{\Gamma}(s,x)ds + \int_0^t \left[\Delta^S \overline{\mathbf{S}}\right](s,x)ds, \\ \overline{\mathbf{F}}(t,x) = \overline{\lambda}_0(t)\left(T_I(t)\overline{\mathbf{I}}(0)\right)(x) + \displaystyle\int_0^t \overline{\lambda}(t-s)T_I(t-s)\left(\overline{\mathbf{S}}(s)\overline{\Gamma}(s)\right)(x)ds, \\ \overline{\mathbf{I}}(t,x) = \overline{\mathbf{I}}(0,x) + \displaystyle\int_0^t \overline{\mathbf{S}}(s,x)\overline{\Gamma}(s,x)ds + \int_0^t \left[\Delta^I \overline{\mathbf{I}}\right](s,x)ds, \\ \quad \text{with } \overline{\mathbf{S}}(t,x)\overline{\Gamma}(t,x) = \dfrac{\overline{\mathbf{S}}(t,x)}{\left[\overline{\mathbf{B}}(t,x)\right]^\gamma}\displaystyle\int_{\mathbb{T}^d} \overline{\mathbf{F}}(t,y)\beta(x,dy), \ t\geq 0, \ \ x\in\mathbb{T}^d. \end{cases} \tag{4.1}$$

where T_I denotes the semigroup generated by $\nu_I\Delta$.

Before proving this theorem, we first establish two Propositions.

Proposition 4.1 *Let $T > 0$. If $(\overline{\mathbf{S}}, \overline{\mathbf{F}}, \overline{\mathbf{I}})$ is a solution of (4.1), then for all $0 \leq t \leq T$, there exists C, $c > 0$ such that $\left\|\overline{\mathbf{S}}(t)\right\|_\infty \leq C$, $\left\|\overline{\mathbf{I}}(t)\right\|_\infty \leq C$ and $\overline{\mathbf{B}}(t,x) \geq c$, for all $x \in \mathbb{T}^d$.*

Proof. The arguments used in the proof of 4.1 and 4.2 are easy to transpose to the present situation. □

Remark 4.1 *Let $\mathscr{H}\left(\overline{\mathbf{S}}, \overline{\mathbf{I}}, \overline{\mathbf{F}}\right)(t,x) := \dfrac{\left[\overline{\mathbf{S}}(t,x)\vee 0\right]\wedge C}{\left[\overline{\mathbf{B}}(t,x)\vee c\right]^\gamma}\displaystyle\int_{\mathbb{T}^d}\beta_t(x,dy)\left[\overline{\mathbf{F}}(t,y)\wedge\lambda^* C\right]$ where C is the upper bound in Lemma 4.1, and c the lower bound in Lemma 4.2. Note $\left(\overline{\mathbf{S}}, \overline{\mathbf{I}}, \overline{\mathbf{F}}\right)$ is a solution of (4.1) iff it is a solution of the following system*

$$\begin{cases} \overline{\mathbf{S}}(t,x) = \Big(T_S(t)\overline{\mathbf{S}}(0)\Big)(x) - \displaystyle\int_0^t \Big(T_S(t-s)\mathscr{H}\left(\overline{\mathbf{S}}(s), \overline{\mathbf{I}}(s), \overline{\mathbf{F}}(s)\right)\Big)(x)ds, \\ \overline{\mathbf{F}}(t,x) = \overline{\lambda}_0(t)\Big(T_I(t)\overline{\mathbf{I}}(0)\Big)(x) + \displaystyle\int_0^t \overline{\lambda}(t-s)\Big(T_I(t-s)\mathscr{H}\left(\overline{\mathbf{S}}(s), \overline{\mathbf{I}}(s), \overline{\mathbf{F}}(s)\right)\Big)(x)ds, \\ \overline{\mathbf{I}}(t,x) = \Big(T_I(t)\overline{\mathbf{I}}(0)\Big)(x) + \displaystyle\int_0^t \Big(T_I(t-s)\mathscr{H}\left(\overline{\mathbf{S}}(s), \overline{\mathbf{I}}(s), \overline{\mathbf{F}}(s)\right)\Big)(x)ds, \ 0\leq t\leq T, \ \ x\in\mathbb{T}^d. \end{cases} \tag{4.2}$$

Note also that the map $\mathscr{H} : \left(L^\infty(\mathbb{T}^d)\right)^3 \longrightarrow L^\infty(\mathbb{T}^d)$ is bounded and globally Lispchitz.

Proposition 4.2 *The system of equations (4.2) has a unique solution.*

Proof. The uniqueness of the solution uses the contraction character of the semigroups T_S and T_I on $L^\infty(\mathbb{T}^d)$, and the fact that the map $\mathscr{H}$ is bounded and globally Lispchitz. The existence of the solution can be proved using the Picard iteration procedure.

□

We introduce the canonical projection $\mathrm{P}_\varepsilon : L^2(\mathbb{T}^d) \longrightarrow \mathrm{H}^\varepsilon$ given by

$$\varphi \longmapsto \mathrm{P}_\varepsilon\varphi(x) = \varepsilon^{-d}\int_{V_\varepsilon(x_\varepsilon)}\varphi(y)dy \quad \text{ if } x\in V_\varepsilon(x_\varepsilon).$$

Proof of Theorem 4.1.
Using the fact that the map $\mathscr{H}$ is bounded and globally Lispchitz, we have, provided that $\varepsilon \le \varepsilon_0$,

$$\left\|\overline{\mathbf{X}}^{\varepsilon}(t)-\overline{\mathbf{X}}(t)\right\|_{\infty} \le C(\lambda^*,\beta^*)\int_0^t \left\|\overline{\mathbf{X}}^{\varepsilon}(s)-\overline{\mathbf{X}}(s)\right\|_{\infty} ds+\pi_\varepsilon(t),$$

where $\pi_\varepsilon(t)=\pi_\varepsilon^S(t)+\pi_\varepsilon^I(t)+\pi_\varepsilon^{\mathfrak{F}}(t)$, with

$$\begin{aligned}\pi_\varepsilon^S(t) &= \left\| T_{S,\varepsilon}(t)\overline{\mathbf{S}}^{\varepsilon}(0)-T_S(t)\overline{\mathbf{S}}(0)\right\|_{\infty}\\ &+\int_0^t \left\| \mathrm{P}_\varepsilon\left(\frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma}\int_{\mathbb{T}^d}\overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right)-\frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma}\int_{\mathbb{T}^d}\overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right\|_{\infty} ds\\ &+\int_0^t \left\| T_{S,\varepsilon}(t-s)\mathrm{P}_\varepsilon\left(\frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma}\int_{\mathbb{T}^d}\overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right)-T_S(t-s)\left(\frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right)^\gamma}\int_{\mathbb{T}^d}\overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right)\right\|_{\infty} ds,\end{aligned}$$

$\pi_\varepsilon^I(t)$ is a quantity similar to $\pi_\varepsilon^S(t)$, with $T_{I,\varepsilon}$ (resp. T_I, $\overline{\mathbf{I}}^{\varepsilon}$ and $\overline{\mathbf{I}}$) in place of $T_{S,\varepsilon}$ (resp. T_S, $\overline{\mathbf{S}}^{\varepsilon}$ and $\overline{\mathbf{S}}$), and

$$\begin{aligned}\pi_\varepsilon^{\mathfrak{F}}(t) &= \lambda^*\left\| T_{I,\varepsilon}(t)\overline{\mathbf{I}}^{\varepsilon}(0)-T_I(t)\overline{\mathbf{I}}(0)\right\|_{\infty}\\ &+\int_0^t \left\| \mathrm{P}_\varepsilon\left(\frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma}\int_{\mathbb{T}^d}\overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right)-\frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma}\int_{\mathbb{T}^d}\overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right\|_{\infty} ds\\ &+\int_0^t \left\| T_{I,\varepsilon}(t-s)\mathrm{P}_\varepsilon\left(\frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma}\int_{\mathbb{T}^d}\overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right)-T_I(t-s)\left(\frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma}\int_{\mathbb{T}^d}\overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right)\right\|_{\infty} ds.\end{aligned}$$

Then from Gronwall's lemma, $\sup_{0\le t\le T}\left\|\overline{\mathbf{X}}^{\varepsilon}(t)-\overline{\mathbf{X}}(t)\right\|_{\infty}\to 0$ follows from $\sup_{0\le t\le T}\pi_\varepsilon(t)\to 0$.
Since the maps $x\longmapsto \overline{\mathbf{S}}(0,x)$, $x\longmapsto\overline{\mathbf{I}}(0,x)$ and $x\longmapsto \frac{\overline{\mathbf{S}}(t,x)}{\left[\overline{\mathbf{B}}(t,x)\right]^\gamma}\int_{\mathbb{T}^d}\overline{\mathbf{F}}(t,y)\beta_t(x,dy)$ are continuous on $\mathbb{T}^d$, and the fact that $T_{S,\varepsilon}\longrightarrow T_S$ and $T_{I,\varepsilon}\longrightarrow T_I$ in L^∞ as $\varepsilon\to 0$, then $\sup\limits_{0\le t\le T}\pi_\varepsilon(t)\longrightarrow 0$, as $\varepsilon\to 0$ (see Kato [5], chapter 9, Section 3, Example 3.10).

□

5. Limit as $N\to\infty$ and $\varepsilon\to 0$

In this section, we extend our stochastic model on the whole space $\mathbb{T}^d$ and let both $N\to\infty$ and $\varepsilon\to 0$ in such a way that $N\varepsilon^d\to\infty$. Before stating the main theorem of this section, we first prove some lemmas and propositions.

Lemma 5.1 *There exist two constants $0<c<C$ such that for all $t\ge 0$, $\varepsilon>0$ and $x_\varepsilon\in \mathrm{D}_\varepsilon$,*

$$c\varepsilon^d\le \mathbb{P}(X(t)=x_\varepsilon)\le C\varepsilon^d.$$

Proof. Define $u^\varepsilon(t,x_\varepsilon):=\mathbb{P}(X(t)=x_\varepsilon)$. We have that $u^\varepsilon(t,x_\varepsilon)=\left(e^{t\left[\Delta_\varepsilon^S\right]^*}u_0^\varepsilon\right)(x_\varepsilon)$. Using the assumption on the initial condition $\mathbb{P}(X(0)=x_\varepsilon)$, then $0<c\varepsilon^d\le u^\varepsilon(0,x_\varepsilon)\le C\varepsilon^d$, from which we deduce that $0<c\varepsilon^d\le e^{t\left[\Delta_\varepsilon^S\right]^*}u^\varepsilon(0,x_\varepsilon)\le C\varepsilon^d$, hence the result.

Lemma 5.2 *There exits a positive constant C such that for all $0\le s\le t$, $\varepsilon>0$ and $x_\varepsilon\in\mathrm{D}_\varepsilon$*

$$\sum_{y_\varepsilon} q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t)=1 \quad \textit{and} \quad \mathbb{P}\left(Y_j(t)=x_\varepsilon\right)\le C\varepsilon^d.$$

Proof. The uniform distribution on D_ε is invariant for the process $Y(t)$. So if we start Y at time s with the uniform distribution i.e. $\mathbb{P}\left(Y(s)=x_\varepsilon\right)=\varepsilon^d$, the law of Y at time t is also the uniform law. But

$$\mathbb{P}\left(Y(t)=x_\varepsilon\right)=\sum_{y_\varepsilon}\mathbb{P}\left(Y(s)=y_\varepsilon\right)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t) \text{ i.e } \varepsilon^d=\varepsilon^d\sum_{y_\varepsilon}q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t),$$

thus $\displaystyle\sum_{y_\varepsilon}q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t)=1$. Finally

$$\begin{aligned}\mathbb{P}\left(Y_j(t)=x_\varepsilon\right)&=\sum_{y_\varepsilon}\mathbb{P}\left(Y_j(0)=y_\varepsilon\right)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\\&\le\sup_{y_\varepsilon}\mathbb{P}\left(Y_j(0)=y_\varepsilon\right)\sum_{y_\varepsilon}q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t).\end{aligned}$$

Hence the second result follows from the first one and Assumption 3.1 (**ii**) and (**iii**). □

Let define $\overline{\mathfrak{F}}_0^{N,\varepsilon}(t,x_\varepsilon):=\displaystyle\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\lambda_{-j}(t)\mathbb{1}_{Y_j(t)=x_\varepsilon}$ and $\overline{\mathfrak{F}}_0^{\varepsilon}(t,x_\varepsilon):=\overline{\lambda}_0(t)\displaystyle\sum_{y_\varepsilon}\overline{I}^\varepsilon(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)$.

We have the

Lemma 5.3 *Let us assume that $(N,\varepsilon)\to(\infty,0)$, in such a way that $N\varepsilon^d\to\infty$. Then for all $T>0$,*

$$\sup_{0\le t\le T}\mathbb{E}\left(\left\|\overline{\mathfrak{F}}_0^{N,\varepsilon}(t)-\overline{\mathfrak{F}}_0^{\varepsilon}(t)\right\|_\infty^2\right)\longrightarrow 0,\quad \text{as } (N\,,\,\varepsilon)\to(\infty\,,\,0).$$

Proof. $\overline{\mathfrak{F}}_0^{N,\varepsilon}(t,x_\varepsilon)$ can be decomposed as follows

$$\overline{\mathfrak{F}}_0^{N,\varepsilon}(t,x_\varepsilon)=\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\left(\lambda_{-j}(t)-\overline{\lambda}_0(t)\right)\mathbb{1}_{Y_j(t)=x_\varepsilon}+\overline{\lambda}_0(t)\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{1}_{Y_j(t)=x_\varepsilon}.$$

Let consider the first term. Since $\left(\lambda_{-j}(t)\right)_j$ are independent and identically distributed and independent of $Y_j(t)$, then

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\left(\lambda_{-j}(t)-\overline{\lambda}_0(t)\right)\mathbb{1}_{Y_j(t)=x_\varepsilon}\right)^2\right]&=\frac{1}{N^2}\sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{E}\left[\left|\lambda_{-j}(t)-\overline{\lambda}_0(t)\right|^2\mathbb{1}_{Y_j(t)=x_\varepsilon}\right]\\&\le\frac{1}{N^2}C(\lambda^*)I^{N,\varepsilon}(0)\mathbb{P}\left(Y_1(t)=x_\varepsilon\right)\le\frac{C(\lambda^*)}{N}.\end{aligned}$$

Now, since

$$\begin{aligned}\mathbb{E}\left[\sup_{x_\varepsilon\in\mathrm{D}_\varepsilon}\left(\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\left(\lambda_{-j}(t)-\overline{\lambda}_0(t)\right)\mathbb{1}_{Y_j(t)=x_\varepsilon}\right)^2\right]&\le\sum_{x_\varepsilon}\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\left(\lambda_{-j}(t)-\overline{\lambda}_0(t)\right)\mathbb{1}_{Y_j(t)=x_\varepsilon}\right)^2\right]\\&\le\frac{C(\lambda^*)}{N}\varepsilon^{-d}\quad\to 0,\end{aligned}\tag{5.1}$$

provided $N\varepsilon^d\to\infty$. It remains to show that

$$\sup_{x_\varepsilon\in\mathrm{D}_\varepsilon}\left|\overline{\lambda}_0(t)\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{1}_{Y_j(t)=x_\varepsilon}-\overline{\lambda}_0(t)\sum_{y_\varepsilon}\overline{I}^\varepsilon(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\right|\longrightarrow 0,\ \text{as } (N,\varepsilon)\longrightarrow(\infty,0).$$

We have

$$\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{1}_{Y_j(t)=x_\varepsilon} = \frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \left[\mathbb{1}_{Y_j(t)=x_\varepsilon} - \mathbb{P}\left(Y_j(t)=x_\varepsilon\right)\right] + \frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{P}\left(Y_j(t)=x_\varepsilon\right).$$

$$\begin{aligned}\mathbb{E}\left\{\left(\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \left[\mathbb{1}_{Y_j(t)=x_\varepsilon} - \mathbb{P}\left(Y_j(t)=x_\varepsilon\right)\right]\right)^2\right\} &= \frac{1}{N^2}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{E}\left(\left|\mathbb{1}_{Y_j(t)=x_\varepsilon} - \mathbb{P}\left(Y_j(t)=x_\varepsilon\right)\right|^2\right) \\ &\leq \frac{C}{N}.\end{aligned}$$

It follows that

$$\mathbb{E}\left\{\sup_{x_\varepsilon\in \mathrm{D}_\varepsilon}\left(\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \left[\mathbb{1}_{Y_j(t)=x_\varepsilon} - \mathbb{P}\left(Y_j(t)=x_\varepsilon\right)\right]\right)^2\right\} \leq \frac{C}{N\varepsilon^d} \quad \to 0, \tag{5.2}$$

provided $N\varepsilon^d \to 0$.

Since $\overline{\lambda}_0(t)$ is bounded, it remains to evaluate the quantity $\displaystyle\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{P}\left(Y_j(t)=x_\varepsilon\right) - \sum_{y_\varepsilon}\overline{I}^\varepsilon(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)$.

We have

$$\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{P}\left(Y_j(t)=x_\varepsilon\right) = \frac{1}{N}\sum_{y_\varepsilon}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{P}\left(Y_j(0)=y_\varepsilon\right) q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t), \text{ thus}$$

$$\begin{aligned}\sup_{x_\varepsilon}\left|\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{P}\left(Y_j(t)=x_\varepsilon\right) - \sum_{y_\varepsilon}\overline{I}^\varepsilon(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\right| &\leq \frac{1}{N}\sup_{x_\varepsilon}\sum_{y_\varepsilon} q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\left|\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{P}\left(Y_j(0)=y_\varepsilon\right) - N\overline{I}^\varepsilon(0,y_\varepsilon)\right| \\ &\leq \frac{1}{N}\sup_{x_\varepsilon}\sum_{y_\varepsilon} q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\frac{\overline{I}^\varepsilon(0,y_\varepsilon)}{\overline{I}^\varepsilon(0)}\left|I^{N,\varepsilon}(0) - N\overline{I}^\varepsilon(0)\right| \\ &\leq \frac{C}{N} \quad \longrightarrow 0. \end{aligned} \tag{5.3}$$

Combining (5.1), (5.2) and (5.3), we finally have

$$\sup_{0\leq t\leq T}\mathbb{E}\left(\sup_{x_\varepsilon\in \mathrm{D}_\varepsilon}\left|\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \lambda_{-j}(t)\mathbb{1}_{Y_j(t)=x_\varepsilon} - \overline{\lambda}_0(t)\sum_{y_\varepsilon}\overline{I}^\varepsilon(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\right|^2\right) \longrightarrow 0, \tag{5.4}$$

provided $N\varepsilon^d \to +\infty$.

□

Let $\sigma^{N,\varepsilon}$ be the stopping time defined by

$$\sigma^{N,\varepsilon}(\omega) := \inf\left\{t>0\,, \omega \notin A_{t,\delta}\cap B_{t,\delta}\right\}, \tag{5.5}$$

where for all $t \leq T$, $\delta > 0$,

$$A_{t,\delta} = \left\{\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_S^{N,\varepsilon}(s)\right\|_\infty \leq \delta\right\}, \quad B_{t,\delta} = \left\{\left\|\int_0^t T_{I,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_I^{N,\varepsilon}(s)\right\|_\infty \leq \delta\right\},$$

with

$$\mathscr{M}_S^{N,\varepsilon}(t)=\sum_{y_\varepsilon\sim x_\varepsilon}\frac{1}{N}M_S^{y_\varepsilon,x_\varepsilon}\left(N\int_0^t\frac{\nu_S}{\varepsilon^2}\overline{S}^{N,\varepsilon}(s,y_\varepsilon)ds\right)-\sum_{y_\varepsilon\sim x_\varepsilon}\frac{1}{N}M_S^{x_\varepsilon,y_\varepsilon}\left(N\int_0^t\frac{\nu_S}{\varepsilon^2}\overline{S}^{N,\varepsilon}(s,x_\varepsilon)ds\right),$$

$$\widetilde{\mathscr{M}}_I^{N,\varepsilon}(t)=\mathscr{M}_I^{N,\varepsilon}(t)+\mathscr{M}_{SI}^{N,\varepsilon}(t),\quad \text{where}$$

$$\mathscr{M}_I^{N,\varepsilon}(t)=\sum_{y_\varepsilon\sim x_\varepsilon}\frac{1}{N}M_I^{y_\varepsilon,x_\varepsilon}\left(N\int_0^t\frac{\nu_I}{\varepsilon^2}\overline{I}^{N,\varepsilon}(s,y_\varepsilon)ds\right)-\sum_{y_\varepsilon\sim x_\varepsilon}\frac{1}{N}M_I^{x_\varepsilon,y_\varepsilon}\left(N\int_0^t\frac{\nu_I}{\varepsilon^2}\overline{I}^{N,\varepsilon}(s,x_\varepsilon)ds\right),$$

$$\mathscr{M}_{SI}^{N,\varepsilon}(t)=\frac{1}{N}\int_0^t\int_0^\infty \mathbb{1}_{u\le S^{N,\varepsilon}(s^-,x_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s^-,x_\varepsilon)}\overline{Q}^{x_\varepsilon}(ds,du).$$

$\overline{Q}^{x_\varepsilon}(ds,du):=Q^{x_\varepsilon}(ds,du)-dsdu$ is the compensated PRM associated with $Q_\varepsilon^{x_\varepsilon}(ds,du)$, and we have used the notations

$$M_S^{x_\varepsilon,y_\varepsilon}(t)=P_S^{x_\varepsilon,y_\varepsilon}(t)-t,\quad M_I^{x_\varepsilon,y_\varepsilon}(t)=P_I^{x_\varepsilon,y_\varepsilon}(t)-t.$$

Let $\bar{\bar{c}}:=\dfrac{\lambda^*\beta^*\left\|\overline{I}^{N,\varepsilon}(t)\right\|_\infty}{c^\gamma}$, where c stands for the bound in 4.2. We define the stopping time

$$\tau^{N,\varepsilon}=\inf\left\{t>0\,,\left\|\int_0^t e^{(t-s)\left(\Delta_\varepsilon^S-\bar{\bar{c}}I_d\right)}d\widetilde{\mathscr{M}}_S^{N,\varepsilon}(s)\right\|_\infty\ge\frac{c}{8}\right\},$$

where I_d is the identity operator on H^ε, and $\widetilde{\mathscr{M}}_S^{N,\varepsilon}(t,x_\varepsilon):=\mathscr{M}_S^{N,\varepsilon}(t,x_\varepsilon)-\mathscr{M}_{SI}^{N,\varepsilon}(t,x_\varepsilon)$.
In the proof of the next Proposition, we shall need the following Lemma.

Lemma 5.4 *As $(N,\varepsilon)\to(\infty,0)$ in such way that $N\varepsilon^d\to\infty$, $\left\|\overline{S}^{N,\varepsilon}(0,.)-\overline{S}^\varepsilon(0,.)\right\|_\infty\longrightarrow 0$ in $L^2(\Omega)$.*

Proof. We have

$$\begin{aligned}\overline{S}^{N,\varepsilon}(0,x_\varepsilon)-\overline{S}^\varepsilon(0,x_\varepsilon) &= \frac{1}{N}\sum_{j=1}^{S^{N,\varepsilon}(0)}\mathbb{1}_{X_j=x_\varepsilon}-\mathbb{P}\left(X=x_\varepsilon\right)\overline{S}^\varepsilon(0)\\ &= \overline{S}^\varepsilon(0)\frac{1}{N\overline{S}^\varepsilon(0)}\sum_{j=1}^{S^{N,\varepsilon}(0)}\left[\mathbb{1}_{X_j=x_\varepsilon}-\mathbb{P}\left(X=x_\varepsilon\right)\right]+\frac{\mathbb{P}\left(X=x_\varepsilon\right)}{N}\left[\overline{S}^{N,\varepsilon}(0)-N\overline{S}^\varepsilon(0)\right].\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[\left|\overline{S}^{N,\varepsilon}(0,x_\varepsilon)-\overline{S}^\varepsilon(0,x_\varepsilon)\right|^2\right] &\le \frac{2}{N^2}\sum_{j=1}^{S^{N,\varepsilon}(0)}Var\left[\mathbb{1}_{X=x_\varepsilon}\right]+\frac{2\left[\mathbb{P}\left(X=x_\varepsilon\right)\right]^2}{N^2}\\ &\le \frac{\overline{S}^\varepsilon(0)}{N}\frac{C}{c}\varepsilon^d+\frac{C\varepsilon^{2d}}{N^2}\le\frac{C'}{N}+\frac{C\varepsilon^{2d}}{N^2}.\end{aligned}$$

Then

$$\mathbb{E}\left[\sup_{x_\varepsilon\in\mathrm{D}_\varepsilon}\left|\overline{S}^{N,\varepsilon}(0,x_\varepsilon)-\overline{S}^\varepsilon(0,x_\varepsilon)\right|^2\right]\le\frac{C'}{N\varepsilon^d}+\frac{C\varepsilon^d}{N^2}.$$

The result follows. □

Proposition 5.1 *For all $T>0$, there exists C such that for N large enough if $t\le\sigma^{N,\varepsilon}\wedge T$, then $\left\|\overline{S}^{N,\varepsilon}(t)\right\|_\infty\le C$ and $\left\|\overline{I}^{N,\varepsilon}(t)\right\|_\infty\le C$, for all $\varepsilon>0$. Moreover there exists $\varepsilon_0>0$ and $c_0>0$ such that if $t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T$, $\overline{B}^{N,\varepsilon}(t,x_\varepsilon)\ge c_0$, for all $x_\varepsilon\in\mathrm{D}_\varepsilon$, provided $\varepsilon\le\varepsilon_0$.*

Proof. Let first treat the term $\|\overline{S}^{N,\varepsilon}(t)\|_\infty$.
Using the Duhamel formula, we have

$$\overline{S}^{N,\varepsilon}(t,x_\varepsilon) \le \left(T_{S,\varepsilon}(t)\overline{S}^{N,\varepsilon}(0,.)\right)(x_\varepsilon) + \int_0^t \left(T_{S,\varepsilon}(t-s)d\mathscr{M}_S^{N,\varepsilon}(s,.)\right)(x_\varepsilon).$$

Since $\overline{S}^{N,\varepsilon}(0,x_\varepsilon) \le C$, for all $x_\varepsilon \in \mathrm{D}_\varepsilon$, we obtain that for $t \le \sigma^{N,\varepsilon} \wedge T$,

$$\|\overline{S}^{N,\varepsilon}(t)\|_\infty \le C + \delta\,.$$

We now consider the term $\left\|\overline{I}^{N,\varepsilon}(t)\right\|_\infty$. Arguing as in the proof of Lemma 4.1, we have for $t \le \sigma^{N,\varepsilon} \wedge T$,

$$\begin{aligned}\left\|\overline{I}^{N,\varepsilon}(t)\right\|_\infty &\le e^{Ct}\left(C + \sup_{0\le t\le T}\left\|\int_0^t T_{I,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_I^{N,\varepsilon}(s)\right\|_\infty\right)\\ &\le (C+\delta)\,e^{CT}.\end{aligned}$$

We finally consider the term $\overline{B}^{N,\varepsilon}(t,x_\varepsilon)$. It follows from Lemma 5.4 that $\|\overline{S}^{N,\varepsilon}(0,.)-\overline{S}^{\varepsilon}(0,.)\|_\infty \longrightarrow 0$ and from Lemma 4.2 that $\overline{S}^{\varepsilon}(0,x_\varepsilon) \ge c$, for all $x_\varepsilon \in \mathrm{D}_\varepsilon$, then for N large enough, $\mathbb{P}\left(\inf_{x_\varepsilon}\overline{S}^{N,\varepsilon}(0,x_\varepsilon) \ge \frac{c}{2}\right)$ is close to 1. Let $T_c^{N,\varepsilon} = \inf\left\{t\,, \inf_{x_\varepsilon}\overline{S}^{N,\varepsilon}(t,x_\varepsilon) < \dfrac{c}{4}\right\}$. On the interval $[0\,, T_c^{N,\varepsilon})$, $\overline{S}^{N,\varepsilon}(t,x_\varepsilon) \ge \dfrac{c}{4}$, $\forall x_\varepsilon \in \mathrm{D}_\varepsilon$. For all $t \le T_c^{N,\varepsilon} \wedge \sigma^{N,\varepsilon} \wedge T$, we have

$$\overline{\Gamma}^{N,\varepsilon}(t,x_\varepsilon) = \frac{1}{\left[\,\overline{B}^{N,\varepsilon}(t,x_\varepsilon)\right]^\gamma}\sum_{y_\varepsilon}\beta_\varepsilon^{x_\varepsilon,y_\varepsilon}(t)\overline{\mathfrak{F}}^{N,\varepsilon}(t,y_\varepsilon) \le \frac{4^\gamma\lambda^*\beta^*\left\|\overline{I}^{N,\varepsilon}(t)\right\|_\infty}{c^\gamma} = \bar{\bar{c}}$$

and then, if moreover $t \le \tau^{N,\varepsilon}$,

$$\begin{aligned}\overline{S}^{N,\varepsilon}(t,x_\varepsilon) &\ge \left(e^{(\Delta_\varepsilon^S-\bar{\bar{c}}I_d)t}\overline{S}^{N,\varepsilon}(0)\right)(x_\varepsilon) + \int_0^t\left(e^{(t-s)(\Delta_\varepsilon^S-\bar{\bar{c}}I_d)}d\widetilde{\mathscr{M}}_S^{N,\varepsilon}(s)\right)(x_\varepsilon)\\ &\ge \frac{c}{2}e^{-\bar{\bar{c}}t} - \frac{c}{8}.\end{aligned}\tag{5.6}$$

We note that $\frac{c}{2}e^{-\bar{\bar{c}}t} \ge \dfrac{c}{4}$ iff $t \le \frac{\log 2}{\bar{\bar{c}}} = T_{\bar{\bar{c}}}$.
So, on the event $\tau^{N,\varepsilon} \wedge \sigma^{N,\varepsilon} \wedge T \ge T_{\bar{\bar{c}}}$, $\overline{S}^{N,\varepsilon}(t,x_\varepsilon) \ge \dfrac{c}{8}$, $\forall\, 0 \le t \le T_{\bar{\bar{c}}}$.

$$\text{For } t > T_{\bar{\bar{c}}},\quad \overline{I}^{N,\varepsilon}(t,x_\varepsilon) \ge \left(T_{I,\varepsilon}(t)\overline{I}^{N,\varepsilon}(0)\right)(x_\varepsilon) + \int_0^t\left(T_{I,\varepsilon}(t-s)d\mathscr{M}_I^{N,\varepsilon}(s)\right)(x_\varepsilon).$$

We choose $T > T_{\bar{\bar{c}}}$ arbitrary. We know from the proof of Lemma 4.2 that there exists ε_0 and $\underline{\underline{c}}$ such that $\overline{I}^{\varepsilon}(t,x_\varepsilon) \ge \underline{\underline{c}}$ for all $\varepsilon \le \varepsilon_0$, $x_\varepsilon \in \mathrm{D}_\varepsilon$ and $\dfrac{\log 2}{\bar{\bar{c}}} \le t \le T$. If we now choose $\delta = \dfrac{\underline{\underline{c}}}{2}$ in the definition of $\sigma^{N,\varepsilon}$, we deduce that for any $\varepsilon \le \varepsilon_0$, $x_\varepsilon \in \mathrm{D}_\varepsilon$, $T_{\bar{\bar{c}}} \le t \le \sigma^{N,\varepsilon} \wedge T$, $\overline{I}^{N,\varepsilon}(t,x_\varepsilon) \ge \dfrac{\underline{\underline{c}}}{2}$.

□

From now, we decree that $\sigma^{N,\varepsilon} = 0$ whenever $\inf_{x_\varepsilon}\overline{S}^{N,\varepsilon}(0,x_\varepsilon) < \dfrac{c}{2}$, or $\varepsilon > \varepsilon_0$.

Lemma 5.5 *Given $T>0$, there exists $C>0$ such that for any $t<\tau^{N,\varepsilon}\wedge\sigma^{N,\varepsilon}$, we have*

$$\left\|\overline{S}^{N,\varepsilon}(t)\overline{\Gamma}^{N,\varepsilon}(t)-\overline{S}^{\varepsilon}(t)\overline{\Gamma}^{\varepsilon}(t)\right\|_{\infty}\le C\Bigg(\left\|\overline{S}^{N,\varepsilon}(t)-\overline{S}^{\varepsilon}(t)\right\|_{\infty} + \left\|\overline{\mathfrak{F}}^{N,\varepsilon}(t)-\overline{\mathfrak{F}}^{\varepsilon}(t)\right\|_{\infty}+\left\|\overline{I}^{N,\varepsilon}(t)-\overline{I}^{\varepsilon}(t)\right\|_{\infty}\Bigg). \tag{5.7}$$

Proof. Note that, using the map $\mathscr{H}$ defined in Remark 4.1, with a slight modification of the constants, we have

$$\overline{S}^{N,\varepsilon}(t,x_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(t,x_\varepsilon)-\overline{S}^{\varepsilon}(t,x_\varepsilon)\overline{\Gamma}^{\varepsilon}(t,x_\varepsilon)=\mathscr{H}\left(\overline{S}^{N,\varepsilon},\overline{I}^{N,\varepsilon},\overline{\mathfrak{F}}^{N,\varepsilon}\right)(t,x_\varepsilon)-\mathscr{H}\left(\overline{S}^{\varepsilon},\overline{I}^{\varepsilon},\overline{\mathfrak{F}}^{\varepsilon}\right)(t,x_\varepsilon),$$

and the result the follows from the fact that $\mathscr{H}$ is bounded and globally Lipschitz.

□

We define $\omega^{N,\varepsilon}(t)=\omega_S^{N,\varepsilon}(t)+\omega_I^{N,\varepsilon}(t)+\omega_{\mathfrak{F}}^{N,\varepsilon}(t)$, with

$$\begin{aligned}
\omega_S^{N,\varepsilon}(t)&=\left\|\overline{S}^{N,\varepsilon}(0)-\overline{S}^{\varepsilon}(0)\right\|_{\infty}+\left\|\int_0^t T_{S,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_S^{N,\varepsilon}(s)\right\|_{\infty},\\
\omega_I^{N,\varepsilon}(t)&=\left\|\bar{I}^{N,\varepsilon}(0)-\overline{I}^{\varepsilon}(0)\right\|_{\infty}+\left\|\int_0^t T_{I,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_I^{N,\varepsilon}(s)\right\|_{\infty},\\
\omega_{\mathfrak{F}}^{N,\varepsilon}(t)&=\left\|\overline{\mathfrak{F}}_0^{N,\varepsilon}(t)-\overline{\mathfrak{F}}_0^{\varepsilon}(t)\right\|_{\infty}+\left\|\mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}(t)\right\|_{\infty},
\end{aligned} \tag{5.8}$$

where

$$\mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon)=\frac{1}{N}\sum_{y_\varepsilon}\int_0^t\int_0^\infty\int_{\mathbf{D}}\int_{\mathbf{D}}\lambda(t-s)\mathbb{1}_{u\le S^{N,\varepsilon}(s^-,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s^-,y_\varepsilon)}\mathbb{1}_{Y^{s,y_\varepsilon}(t)=x_\varepsilon}\overline{Q}^{y_\varepsilon}(ds,du,d\lambda,dY).$$

Note that $\mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}$ is not a martingale.

Lemma 5.6 *As $(N,\varepsilon)\to(\infty,0)$, in such a way that $N\varepsilon^d\to\infty$,*

$$\sup_{0\le t\le T}\mathbb{E}\left(\mathbb{1}_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\,[\omega^{N,\varepsilon}(t)]^2\right)\to 0.$$

Proof. We shall use the following notation

$$\left\|\Phi^\varepsilon\right\|_{\mathtt{H}^\varepsilon}:=\left[\sum_{x_\varepsilon}\left|\Phi^\varepsilon_{x_\varepsilon}\right|^2\right]^{1/2},$$

for any step function Φ^ε ($\Phi^\varepsilon_{x_\varepsilon}$ denoting the value of Φ^ε on the cell $V_\varepsilon(x_\varepsilon)$).
Thanks to Theorem 2.1 in P. Kotelenez [7], we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s)\right\|^2_{\mathtt{H}^\varepsilon}\right]&\le C\mathbb{E}\left[\left\|\mathscr{M}_{SI}^{N,\varepsilon}(\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T)\right\|^2_{\mathtt{H}^\varepsilon}\right]\\
&\le\frac{C}{N}\sum_{x_\varepsilon}\mathbb{E}\left(\int_0^T\overline{S}^{N,\varepsilon}(s\wedge\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon},x_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s\wedge\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon},x_\varepsilon)ds\right).
\end{aligned}$$

Provided $t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T$, $\overline{\Gamma}^{N,\varepsilon}(t,x_\varepsilon)\le C(\lambda^*,\beta^*)$ and $\overline{S}^{N,\varepsilon}(t,x_\varepsilon)\le C$. Then

$$\mathbb{E}\left[\sup_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s)\right\|^2_{\mathtt{H}^\varepsilon}\right]\le C(\lambda^*,\beta^*)\frac{1}{N\varepsilon^d}.$$

Since the L^∞ norm is bounded by the $\mathtt{H}^{\varepsilon}$ horm, as $(N,\varepsilon)\to(\infty,0)$, provided $N\varepsilon^d\to 0$,

$$\mathbb{E}\left[\sup_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s)\right\|_\infty^2\right]\longrightarrow 0. \tag{5.9}$$

The same argument can be used for the term $\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_{S}^{N,\varepsilon}(s)\right\|_\infty$. We conclude that as $(N,\varepsilon)\longrightarrow(\infty,0)$, in such a way that $N\varepsilon^d\to 0$,

$$\sup_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\omega_S^{N,\varepsilon}(t)\longrightarrow 0 \text{ in } L^2(\Omega)\,. \tag{5.10}$$

A similar proof establishes that

$$\sup_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\omega_I^{N,\varepsilon}(t)\longrightarrow 0 \text{ in } L^2(\Omega)\,. \tag{5.11}$$

We now consider $\omega_{\mathfrak{F}}^{N,\varepsilon}(t)$ The convergence to zero of the first term has been established in Lemma 5.3. We now consider the second term. We have

$$\begin{aligned}
&\sup_{t\le T}\mathbb{E}\left(\mathbb{1}_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\sup_{x_\varepsilon}\left|\mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon)\right|^2\right)\\
&\quad=\frac{1}{N^2}\sup_{t\le T}\mathbb{E}\Bigg[\mathbb{1}_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\sup_{x_\varepsilon}\Bigg(\sum_{y_\varepsilon}\int_0^t\int_0^\infty\int_{\mathbf{D}}\int_{\mathbf{D}}\lambda(t-s)\mathbb{1}_{u\le S^{N,\varepsilon}(s^-,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s^-,y_\varepsilon)}\\
&\quad\quad\times\mathbb{1}_{Y^{s,y_\varepsilon}(t)=x_\varepsilon}\overline{Q}^{y_\varepsilon}(ds,du,d\lambda,dY)\Bigg)^2\Bigg]\\
&\quad\le\frac{1}{N^2}\sum_{x_\varepsilon,y_\varepsilon}\mathbb{E}\int_0^{\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\lambda^2(t-s)S^{N,\varepsilon}(s,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t)ds\\
&\quad\le\frac{(\lambda^*)^2}{N}\sum_{x_\varepsilon}\mathbb{E}\left[\int_0^{\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\sup_{y_\varepsilon}\left|\overline{S}^{N,\varepsilon}(s,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s,y_\varepsilon)\right|\sum_{y_\varepsilon}q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t)ds\right]\\
&\quad\le C(\lambda^*)\frac{T}{N\varepsilon^d}.
\end{aligned}\tag{5.12}$$

The result follows. Note that since $\mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon)$ is not a martingale, the result for $\omega_{\mathfrak{F}}^{N,\varepsilon}(t)$ is weaker than (5.10) and (5.11). □

Lemma 5.6 clearly implies

Lemma 5.7 *As $(N,\varepsilon)\longrightarrow(\infty,0)$ in such way that $N\varepsilon^d\to\infty$, $\mathbb{1}_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\int_0^t\omega^{N,\varepsilon}(s)ds\longrightarrow 0$ in probability.*

It remains to establish the next result.

Lemma 5.8 *As $(N,\varepsilon)\to(\infty,0)$, $\mathbb{P}\left(\sigma^{N,\varepsilon}<T\right)\longrightarrow 0$ and $\mathbb{P}\left(\tau^{N,\varepsilon}<T\right)\longrightarrow 0$.*

Proof. We have

$$\begin{aligned}
\mathbb{P}\left(\sigma^{N,\varepsilon}<T\right)\le\mathbb{P}&\left(\sup_{t\le\sigma^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_S^{N,\varepsilon}(s)\right\|_\infty\ge\delta/2\right)\\
&+\mathbb{P}\left(\sup_{t\le\sigma^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{I,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_I^{N,\varepsilon}(s)\right\|_\infty\ge\delta/2\right).
\end{aligned}\tag{5.13}$$

We consider the second term only. The first one is treated similarly.

$$\left\|\int_0^t T_{I,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_I^{N,\varepsilon}(s)\right\|_\infty \le \left\|\int_0^t T_{I,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s)\right\|_\infty + \left\|\int_0^t T_{I,\varepsilon}(t-s)d\mathscr{M}_I^{N,\varepsilon}(s)\right\|_\infty,$$

from Proposition 3.2 of [8], we have

$$\mathbb{P}\left(\sup_{t\le\sigma^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{I,\varepsilon}(t-s)d\mathscr{M}_I^{N,\varepsilon}(s)\right\|_\infty \ge \frac{\delta}{2}\right) \le 4\varepsilon^{-d-2}\exp\left(-\mathtt{a}\frac{\delta^2}{16}N\right) \tag{5.14}$$

Since we assume that $N\varepsilon^d \longrightarrow 0$, the right side, hence also the left hand side of (5.14) tends to 0. By Chebyshev's inequality, we have

$$\mathbb{P}\left(\sup_{t\le\sigma^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{I,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s)\right\|_{\mathtt{H}^\varepsilon} \ge \frac{\delta}{2}\right) \le \frac{4}{\delta^2}\mathbb{E}\left[\sup_{t\le\sigma^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{I,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s)\right\|_{\mathtt{H}^\varepsilon}^2\right].$$

The right hand side tends to 0 as shown in the proof of Lemma 5.6. Since the L^∞ norm is bounded by the $\mathtt{H}^\varepsilon$ norm, this finishes the proof that $\mathbb{P}\left(\sigma^{N,\varepsilon} < T\right) \to 0$. A similar proof establishes the same result for $\tau^{N,\varepsilon}$.

□

We now extend our stochastic process to the whole space $\mathbb{T}^d$. So, we define

$$\overline{\mathbf{S}}^{N,\varepsilon}(t,x) := \sum_{x_\varepsilon}\overline{S}^\varepsilon(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x), \quad \overline{\mathbf{I}}^{N,\varepsilon}(t,x) := \sum_{x_\varepsilon}\overline{I}^\varepsilon(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x)$$
$$\overline{\mathbf{B}}^{N,\varepsilon}(t,x) := \sum_{x_\varepsilon}\overline{\mathbf{B}}^\varepsilon(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x), \quad \overline{\mathbf{F}}^{N,\varepsilon}(t,x) := \sum_{x_\varepsilon}\overline{\mathfrak{F}}^\varepsilon(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x)$$

and set $\overline{\mathbf{X}}^{N,\varepsilon} := (\overline{\mathbf{S}}^{N,\varepsilon}, \overline{\mathbf{F}}^{N,\varepsilon}, \overline{\mathbf{I}}^{N,\varepsilon})$.

Let us recall the following Gronwall's lemma.

Lemma 5.9 *Let ϕ and ψ be two nonegative Borel measurable locally bounded functions on an interval $[0,T)$, with $T<\infty$ and C a non-negative constant. If for all $t\in[0,T)$, the following inequality is satisfied :*

$$\phi(t) \le C\int_0^t \phi(s)ds + \psi(t), \tag{5.15}$$

then $\phi(t) \le C\int_0^t e^{C(t-s)}\psi(s)ds + \psi(t)$ for all $t\le T$.

Theorem 5.1 *Let us assume that $(N,\varepsilon)\to(\infty,0)$, in such a way that $N\varepsilon^d\to\infty$. Then $(N,\varepsilon)\to(\infty,0)$*

$$\left\|\overline{\mathbf{X}}^{N,\varepsilon}(t) - \overline{\mathbf{X}}^\varepsilon(t)\right\|_\infty \longrightarrow 0, \text{ in probability}, \ \forall\, t\ge 0. \tag{5.16}$$

Proof. Since $\left\|\overline{\mathbf{X}}^{N,\varepsilon}(t) - \overline{\mathbf{X}}^\varepsilon(t)\right\|_\infty = \left\|\overline{X}^{N,\varepsilon}(t) - \overline{X}^\varepsilon(t)\right\|_\infty$, it suffices to show that

$$\left\|\overline{X}^{N,\varepsilon}(t) - \overline{X}^\varepsilon(t)\right\|_\infty \longrightarrow 0, \ \text{ in probability, for all } t\ge 0.$$

We first consider

$$\overline{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon) = \frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\lambda_{-j}(t)\mathbb{1}_{Y_j(t)=x_\varepsilon} + \sum_{y_\varepsilon}\int_0^t\overline{\lambda}(t-s)\overline{S}^{N,\varepsilon}(s,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t)ds + \mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon),$$

$$\overline{\mathfrak{F}}^{\varepsilon}(t,x_\varepsilon)=\overline{\lambda}_0(t)\sum_{y_\varepsilon}\overline{I}^{\varepsilon}(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)+\sum_{y_\varepsilon}\int_0^t\overline{\lambda}(t-s)\overline{S}^{\varepsilon}(s,y_\varepsilon)\overline{\Gamma}^{\varepsilon}(s,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t)ds.$$

Exploiting Lemma 5.5, we have the following: for all $t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}$

$$\begin{aligned}\left\|\overline{\mathfrak{F}}^{N,\varepsilon}(t)-\overline{\mathfrak{F}}^{\varepsilon}(t)\right\|_\infty\le\omega_{\mathfrak{F}}^{N,\varepsilon}(t)+C\int_0^t\Bigg(&\left\|\overline{S}^{N,\varepsilon}(s)-\overline{S}^{\varepsilon}(s)\right\|_\infty+\left\|\overline{\mathfrak{F}}^{N,\varepsilon}(s)-\overline{\mathfrak{F}}^{\varepsilon}(s)\right\|_\infty\\&+\left\|\overline{I}^{N,\varepsilon}(s)-\overline{I}^{\varepsilon}(s)\right\|_\infty\Bigg)ds.\end{aligned}\tag{5.17}$$

By writing $\overline{S}^{N,\varepsilon}(t,x_\varepsilon)-\overline{S}^{\varepsilon}(t,x_\varepsilon)$ and $\overline{I}^{N,\varepsilon}(t,x_\varepsilon)-\overline{I}^{\varepsilon}(t,x_\varepsilon)$ in their mild semigroup form, and using estimates in Lemmas 4.1, 4.2, 5.1 and 5.5, we obtain, for $t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T$

$$\left\|\overline{X}^{N,\varepsilon}(t)-\overline{X}^{\varepsilon}(t)\right\|_\infty\le C\int_0^t\left\|\overline{X}^{N,\varepsilon}(s)-\overline{X}^{\varepsilon}(s)\right\|_\infty ds+\omega^{N,\varepsilon}(t).\tag{5.18}$$

Then, it follows from Gronwall's Lemma (5.9) that

$$\begin{aligned}\left\|\overline{X}^{N,\varepsilon}(t)-\overline{X}^{\varepsilon}(t)\right\|_\infty&\le C\int_0^t e^{C(t-s)}\omega^{N,\varepsilon}(s)ds+\omega^{N,\varepsilon}(t)\\&\le Ce^{Ct}\int_0^t\omega^{N,\varepsilon}(s)ds+\omega^{N,\varepsilon}(t),\quad\forall\,t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}.\end{aligned}\tag{5.19}$$

Consequently using lemmas 5.6, 5.7 and 5.8, for any $t>0$, as $(N,\varepsilon)\to(\infty,0)$, in such a way that $N\varepsilon^d\longrightarrow\infty$,

$$\left\|\overline{X}^{N,\varepsilon}(t)-\overline{X}^{\varepsilon}(t)\right\|_\infty\longrightarrow 0\quad\text{in probability},\ \forall t\ge0.$$

□

Theorem 5.2 *For all $T>0$, as $(N,\varepsilon)\longrightarrow(\infty,0)$ in such a way that $N\varepsilon^d\to\infty$, we have,*

$$\sup_{0\le t\le T}\left(\left\|\overline{\mathbf{S}}^{N,\varepsilon}(t)-\overline{\mathbf{S}}^{\varepsilon}(t)\right\|_\infty+\left\|\overline{\mathbf{I}}^{N,\varepsilon}(t)-\overline{\mathbf{I}}^{\varepsilon}(t)\right\|_\infty\right)\longrightarrow 0\ \textit{in probability.}$$

Proof. In the proof of the theorem 5.1, we have established the following:

$$\begin{aligned}\left\|\overline{S}^{N,\varepsilon}(t)-\overline{S}^{\varepsilon}(t)\right\|_\infty&\le C\int_0^t\left\|\overline{X}^{N,\varepsilon}(s)-\overline{X}^{\varepsilon}(s)\right\|_\infty ds+\omega_S^{N,\varepsilon}(t)\\\left\|\overline{I}^{N,\varepsilon}(t)-\overline{I}^{\varepsilon}(t)\right\|_\infty&\le C\int_0^t\left\|\overline{X}^{N,\varepsilon}(s)-\overline{X}^{\varepsilon}(s)\right\|_\infty ds+\omega_I^{N,\varepsilon}(t).\end{aligned}\tag{5.20}$$

It follows that

$$\begin{aligned}\sup_{0\le t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\left\|\overline{S}^{N,\varepsilon}(t)-\overline{S}^{\varepsilon}(t)\right\|_\infty\le&\sup_{0\le t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}C\int_0^t\left\|\overline{X}^{N,\varepsilon}(s)-\overline{X}^{\varepsilon}(s)\right\|_\infty ds\\&+\sup_{0\le t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\omega_S^{N,\varepsilon}(t).\end{aligned}$$

On the other hand, from (5.19), for all $t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}$,

$$\left\|\overline{X}^{N,\varepsilon}(t)-\overline{X}^{\varepsilon}(t)\right\|_\infty\le Ce^{Ct}\int_0^t\omega^{N,\varepsilon}(s)ds+\omega^{N,\varepsilon}(t).\tag{5.21}$$

So we deduce from Lemmas 5.6, 5.7 and 5.8 and (5.10) that

$$\sup_{0\le t\le T}\left\|\overline{S}^{N,\varepsilon}(t)-\overline{S}^{\varepsilon}(t)\right\|_\infty\longrightarrow 0\ \text{ in probability as }(N,\varepsilon)\longrightarrow(\infty,0),$$

and the same is true for $\overline{I}^{N,\varepsilon}(t) - \overline{I}^{\varepsilon}(t)$. Thus the claim follows.

□

We can now state our main result.

Theorem 5.3 *For all $T > 0$, as $(N,\varepsilon) \longrightarrow (\infty, 0)$ in such a way that $N\varepsilon^d \to \infty$, we have,*

$$\forall\, t \in [0,T], \quad \left\| \overline{\mathbf{F}}^{N,\varepsilon}(t) - \overline{\mathbf{F}}(t) \right\|_\infty \longrightarrow 0, \quad \textit{in probability},$$

and

$$\sup_{0\le t\le T} \left(\left\| \overline{\mathbf{S}}^{N,\varepsilon}(t) - \overline{\mathbf{S}}(t) \right\|_\infty + \left\| \overline{\mathbf{I}}^{N,\varepsilon}(t) - \overline{\mathbf{I}}(t) \right\|_\infty \right) \longrightarrow 0 \textit{ in probability}$$

as $(N,\varepsilon) \to (\infty, 0)$ in such a way that $N\varepsilon^d \to \infty$.

Proof. By using the triangle inequality, the first statement follows from Theorem 4.1 and Theorem 5.1, and the second statement from Theorem 4.1 and Theorem 5.2.

□

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