Random walk delayed on percolation clusters

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April 18, 2010

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Abstract

We study a continuous time random walk on the d-dimensional lattice, subject to a drift and an attraction to large clusters of a subcritical Bernoulli site percolation. We find two distinct regimes: a ballistic one, and a subballistic one taking place when the attraction is strong enough. We identify the speed in the former case, and the algebraic rate of escape in the latter case. Finally, we discuss the diffusive behavior in the case of zero drift and weak attraction.

Short Title: Random walk on percolation clusters

Keywords: Random walk in random environment, subcritical percolation, anomalous transport, anomalous diffusion, environment seen from the particle, coupling

AMS 2000 subject classifications: 60K37; 60F15, 60K35, 82D30

1 Model and results

Consider the graph of nearest neighbors on $\mathbb{Z}^d$, $d \geqslant 1$, and write $x \sim y$ when $\|x-y\|_1 = 1$. Here, $\|\cdot\|_1$ is the ℓ_1-norm, though $|\cdot|$ denotes the Euclidean norm.

An environment is an element ω of $\Omega = \{0,1\}^{\mathbb{Z}^d}$. Environments are used to construct the independent identically distributed (i.i.d.) Bernoulli site percolation on the lattice. We consider the product σ-field on Ω and for $p \in (0,1)$, the probability $\mathbb{P} = \mathcal{B}(p)^{\otimes \mathbb{Z}^d}$, where $\mathcal{B}(p)$ denotes the Bernoulli law with parameter p. A site x in $\mathbb{Z}^d$ is said open if $\omega_x = 1$, and closed otherwise. Consider the open connected components (so-called clusters) in the percolation graph. The cluster of an open site $x \in \mathbb{Z}^d$ is the union of $\{x\}$ with the set of all $y \in \mathbb{Z}^d$ which are connected to x by a path with all vertices open. The cluster of a closed site is empty. We denote by C_x the cardinality of the cluster of x.

It is well known that there exists a critical $p_c = p_c(d)$ such that for $p < p_c$, $\mathbb{P}$-almost surely, all connected open components (clusters) of ω are finite, though for $p > p_c$, there a.s. exists an infinite cluster. Moreover, it follows from [1], [10] that, in the first case, the

*Partially supported by CNRS (UMR 7599 "Probabilités et Modèles Aléatoires")

clusters size has an exponential tail: For any $p < p_c$, there exists $\xi = \xi(p) > 0$ such that for all x,

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}(C_x \geqslant n) = -\xi \ .$$

In this paper, we fix $p < p_c$. Let $\ell = (\ell_k; 1 \leqslant k \leqslant d)$ be a unit vector, λ and β two non-negative number. For every environment ω, let P_ω be the law of the continuous time Markov chain $Y = (Y_t)_{t\,\geqslant 0}$ on $\mathbb{Z}^d$ starting at 0 with generator L given for continuous bounded functions f by

$$Lf(x) = K \sum_{e\sim 0} e^{\lambda \ell\cdot e - \beta C_x} \Big[f(x+e) - f(x) \Big] \ ,$$

where we chose the normalizing constant K as $K = \big(\sum_{e\sim 0} e^{\lambda \ell \cdot e} \big)^{-1}$ for simplicity. Given ω, define the measure μ on $\mathbb{Z}^d$ by

$$\mu(x) = e^{2\lambda \ell \cdot x + \beta C_x} \ . \tag{1}$$

The random measure μ combines a shift in the direction ℓ together with an attraction to large clusters. Observe that the process Y admits μ as invariant, reversible measure. Markov processes having μ as invariant measure are of natural interest in the context of random walks in random environment. They describe random walks which have a tendency to live on large clusters, the attraction becoming stronger as β is increased. The isotropic case, $\lambda = 0$, has been considered in [14] with a different, discrete-time dynamics. There, the authors proved that the walk is diffusive for small β, and subdiffusive for large β. The investigation of slowdowns in the anisotropic case is then natural. In [16], a random resistor network is considered with a invariant reversible measure of the form $C(x,\omega)e^{2\lambda\ell\cdot x}$ where the random field $(C(x,\omega); x \in \mathbb{Z}^d)$ is stationary ergodic and bounded away from 0 and $+\infty$: in this case, the random walks in random environment is ballistic for all positive λ.

The study of a general dynamics in the presence of a drift contains many difficult questions, and the advantage of the particular process Y considered here is that we can push the analysis farther. We could as well handle the discrete time analogous of Y, i.e. the random walks in random environment with geometric holding times instead of exponential ones, which falls in the class of marginally nestling walks in the standard classification (e.g., [20]). The Markov process Y can also be described with it skeleton and its jump rates. The skeleton $X = (X_n)_{n\in\mathbb{N}}$ is defined as the sequence of distinct consecutive locations visited by Y. Then, X is a discrete time Markov chain with transition probabilities $\widetilde{P}$, given for $x \in \mathbb{Z}^d$ and $e \sim 0$ by

$$\forall x \in \mathbb{Z}^d,\ \forall e \sim 0, \quad \widetilde{P}(X_{n+1} = x + e | X_n = x) = \frac{e^{\lambda\ell\cdot e}}{\sum_{e'\sim 0} e^{\lambda \ell \cdot e'}} =: \widetilde{p}_e \ ,$$

and $\widetilde{P}(X_{n+1} = y|X_n = x) = 0$ if y is not a nearest neighbor of x. The jump rate of Y at site x is equal to $\exp -\beta C_x$, and the holding times are independent, exponentially distributed with mean $\exp \beta C_x$. The Markov chain X is quite simple, it is the random walk on $\mathbb{Z}^d$ with drift

$$d(\lambda) = \frac{1}{\sum_{k=1}^d \cosh(\lambda \ell_k)} \Big(\sinh(\lambda \ell_k) \Big)_{1\,\leqslant\, k\,\leqslant\, d}. \tag{2}$$

It is plain that for the random walk,

$$\frac{X_n}{n} \longrightarrow d(\lambda) \qquad \widetilde{P} - a.s., \tag{3}$$

so directional transience is clear, and the law of large number for Y boils down to studying the clock process which takes care of the jump times. As can be seen from formula (6), the process considered here is a generalization of the so-called random walk in a random scenery, or the random walk subordinated to a renewal process, which are used as effective models for anomalous diffusions. The difference is essentially that the environment (i.e., the field of jump rates) has here some space correlations, which are short-range. It is also related to the trap model considered in the analysis of the aging phenomenon introduced in [3]: the aging of this model has been studied in details, see [4] for a recent review.

For a fixed ω, P_ω is called the quenched law and we define the annealed law P by

$$P = \mathbb{P} \times P_\omega.$$

Of course, statements which hold P-a.s., equivalently hold P_ω-a.s. for $\mathbb{P}$-a.e. environment.

Finally, we stress that we assume $d \geqslant 1$ in this paper. The case $d = 1$ is special since the critical threshold $p_c(1) = 1$. Moreover, specific techniques are available in one dimension, e.g. [20] for a survey, however we will stick as much as possible to techniques applying for all d.

Our first result is the law of large numbers.

Theorem 1. (Law of large numbers) *For any $\lambda \geqslant 0$ and any $\beta \geqslant 0$,*

$$\frac{Y_t}{t} \xrightarrow[t\to+\infty]{} v(\lambda, \beta), \quad P - a.s.,$$

where

$$v(\lambda, \beta) = \left(\mathbb{E}e^{\beta C_0}\right)^{-1} d(\lambda) \ . \tag{4}$$

In particular, $v(\lambda, \beta) = 0$ if $\beta > \xi$ or $\lambda = 0$ though $v(\lambda, \beta) \cdot \ell > 0$ if $\beta < \xi$ and $\lambda \neq 0$.

As in the case $\lambda = 0$ considered in [14], slowdowns occur for large disorder intensity β, when the walk gets trapped on large percolation clusters. This behavior is reminiscent of the biased random walk on the supercritical percolation infinite cluster [19], [2] where ballistic or subballistic regimes take place according to the parameters values. The slowdowns in our paper have a similar nature to those in some one dimensional random walks in random environment, see [18], [9] and [17]. Moreover, as in the one dimensional case, we obtain here explicit values for the rate of escape, a rather unusual fact in larger dimension. More drastic (logarithmic) slowdowns were also found for an unbiased walker in a moon craters landscape in [6], [7], or diffusions in random potentials [11], but in these models the behavior at small disorder is qualitatively different from the behavior without disorder.

The next result contains extra information on the subballistic behavior.

Theorem 2. (Subballistic regime) *Let $\beta \geqslant \xi$.*

1. *For any $d \geqslant 1$ and $\lambda > 0$,*

$$\frac{\ln |Y_t|}{\ln t} \xrightarrow[t\to+\infty]{} \frac{\xi}{\beta} \qquad P - a.s.$$

2. If $\lambda = 0$, for any $d \geqslant 2$ we have

$$\limsup_{t \to +\infty} \frac{\ln |Y_t|}{\ln t} = \frac{\xi}{2\beta} \qquad P-a.s.$$

3. If $d = 1$ and $\lambda = 0$ we have

$$\limsup_{t \to +\infty} \frac{\ln |Y_t|}{\ln t} = \frac{1}{2}\Big(\frac{\beta}{2\xi} + \frac{1}{2}\Big)^{-1} \qquad P-a.s.$$

Hence, the spread of the random walks in random environment scales algebraically with time in all cases. Note that in the isotropic case $\lambda = 0$, the slowdown is larger for $d = 1$ than for $d \geqslant 2$. This will appear in the proof as a consequence of the strong recurrence of the simple random walk X in the one-dimensional case. Note that our results are only in the logarithmic scale, though the scaling limit has been obtained for the isotropic trap model, in dimension $d = 1$ (e.g., [4]), and $d \geqslant 2$ [5] with limit given, if the disorder is strong, by the time change of a Brownian motion by the inverse of a stable subordinator (fractional kinetics). Though we believe that the scaling limit of our model without drift ($\lambda = 0$) is the same, we could not get finer results because of the presence of correlations in the medium. Moreover, the case of a drift $\lambda \neq 0$ has not been considered in the literature, except for $d = 1$ with renormalization group arguments [13].

To complete the picture, we end by the diffusive case. (Recall that $\beta < \xi$ is sufficient for $\mathbb{E}(e^{\beta C_0}) < \infty$.)

Theorem 3. (Diffusive case regime) *Assume $\lambda = 0$, and $\mathbb{E}(e^{\beta C_0}) < \infty$. Then, we have a quenched invariance principle for the rescaled process $Z^\epsilon = (Z^\epsilon_t)_{t \geqslant 0}$, $Z^\epsilon_t = \epsilon^{1/2} Y_{\epsilon^{-1}t}$: For almost every ω, as $\epsilon \searrow 0$, the family of processes Z^ϵ converges in law under P_ω in the Skorohod topology to the d-dimensional Brownian motion with diffusion matrix $\Sigma = \big(d \times \mathbb{E}(e^{\beta C_0})\big)^{-1} I_d$. Moreover,*

$$\limsup_{t \to +\infty} \frac{\ln |Y_t|}{\ln t} = \frac{1}{2} \qquad a.s. \tag{5}$$

For the proof of our results we will take the point of view of the environment seen from the walker. It turns out that the "static" environmental distribution is invariant for the dynamics. Hence the environment is always at equilibrium.

The paper is organized as follows. In the next section, we introduce the basic ingredients for our analysis and we prove the law of large numbers of Theorem 1. The last section is devoted to the subballistic regime and contains the proofs of Theorem 2 and 3.

2 Preliminaries and the proof of Theorem 1

For $x \in \mathbb{Z}^d$, T^x will denote the space shift with vector x. We will consider also the time shift θ.

Skeleton and clock process of Y. The sequence $(S_n; n \geqslant 0)$ of jump times of the Markov process Y with right-continuous paths is defined by $S_0 = 0 < S_1 < S_2 < \ldots$, $Y_t = Y_{S_n}$ for $t \in [S_n, S_{n+1})$, $Y_{S_{n+1}} \neq Y_{S_n}$. The skeleton of Y is the sequence X given by $X_n = Y_{S_n}, n \geqslant 0$. As mentioned above, the skeleton X of Y is the simple random walk with drift. For any x in $\mathbb{Z}^d$, the jump rate of $(Y_t)_{t \geqslant 0}$ at x is $e^{-\beta C_x}$. Hence the time S_n of the n-th jump is the sum of n independent random variables with exponential distribution with mean $e^{\beta C_{X_i}}, i = 1, \ldots n$. This means that the sequence $\mathcal{E} = (\mathcal{E}_i)_{i \in \mathbb{N}}$, with $\mathcal{E}_i = e^{-\beta C_{X_i}}(S_{i+1} - S_i)$, is, under the quenched law and then also under the annealed law, a sequence of i.i.d. exponential variables with mean 1, with $\mathcal{E}$ and X independent. The law of this sequence will be denoted by Q ($Q = \mathcal{E}xp(1)^{\otimes \mathbb{N}}$, with $\mathcal{E}xp(1)$ the mean 1, exponential law). For any n in $\mathbb{N}$, the time S_n of the n-th jump is given by

$$S_n = \sum_{i=0}^{n-1} \mathcal{E}_i e^{\beta C_{X_i}}. \tag{6}$$

This sequence can be view as a step function $S_t := S_{[t]}$, where $[\cdot]$ is the integer part, and we also define its generalized inverse S^{-1}: for any $t \geqslant 0$,

$$S^{-1}(t) = n \iff S_n \leqslant t < S_{n+1} .$$

We observe that $S_n \to \infty$ as $n \to \infty$ P_ω-a.s. for all ω, making the function S^{-1} defined on the whole of $\mathbb{R}_+$. Then, P_ω-a.s.,

$$X_{S^{-1}(t)} = Y(t) , \quad \forall t \geqslant 0 . \tag{7}$$

and therefore, the process S^{-1} is called the clock process.

Conversely, let $\mathcal{E}, X$ and ω be independent, with distribution $Q, \widetilde{P}$ and $\mathbb{P}$ respectively, defined on some new probability space. Then, fixing λ and viewing β as a parameter, by (6) and (7) we construct, on this new probability space, a coupling of the processes $Y = Y^{(\beta)}$ for all $\beta \in \mathbb{R}$. The coupling has the properties that the skeleton is the same for all β, and that the clock processes are such that for $\beta \geqslant \beta'$ and $t \geqslant 0$,

$$S^{-1}(\beta; t) \leqslant S^{-1}(\beta'; t). \tag{8}$$

The environment seen from the walker. Depending on the time being discrete or continuous, we consider the processes $(\widetilde{\omega}_n)_{n \in \mathbb{N}}$ and $(\hat{\omega}_t)_{t \geqslant 0}$ defined by

$$\widetilde{\omega}_n = T^{X_n} \omega , \quad \hat{\omega}_t = T^{Y_t} \omega = \widetilde{\omega}_{S^{-1}(t)}$$

for $n \geqslant 0, t \geqslant 0$. We start with the case of discrete time.

Lemma 1. *Under P, $(\widetilde{\omega}_i)_{i \in \mathbb{N}}$ is a stationary ergodic Markov chain. The same holds for $(\widetilde{\omega}_i, \mathcal{E}_i)_{i \in \mathbb{N}}$.*

Proof of Lemma 1. As $(\mathcal{E}_i)_{i \in \mathbb{N}}$ is an i.i.d. sequence of variables independent of $\widetilde{\omega}$, it is enough to prove Lemma 1 for the process $(\widetilde{\omega}_i)_{i \in \mathbb{N}}$. Under P (resp P_ω) $(\widetilde{\omega}_i)_{i \in \mathbb{N}}$ is markovian with transition kernel R defined for any bounded function f by

$$Rf(\omega) = \sum_{e \sim 0} \widetilde{p}_e f(T^e \omega) \quad \forall \omega \in \Omega,$$

and initial distribution $\mathbb{P}$ (resp δ_ω). The transitions of $(\widetilde{\omega}_i)_{i\in\mathbb{N}}$ does not depend on ω like those of X and, in this sense, the sequence is itself a random walk. Since $\mathbb{P}$ is invariant by translation,

$$E[f(\widetilde{\omega}_1)] = \int \sum_{e\sim 0} \widetilde{p}_e f(T^e\omega) d\mathbb{P} = \sum_{e\sim 0} \widetilde{p}_e \int f(T^e\omega) d\mathbb{P} = \mathbb{E}[f(\omega)],$$

showing that $\mathbb{P}$ is an invariant measure for $(\widetilde{\omega}_i)_{i\in\mathbb{N}}$.

We will use $\mathcal{F}$ to denote the product σ-field on $\Omega^{\mathbb{N}}$, and for any $k \geqslant 0$, $\mathcal{F}_k$ will denote the σ-field generated by the k first coordinates. Note that θ is measurable and preserves the law of $\widetilde{\omega}$ under P. We have to prove that the invariant σ-field $\Sigma := \{A \in \mathcal{F}, 1_A(\widetilde{\omega}) = 1_A(\theta\widetilde{\omega}),\ P\text{-a.s.}\}$ is trivial. Let Y be a Σ-measurable bounded random variable on $\Omega^{\mathbb{N}}$, we have to show that it is P-a.s. constant.

Define for all ω in Ω, $h_Y(\omega) := E_\omega[Y]$. We will study this function with standard arguments e.g. chapter 17.1.1 of [12]. Using Markov property and the θ-invariance of Y, we can show that,

$$h_Y(\widetilde{\omega}_k) = E[Y|\mathcal{F}_k] \qquad \forall k \in \mathbb{N},\ P\text{-a.s.} \tag{9}$$

As a consequence, under P, $(h_Y(\widetilde{\omega}_k))_{k\,\geqslant\,0}$ is both a stationary process and an a.s. convergent martingale, and hence it is a.s. constant. In particular,

$$Y = h_Y(\widetilde{\omega}_0) \qquad P\text{-a.s.},$$

what means that Y can be consider as a function of the first coordinate alone. The next step is to show that h_Y is $\mathbb{P}$-a.s. harmonic, that is

$$Rh_Y(\widetilde{\omega}_0) = h_Y(\widetilde{\omega}_0), \quad P\text{-a.s.}$$

It is a consequence of the following computation,

$$\begin{aligned} Rh_Y(\widetilde{\omega}_0) &= E[h_Y(\widetilde{\omega}_1)|\mathcal{F}_0] \qquad P\text{-a.s.} \\ &= E[E[Y|\mathcal{F}_1]|\mathcal{F}_0] \qquad P\text{-a.s.} \\ &= h_Y(\widetilde{\omega}_0) \qquad P\text{-a.s.}, \end{aligned}$$

where the second equality is true because of (9). We will now show that Y is invariant by translation in space. By invariance of $\mathbb{P}$ and harmonicity of h_Y, it is true that

$$\sum_{e\sim 0} \int \widetilde{p}_e (Y - Y\circ T^e)^2 d\mathbb{P} = 0.$$

For every e neighbour of 0, $\widetilde{p}_e > 0$, and the previous equation implies that, $\mathbb{P}$ almost surely $Y = Y \circ T^e$ for any $e \sim 0$. Together with the ergodicity of $\mathbb{P}$, this shows that Y is P-a.s. constant, and completes the proof. □

As a consequence of Lemma 1 and Birkhoff's ergodic theorem, for any function f in $L_1(\Omega^{\mathbb{N}})$ (or f non negative),

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\theta^k\widetilde{\omega}) \xrightarrow{n\to+\infty} E[f] \qquad P\text{- a.s.}$$

Now, we turn to the time continuous case, and we consider the empirical distribution $\frac{1}{t}\int_0^t \delta_{\hat{\omega}_s} ds$ of the environment seen from the walker up to time t. Our next result is a law of large numbers for this random probability measure. For small β, the empirical distribution converges to some limit $\mathbb{P}^0$, which is then an invariant measure for $(\hat{\omega}_t)_{t\geqslant 0}$.

Corollary 1. *If $\beta < \xi$ then P-almost surely, the empirical distribution of the environment seen from the walker, $\frac{1}{t}\int_0^t \delta_{\hat{\omega}_s} ds$, converges weakly to $\mathbb{P}^0$ defined by $d\mathbb{P}^0 = \frac{e^{\beta C}}{\mathbb{E}[e^{\beta C}]} d\mathbb{P}$.*

Proof of Corollary 1. We need to show that $t^{-1}\int_0^t f(\hat{\omega}_s)ds \to \int f d\mathbb{P}^0$ as $t \to \infty$, for all real bounded continuous function f on Ω. Since $e^{\beta C_0}$ is integrable when $\beta < \xi$, this follows from the convergence along the sequence $t = S_n$, $n \to \infty$. By (6), this is equivalent to

$$\frac{n^{-1}\sum_{i=0}^{n-1}\mathcal{E}_i e^{\beta C_{X_i}} f(\widetilde{\omega}_i)}{n^{-1}\sum_{i=0}^{n-1}\mathcal{E}_i e^{\beta C_{X_i}}} \longrightarrow \int_\Omega f d\mathbb{P}^0 \ , \quad n \to \infty.$$

We first study the P-almost sure convergence of the denominator, i.e. of $n^{-1}S_n$. Define the real function g on $(\mathbb{R}^{\mathbb{N}}, \Omega^{\mathbb{N}})$

$$g : ((\mathcal{E}_i)_{i\in\mathbb{N}}, (\widetilde{\omega}_i)_{i\in\mathbb{N}}) \mapsto \mathcal{E}_0 e^{\beta C_0(\widetilde{\omega}_0)}$$

and note that $C_{X_n} = C_0(\widetilde{\omega}_n)$. Applying Lemma 1 and the ergodic theorem to $(\widetilde{\omega}, \mathcal{E})$ and to the non negative function g, we obtain that $n^{-1}S_n$ converges P-almost surely to $\mathbb{E}[e^{\beta C_0}]$. The numerator can be studied with the same arguments, and we obtain the claim since for $\beta < \xi$ both limits are finite. $\square$

With this in hand, we can easily complete the

Proof of Theorem 1. Write

$$\frac{Y_t}{t} = \frac{X_{S^{-1}(t)}}{S^{-1}(t))} \frac{S^{-1}(t)}{S(S^{-1}(t))} \frac{S(S^{-1}(t))}{t} \ .$$

Recall from (3) that the first factor in the right-hand side converge almost surely to $d(\lambda)$ as $t \to \infty$. In the proof of Corollary 1 we have shown that $S(S^{-1}(t))/S^{-1}(t) \to \mathbb{E}[e^{\beta C_0}]$ a.s. for $\beta < \xi$, but clearly the result remains true for all β (the limit is infinite for $\beta > \xi$). For the last factor in the right-hand side we simply observe that

$$\frac{S(S^{-1}(t))}{S(S^{-1}(t)+1)} \leqslant \frac{S(S^{-1}(t))}{t} \leqslant 1 \ , \tag{10}$$

yielding that $S(S^{-1}(t))/t$ converges P-almost surely to 1 if $\mathbb{E}[e^{\beta C_0}] < \infty$: in this case, we then conclude that Y_t/t converges P-almost surely to $v(\lambda, \beta)$ given by (4).

In the case $\mathbb{E}[e^{\beta C_0}] = \infty$, we just use the right inequality in (10) to obtain the P-almost surely convergence of Y_t/t to $v(\lambda, \beta) = 0$. $\square$

3 Subballistic regime, and the proofs of Theorem 2 and 3

We start with a few auxiliary results.

Lemma 2. *Assume $d \geqslant 2$ or $\lambda > 0$. Then, for any $\epsilon > 0$, there exists $\alpha > 0$ such that P-almost surely, we eventually have*

$$\sharp\Big\{i \leqslant n,\ C_{X_i} > (\frac{1}{\xi} - \epsilon)\ln n\Big\} \geqslant n^\alpha$$

with the notation $\sharp A$ for the cardinality of a set A.

Proof of Lemma 2. Define the range R_n as the number of points visited by $(X_i)_{i\in\mathbb{N}}$ during the first n steps. For $\lambda > 0$, there exists a constant $c_1 > 0$ such that $\widetilde{P}$-almost surely eventually $R_n > c_1 n$. For $\lambda = 0$ and $d \geqslant 2$, it is well known (see chapter 21 of [15]) that there exists a constant c_2 such that $\widetilde{P}$-almost surely eventually $R_n > c_2 \frac{n}{\ln n}$ (when $d \geqslant 3$, the walk is transient and the correct order of R_n is n). In all cases, there exists a constant $c_3 > 0$ such that under the assumptions of Lemma 2, we have $\widetilde{P}$-almost surely, eventually, $R_n > c_3 \frac{n}{\ln n}$. For a fixed n in $\mathbb{N}$, we define recursively the time T_i^n by

$$\begin{aligned}
&T_0^n = 0,\\
&T_i^n = \inf\{T_{i-1}^n < k \leqslant n,\ |X_k - X_{T_j^n}| > 2(\frac{1}{\xi} - \epsilon)\ln n,\ \forall j < i\} \qquad \forall i \geqslant 1,\\
&\inf\emptyset = +\infty.
\end{aligned}$$

Note that the balls with center $X_{T_j^n}$ and radius $(\xi^{-1} - \epsilon)\ln n$ are pairwise disjoint, and define K_n the number of such balls, i.e.

$$K_n = \max\{i \geqslant 0 :\ T_i^n < +\infty\}$$

As the cardinality of those ball is $c_4 \ln^d n$ (for some $c_4 > 0$), it follows from the previous discussion on the range that $\widetilde{P}$-almost surely, eventually, $K_n > c\frac{n}{\ln^{d+1} n}$, where c denotes a positive constant. From now on we fix a path $(X_i)_{i\geqslant 0}$ such that $K_n > c\frac{n}{\ln^{d+1} n}$ eventually. In the rest of the proof, we take n large enough so that the inequality holds. Then,

$$\begin{aligned}
&\mathbb{P}\Big(\sharp\{i \leqslant K_n,\ C_{X_{T_i^n}} \leqslant (\frac{1}{\xi} - \epsilon)\ln n\} \geqslant K_n - n^\alpha\Big)\\
&\quad = \mathbb{P}\Big(\exists I \subset \{1, \dots K_n\}, \sharp I = K_n - [n^\alpha] : \forall i \in I,\ C_{X_{T_i^n}} \leqslant (\frac{1}{\xi} - \epsilon)\ln n\ \Big)\\
&\quad \leqslant \sum_{I\subset\{1,\dots K_n\}, \sharp I = K_n - [n^\alpha]} \mathbb{P}\Big(\forall i \in I,\ C_{X_{T_i^n}} \leqslant (\frac{1}{\xi} - \epsilon)\ln n\ \Big)
\end{aligned}$$

For all j such that $0 \leqslant j \leqslant K_n - n^\alpha$, B_i^n denotes the ball with center $X_{T_i^n}$ and radius $(\frac{1}{\xi} - \epsilon)\ln n$. The event $\{C_{X_{T_i^n}} \leqslant (\frac{1}{\xi} - \epsilon)\ln n)\}$ is $\sigma\{\omega_x, x \in B_i^n\}$ measurable. As the balls B_i^n are disjoint and the environment is i.i.d.,

$$\begin{aligned}
&\mathbb{P}\Big(\sharp\{i \leqslant K_n,\ C_{X_{T_i^n}} \leqslant (\frac{1}{\xi} - \epsilon)\ln n\} \geqslant K_n - n^\alpha\Big)\\
&\quad \leqslant \binom{K_n}{[n^\alpha]}\big(1 - \mathbb{P}(C_0 > (\frac{1}{\xi} - \epsilon)\ln n)\big)^{K_n - [n^\alpha]}\\
&\quad \leqslant c_5 n^{n^\alpha}\big(1 - n^{-(1-\epsilon\xi)+o(1)}\big)^{c\frac{n}{\ln^{d+1} n} - n^\alpha},
\end{aligned}$$

for some suitable constant $c_5 > 0$. We now choose $\alpha < \min(1, \epsilon\xi)$, so that

$$\sum_n \mathbb{P}\Big(\sharp\{i \leqslant K_n,\ C_{X_{T_i}} \leqslant (\frac{1}{\xi} - \epsilon)\ln n\} \geqslant K_n - n^\alpha\Big) < \infty$$

We conclude using Borel-Cantelli's lemma. $\square$

Lemma 3. *Assume $\beta > \xi$. For $d \geqslant 2$ or $\lambda > 0$, we have $\liminf_n \frac{\ln S_n}{\ln n} \geqslant \frac{\beta}{\xi}$, P-almost surely.*

Proof of Lemma 3. Let η be a positive real number. With $\epsilon := \eta/\beta$, from Lemma 2, there exists $\alpha > 0$ such that $\widetilde{P} \otimes \mathbb{P}$-almost surely, there exists a natural number $N = N(X, \omega)$ such that for $n > N$, the set $I = \{i \leqslant n,\ C_{X_i} > (\frac{1}{\xi} - \epsilon)\ln n\}$ has cardinality $\sharp I \geqslant n^\alpha$. For $n > N$,

$$\begin{aligned} Q(S_n < n^{\beta/\xi - \eta}) &\leqslant Q(\mathcal{E}_i e^{\beta C_{X_i}} < n^{\beta/\xi-\eta},\ i \in I) \\ &\leqslant Q(\mathcal{E}_1 e^{\beta C_{X_1}} < n^{\beta/\xi-\eta})^{n^\alpha} \\ &\leqslant Q(\mathcal{E}_1 < n^{\beta\epsilon - \eta})^{n^\alpha} \\ &= (1 - e^{-1})^{n^\alpha}. \end{aligned}$$

From previous inequality, we obtain that $Q(S_n < n^{\beta/\xi-\eta})$ is the general term of a convergent series and we can use Borel-Cantelli's Lemma to conclude. $\square$

Lemma 4. *Assume $\beta > \xi$. For $d \geqslant 1$ and $\lambda \geqslant 0$, we have P-almost surely,* $\limsup_n \frac{\ln S_n}{\ln n} \leqslant \frac{\beta}{\xi}$.

Proof of Lemma 4. For any α in $(0,1)$, by subadditivity we have $(u+v)^\alpha \leqslant u^\alpha + v^\alpha$ for all positive u, v, and then

$$S_n^\alpha \leqslant \sum_{i=1}^n \mathcal{E}_i^\alpha e^{\alpha\beta C_{X_i}}.$$

Now, define the function f_α

$$\begin{aligned} f_\alpha : \quad & (\mathbb{R}^{\mathbb{N}}, \Omega^{\mathbb{N}}) && \to \mathbb{R} \\ & ((\mathcal{E}_i)_{i\in\mathbb{N}}, (\widetilde{\omega}_i)_{i\in\mathbb{N}}) && \to \mathcal{E}_0^\alpha e^{\alpha\beta C_0(\widetilde{\omega}_0)}. \end{aligned}$$

Applying Lemma 1 and the ergodic theorem to $(\widetilde{\omega}, \mathcal{E})$ with the non negative function f_α, we obtain that for any α such that $\alpha\beta < \xi$,

$$\limsup_{n\to+\infty} \frac{S_n^\alpha}{n} \leqslant \lim_{n\to+\infty} \frac{\sum_{i=1}^n \mathcal{E}_i^\alpha e^{\alpha\beta C_{X_i}}}{n} = E_Q(\mathcal{E}_1^\alpha) \times \mathbb{E}(e^{\alpha\beta C_0}) < \infty$$

almost surely. Therefore,

$$\limsup_{n\to+\infty} \frac{\ln S_n}{\ln n} < \frac{1}{\alpha}.$$

Since α is arbitrary in $(0, \xi/\beta)$, the proof is complete. $\square$

The two following lemmas deal with the one dimensional case. Notice that when $d = 1$, for all $n > 0$,

$$\mathbb{P}(C \geqslant n) = p \sum_{k=0}^{n-1} p^k p^{n-1-k} = np^n,$$

and as a consequence $\xi = -\ln p$.

Lemma 5. *Assume $\beta > \xi$. For $d = 1$ and $\lambda = 0$, we have P-almost surely,* $\limsup_n \frac{\ln S_n}{\ln n} \leqslant \frac{\beta}{2\xi} + \frac{1}{2}$.

Proof of Lemma 5. Here we need to relabel our sequence of exponential variables $(\mathcal{E}_i; i \geqslant 0)$. For $y \in \mathbb{Z}, k \in \mathbb{N}$, define $\mathcal{E}_{y,k}$ by

$$\mathcal{E}_{y,k} = \mathcal{E}_i \quad \text{with } i \text{ such that} \quad X_i = y,\ \sharp\{j : 0 \leqslant j \leqslant i, X_j = y\} = k\ ,$$

i.e. the exponential corresponding to the k-th passage at y. These new variables are a.s. well defined when $d = 1$ and $\lambda = 0$, and it is not difficult to see that the sequence $(\mathcal{E}_{y,k})_{y \in \mathbb{Z}, k \in \mathbb{N}}$ is i.i.d. with mean 1 exponential distribution, and independent of X and of ω. The number of visits of the walk to a site y at time n will be denoted by $\theta(n, y)$. We can rewrite S_n in the following way,

$$S_n = \sum_{i=0}^{n-1} e^{\beta C_{X_i}} \mathcal{E}_i = \sum_{y \in \mathbb{Z}} e^{\beta C_y} \left(\sum_{k=0}^{\theta(n,y)-1} \mathcal{E}_{y,k} \right). \tag{11}$$

Notice that for any $\eta > 0$,

$$\widetilde{P} - a.s. \text{ for } n \text{ large enough}, \quad \theta(n, y) = 0 \quad \forall y > n^{\frac{1}{2}+\eta} \tag{12}$$

(see for example Theorem 5.7 p.44 in [15]). As a consequence, we obtain that for any positive $\alpha < 1$, $\widetilde{P}$-almost surely for n large enough,

$$S_n^\alpha \leqslant \sum_{y=-n^{-\frac{1}{2}+\eta}}^{n^{\frac{1}{2}+\eta}} e^{\alpha\beta C_y} \left(\sum_{k=0}^{\theta(n,y)-1} \mathcal{E}_{y,k} \right)^\alpha.$$

Here and below, the sum $\sum_{y=a}^{b}$ with real numbers $a < b$, ranges over all $y \in \mathbb{Z}$ with $a \leqslant y \leqslant b$. Notice now that for any $\nu > 0$,

$$\widetilde{P} - a.s. \text{ for } n \text{ large enough}, \quad \sup\{\theta(n, y), y \in \mathbb{Z}\} < n^{\frac{1}{2}+\nu} \tag{13}$$

(see for example Theorem 11.3 p118 in [15]), and we obtain for such n,

$$\frac{1}{2n^{\frac{1}{2}+\eta} n^{(\frac{1}{2}+\nu)\alpha}} S_n^\alpha \leqslant \frac{1}{2n^{\frac{1}{2}+\eta}} \sum_{y=-n^{-\frac{1}{2}+\eta}}^{n^{\frac{1}{2}+\eta}} e^{\alpha\beta C_y} \Big(\frac{1}{n^{\frac{1}{2}+\nu}} \sum_{k=0}^{n^{\frac{1}{2}+\nu}} \mathcal{E}_{y,k} \Big)^\alpha. \tag{14}$$

For any y in $\mathbb{Z}$ and n in $\mathbb{N}$, we define $u_{y,n} = \frac{1}{n^{\frac{1}{2}+\nu}} \sum_{k=0}^{n^{\frac{1}{2}+\nu}} \mathcal{E}_{y,k}$. Fix $\mu > 0$, according to the large deviation principle for i.i.d. sequences, there exists $I_\mu > 0$ such that, for any y in $\mathbb{Z}$ and any n in $\mathbb{N}$,

$$Q(|u_{y,n} - 1| > \mu) \leqslant e^{-I_\mu n^{\frac{1}{2}+\nu}}.$$

Using the independence of the $(\mathcal{E}_{y,k})_{y\in\mathbb{Z},k\in\mathbb{N}}$, it is easy to check that $Q(\exists y \in [-n^{\frac{1}{2}+\eta}, n^{\frac{1}{2}+\eta}], |u_{y,n} - 1| > \mu)$ is the general term of a convergent series and using Borel-Cantelli's lemma we obtain that Q-almost surely, for n large enough and for any $-n^{\frac{1}{2}+\eta} < y < n^{\frac{1}{2}+\eta}$,

$$|u_{y,n} - 1| < \mu. \tag{15}$$

From the ergodicity of the environment, it is true that $\mathbb{P}$-almost surely,

$$\frac{1}{2n^{\frac{1}{2}+\eta}} \sum_{y=-n^{-\frac{1}{2}+\eta}}^{n^{\frac{1}{2}+\eta}} e^{\alpha\beta C_y} \xrightarrow{n\to+\infty} \mathbb{E}[e^{\alpha\beta C}]. \tag{16}$$

Using now (14),(15) and (16), we obtain that for any $\alpha < \xi/\beta$, there exists $M < +\infty$ such that, P-almost surely for n large enough,

$$S_n < M n^{\frac{1}{2\alpha}+\frac{\eta}{\alpha}+\frac{1}{2}+\nu}.$$

Since the last inequality is true for η and μ arbitrary small and α arbitrary close to ξ/β, the proof is complete. $\square$

Lemma 6. *Assume $\beta > \xi$. For $d = 1$ and $\lambda = 0$, we have P-almost surely,* $\liminf_{n\to\infty} \frac{\ln S_n}{\ln n} \geqslant \frac{\beta}{2\xi} + \frac{1}{2}$.

Proof of Lemma 6. Let η and ν be two positive real numbers. As a consequence of (12) and (13), $\widetilde{P}$-almost surely, for n large enough, at least $n^{\frac{1}{2}-\frac{\nu\xi}{4}}$ sites are visited more than $n^{\frac{1}{2}-\eta}$ times, we will denote the set of those sites by O_n. Fix now a path $(X_i)_{i\geqslant 0}$ such that for all $n \geqslant 0$, $\sharp O_n \geqslant n^{\frac{1}{2}-\frac{\nu\xi}{4}}$. As in the proof of Lemma 3, we can choose a family of $\alpha_n := \frac{n^{\frac{1}{2}-\frac{\nu\xi}{4}}}{\frac{1}{2}(\frac{1}{\xi}-\nu)\ln n}$ points $(y_i)_{i\leqslant\alpha_n}$ in O_n such that the intervals $(I_i)_{i\leqslant\alpha_n}$ centered in $(y_i)_{i\leqslant\alpha_n}$ and of length $\frac{1}{2}(\frac{1}{\xi}-\nu)$ are disjoint. If all sites of an interval are open, it will be said open, otherwise it will be said closed. Using the fact that the $(I_i)_{i\leqslant\alpha_n}$ are disjoint, we obtain that,

$$\begin{aligned}\mathbb{P}(I_i \text{ is closed, for all } i \leqslant \alpha_n) &\leqslant (1 - n^{-\frac{1}{2}(1-\nu\xi)+o(1)})^{\alpha_n} \\ &\leqslant e^{-n^{\frac{\nu\xi}{4}+o(1)}}.\end{aligned}$$

As a consequence of Borel-Cantelli's lemma we obtain that P-almost surely, for n large enough, there exists at least one site visited more than $n^{\frac{1}{2}-\eta}$ times and that belongs to a cluster of size greater than $\frac{1}{2}(\frac{1}{\xi}-\nu)\ln n$, we will note this site $\widetilde{y}_n$, and therefore,

$$S_n \geqslant \sum_{i=0}^{n^{\frac{1}{2}-\eta}} n^{\frac{\beta}{2\xi}-\nu\beta} \mathcal{E}_{\widetilde{y}_n,i}.$$

Using the large deviation upper bound similarly to the lines below (14), we obtain from the last inequality that P-almost surely, for n large enough,

$$S_n \geqslant \frac{1}{2} n^{\frac{1}{2}+\frac{\beta}{2\xi}-\nu\beta-\eta}.$$

Since ν and η can be chosen arbitrary small, this last inequality ends the proof. $\square$

Proof of Theorem 2. We first assume that $\beta > \xi$. From Lemma 3 and Lemma 4, we know that under assumptions of parts 1 or 2 of Theorem 2,

$$\lim_{n\to+\infty} \frac{\ln S_n}{\ln n} = \frac{\beta}{\xi} \qquad P - \text{a.s.}$$

From the inequalities

$$\frac{\ln S(S^{-1}(t))}{\ln S^{-1}(t)} \leqslant \frac{\ln t}{\ln S^{-1}(t)} < \frac{\ln S(S^{-1}(t)+1)}{\ln S^{-1}(t)},$$

we deduced that P-almost surely,

$$\lim_{t\to+\infty} \frac{\ln t}{\ln S^{-1}(t)} = \frac{\beta}{\xi}.$$

Applying the same arguments as above, we deduce from Lemma 5 and Lemma 6 that under assumptions of part 3 of Theorem 2,

$$\lim_{t\to+\infty} \frac{\ln t}{\ln S^{-1}(t)} = \frac{\beta}{2\xi} + \frac{1}{2}, \qquad P - \text{a.s.}$$

Write now,

$$\frac{\ln |Y_t|}{\ln t} = \frac{\ln |X_{S^{-1}(t)}|}{\ln S^{-1}(t)} \frac{\ln S^{-1}(t)}{\ln t}.$$

To conclude in the case $\beta > \xi$, note that under assumptions of part 1, $\frac{\ln |X_n|}{\ln n}$ converges $\widetilde{P}$-almost surely to 1 and under assumption of part 2 and 3, $\widetilde{P}$-almost surely, $\limsup_{n\to+\infty} \frac{\ln |X_n|}{\ln n} = \frac{1}{2}$ by the law of iterated logarithm.

To extend the results to the border case $\beta = \xi$, we use the property (8) of the coupling, which implies that the long-time limit of $\frac{\ln |Y_t|}{\ln t}$ is non-increasing in β. This completes the proof of part 1 with $\beta = \xi$. For the other parts, we use the property (5), that we will prove independently below. Again, the results claimed for $\beta = \xi$ in parts 2 and 3 follow from the monotonicity of the coupling. $\square$

Proof of Theorem 3. First observe that when $\lambda = 0$,

$$f(Y_t) - \int_0^t (2d)^{-1} e^{-\beta C_{Y_s}} \sum_{e\sim 0} \Big[f(Y_s+e) - f(Y_s) \Big] ds$$

is a P_ω-martingale for f continuous and bounded. Then, for all ω, the process Y is a square integrable martingale under the quenched law P_ω. Its bracket is the unique process

$\langle Y\rangle$ taking its values in the space of nonnegative symmetric $d \times d$ matrices such that $Y_t Y_t^* - \langle Y\rangle_t$ is a martingale and $\langle Y\rangle_0 = 0$. We easily compute

$$\langle Y\rangle_t = \int_0^t e^{-\beta C_{Y_s}} ds \times d^{-1} I_d$$

By Corollary 1, we see that the bracket Z^ϵ is such that, for all $t \geqslant 0$,

$$\begin{aligned}\langle Z^\epsilon\rangle_t &= \epsilon\langle Y\rangle_{\epsilon^{-1}t} \\ &= \epsilon\int_0^{\epsilon^{-1}t} e^{-\beta C_{Y_s}} ds \times d^{-1} I_d \\ &\longrightarrow t\Sigma \qquad \text{as } \epsilon \searrow 0\end{aligned}$$

P-a.s., and then in P_ω-probability for a.e. ω. Let us fix such an ω, and use the law P_ω. Since the martingale Z^ϵ has jumps of size $\epsilon^{-1/2}$ tending to 0 and since its bracket converges to a deterministic limit, it is well known (e.g. Theorem VIII-3.11 in [8]) that the sequence $(Z^\epsilon, \epsilon > 0)$ converges to the centered Gaussian process with variance $t\Sigma$, yielding the desired invariance principle under P_ω.

We now prove (5). Since $\lambda = 0$ we have $\limsup_n \ln|X_n|/\ln n = 1/2, \widetilde{P}$-a.s., and since $\mathbb{E}e^{\beta C_0} < \infty$ it holds a.s. $\lim_t \ln S^{-1}(t)/\ln t = 1$. This implies the claim. □

Concluding remarks: (i) Part 2 of Theorem 2 deals with the upper limit in the subdiffusive case $\lambda = 0, \beta > \xi$. We comment here on the lower limit. In dimension $d \geqslant 3$, $n^{-1/2}|X_{[ns]}|$ converges to a transient Bessel process, and it is not difficult to see that

$$\limsup_{t\to\infty} \frac{\ln|Y_t|}{\ln t} = \lim_{t\to\infty} \frac{\ln|Y_t|}{\ln t} = \xi/(2\beta)$$

In dimension $d \leqslant 2$, X is recurrent, and then $\liminf_t |Y_t| = 0$ and

$$\liminf_{t\to\infty} \frac{\ln|Y_t|}{\ln t} = -\infty$$

(ii) A natural question is: What does the environment seen from the walker look like in the subballistic case? In fact, the prominent feature is that the size of surrounding cluster is essentially the largest one which was visited so far. Consider for instance the case of positive λ. One can prove that, for $\beta > \xi$ and $\epsilon > 0$,

$$\frac{1}{t}\Big|\left\{s \in [0,t] : (\ln t)^{-1} C_{Y_s} \in [\beta^{-1} - \epsilon, \beta^{-1} + \epsilon]\right\}\Big| \longrightarrow 1$$

P-a.s. as $t \nearrow \infty$.

(iii) We end the paper with a short comment on the case when the environment is a general random field, not necessarily coming from site percolation. It is easy to check that Theorems 1 and 3, together with their proofs, remain valid for a stationary, ergodic random field $(C_x, x \in \mathbb{Z}^d)$. On the contrary, our proof of Theorem 2 uses some independence property specific to the percolation model.

Acknowledgement: We thank Marina Vachkovskaia for stimulating discussions on the model.

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