Occupation time fluctuations of particle systems

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joint work with T. Bojdecki and L. Gorostiza

Particle system in $\mathbb{R}^d$

- initial distribution of particles – given by a Poisson random measure with intensity measure ν (or ν_T)
- particle motion – standard, spherically symmetric α-stable Lévy process
- intensity of branching $V > 0$,

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- $(1+\beta)$ branching mechanism, $0 < \beta \leq 1$, with probability generating function

$$g(s) = s + \frac{1}{1+\beta}(1-s)^{1+\beta}, \ \ 0 < s < 1.$$

branching is critical – mean value $= 1$
if $\beta = 1$ – binary branching
if $\beta < 1$ – infinite variance branching

- particles move and branch independently

Empirical process

N_t: $N_t(A) =$ number of particles in the set A at time t.

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Problem

$$\int_0^{Tt} N_s \qquad ds, \qquad t \geq 0$$

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Rescaled occupation time fluctuations

$$X_T(t) = \frac{1}{F_T} \int_0^{Tt} (N_s - EN_s)ds, \qquad t \geq 0$$

where F_T is a proper norming.

Problem

Rescaled occupation time fluctuations

$$X_T(t) = \frac{1}{F_T} \int_0^{Tt} (N_s - EN_s)ds, \qquad t \geq 0$$

where F_T is a proper norming.

Problem

- find a suitable norming F_T, such that X_T converges in law as $T \to \infty$ to a nontrivial limit
- identification of the limit
- properties of the limit

Two cases

1. initial intensity measure $= \nu$ does not depend on T (only speed up the time)
2. initial intensity measure $= \nu_T = H_T\nu$, $H_T \to \infty$ (speed up the time + high density)

We consider the following initial intensity measures

- $\nu = \lambda$ (Lebesgue measure)
- $H_T\mu$, where μ finite measure, $H_T \to \infty$
- $H_T\mu_\gamma$ with

$$\mu_\gamma(dx) = \frac{dx}{1+|x|^\gamma}, \quad \gamma \geq 0$$

either $H_T \equiv 1$ or $H_T \to \infty$ sufficiently quickly

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History and related problems

- Deuschel, Wang (1994), $\alpha = 2$ (Brownian particles), Lebesgue measure, without branching
- Dawson, Gorostiza, Wakolbinger (2001), convergence of random variable $X_T(1)$, $\beta = 1$, general β only for “large” dimensions
- Iscoe (1986), for superprocesses, Lebesgue, convergence of $X_T(1)$.
- Birkner, Zähle (2007), for branching random walks on a lattice.

Notation

- $\mathcal{S}(\mathbb{R}^d)$ – the Schwartz space of smooth quickly decreasing functions
  $\mathcal{S}'(\mathbb{R}^d)$ – the space of tempered distributions on $\mathbb{R}^d$,
  It is convenient to consider X_T as a process in $\mathcal{S}'(\mathbb{R}^d)$
- p_t – the transition density of the α-stable process,
  $(\hat{p}_t(x) = e^{-t|x|^\alpha})$
  $\mathcal{T}_t$ – transition semigroup $\mathcal{T}_t f = p_t * f$,
- $\underset{c}{\Rightarrow}$ – convergence in law in $C([0,\tau], \mathcal{S}'(\mathbb{R}^d))$, for any $\tau > 0$
  $\underset{f}{\Rightarrow}$ – convergence of finite dimensional distributions

Expected value of the occupation time

By the Poisson property of the initial distribution:
$E\langle N_t,\varphi\rangle = \int_{\mathbb{R}^d} \mathcal{T}_t\varphi(x)\nu(dx)$

Proposition 1

Initial intensity measure $\nu = \mu_\gamma = \frac{dx}{1+|x|^\gamma}$,

$$E\int_0^T \langle N_t,\varphi\rangle\, dt \sim \begin{cases} T^{1-\frac{\gamma}{\alpha}} & \text{if} \quad \gamma < d, \gamma < \alpha \\ \log T & \gamma = \alpha < d \\ T^{1-\frac{d}{\alpha}}\log T & \gamma = d < \alpha \\ (\log T)^2 & \gamma = d = \alpha \\ T^{1-\frac{d}{\alpha}} & \gamma > d, d < \alpha \\ \log T & \gamma > d, d = \alpha \\ 1 & \gamma > \alpha, d > \alpha \end{cases}$$

stable random measure

Let M be independently scattered $(1+\beta)$-stable random measure on $\mathbb{R}^{d+1}$ with control measure λ_{d+1} (Lebesgue) totally skewed to the right, i.e. for any $A \in \mathcal{B}(\mathbb{R}^{d+1})$ such that $0 < \lambda_{d+1}(A) < \infty$, $M(A)$ is $(1+\beta)$-stable, with characteristic function

$$\exp\left\{-\lambda_{d+1}(A)|z|^{1+\beta}\left(1 - i(\operatorname{sgn} z)\tan\frac{\pi}{2}(1+\beta)\right)\right\},\qquad z \in \mathbb{R},$$

M is σ-additive and $M(A_j)$, $j = 1, 2, \ldots$ are independent if A_j are disjoint.

Theorem 2

Assume that the initial intensity measure is $\nu = \lambda$ $(\gamma = 0)$, *then*

(i) $\frac{\alpha}{\beta} < d < \frac{\alpha(1+\beta)}{\beta}$, $F_T^{1+\beta} = T^{(2+\beta-\frac{d}{\alpha}\beta)}$, *then* $X_T \underset{c}{\Rightarrow} K\lambda\xi$, *where*

$$\xi_t = \int_{\mathbb{R}^{d+1}} \left(1\!\!1_{[0,t]}(r) \int_r^t p_{u-r}(x)du \right) M(drdx), t \geq 0,$$

K is a constant.

ξ is well defined for $d < \frac{\alpha(1+\beta)}{\beta}$, since

$$\int_{\mathbb{R}^d} \left(\int_0^\tau p_u(x)du \right)^{1+\beta} dx < \infty, \quad \text{for any } \tau > 0.$$

$$\xi_t = \int_{\mathbb{R}^{d+1}} \left(1\!\!1_{[0,t]}(r) \int_r^t p_{u-r}(x) du \right) M(drdx), t \geq 0.$$

Remark: If $\beta = 1$ (binary branching), then ξ is a centered Gaussian process with covariance function

$$s^h + t^h - \frac{1}{2} \left[(s+t)^h + |s-t|^h \right],$$

$h = 3 - d/\alpha$ $(0 < h < 2)$ (sub-fractional Brownian motion)

Theorem

(ii) $d = \frac{\alpha(1+\beta)}{\beta}$, $F_T^{1+\beta} = T \log T$, *then* $X_T \underset{f}{\Rightarrow} K\lambda\tilde{\xi}$, *where* $\tilde{\xi}$ *is one-dimensional* $(1+\beta)$*-stable process, totally skewed to the right with stationary independent increments.*

$$e^{iz\tilde{\xi}_t} = \exp\left\{-t\,|z|^{(1+\beta)}\left(1 - i(sgnz)\tan\frac{\pi}{2}(1+\beta)\right)\right\}.$$

(iii) $d > \frac{\alpha(1+\beta)}{\beta}$, $F_T^{1+\beta} = T$, *then* $X_T \underset{f}{\Rightarrow} X$, *where* X *is a* $(1+\beta)$*-stable process with values in* $\mathcal{S}'(\mathbb{R}^d)$, *it has stationary independent increments*

$$E\exp\{i\langle X(t),\varphi\rangle\} = \exp\Big\{-Kt\int_{\mathbb{R}^d}\Big[c_\beta\varphi(x)G\varphi(x)$$

$$+\frac{V}{2}|G\varphi(x)|^{1+\beta}\Big(1-i(\mathrm{sgn}G\varphi(x))\tan\frac{\pi}{2}(1+\beta)\Big)\Big]dx\Big\},$$

$$\varphi \in \mathcal{S}(\mathbb{R}^d), t \geq 0,$$

where $\quad G\varphi(x) = \int_0^\infty \mathcal{T}_s\varphi(x)ds = C_{\alpha,d}\int_{\mathbb{R}^d}\frac{1}{|x-y|^{d-\alpha}}\varphi(y)dy,$

$$c_\beta = \begin{cases} 0 & \textit{if } 0 < \beta < 1 \\ 1 & \textit{if } \beta = 1. \end{cases}$$

Remark: If $d < \frac{\alpha}{\beta}$, $\alpha = 2$, then there is a.s. local extinction, i.e.

$$\lim_{t \to \infty} N_t(A) = 0$$

a.s. for any bounded $A \in \mathcal{B}(\mathbb{R}^d)$
Hypothesis: true also for $\alpha < 2$.

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Hypothesis: true also for $\alpha < 2$.
High density

Theorem 2(i')

Assume that $d \leq \frac{\alpha}{\beta}$, $\nu_T = H_T \lambda$, H_T *satisfies*

$$\lim_{T \to \infty} H_T^{-\beta} T^{1-\frac{d\beta}{\alpha}} = 0,$$

$F_T^{1+\beta} = H_T T^{(2+\beta-\frac{d}{\alpha}\beta)}$, *then* $X_T \underset{c}{\Rightarrow} K\lambda\xi$, *where* ξ *as in Theorem 2 (i).*

Remark. If $\beta = 1$, then ξ is a centered Gaussian process with covariance function

- for $d < \frac{\alpha}{\beta}$

  $$-s^h - t^h + \frac{1}{2}[(s+t)^h + |s-t|^h],$$

  where $h = 3 - d/\alpha$ $(2 < h < 2 + \frac{1}{2})$.
  (This function is positive definite for $(2 < h < 4)$).
- for $d = \frac{\alpha}{\beta}$

  $$\frac{1}{2}\left[(s+t)^2 \log(s+t) + (s-t)^2 \log|s-t|\right] - s^2 \log s - t^2 \log t$$

Introducing high density in Theorem 2 does not change the results, only F_T is different.

General case ($\gamma \geq 0$)

initial intensity measure: $H_T \frac{dx}{1+|x|^\gamma}$, $H_T \to \infty$ sufficiently quickly,
in some cases one can take $H_T = 1$
limits: $(1+\beta)$-stable processes X

dimension d	spatial structure	temporal structure
"low"	simple ($X = K\lambda\xi$)	complicated (long range dependence)
"critical"	simple ($X = K\lambda\xi$)	simple (independent increments)
"large"	complicated (space inhomog.)	simple (independent increments)

(ξ varies in different cases)

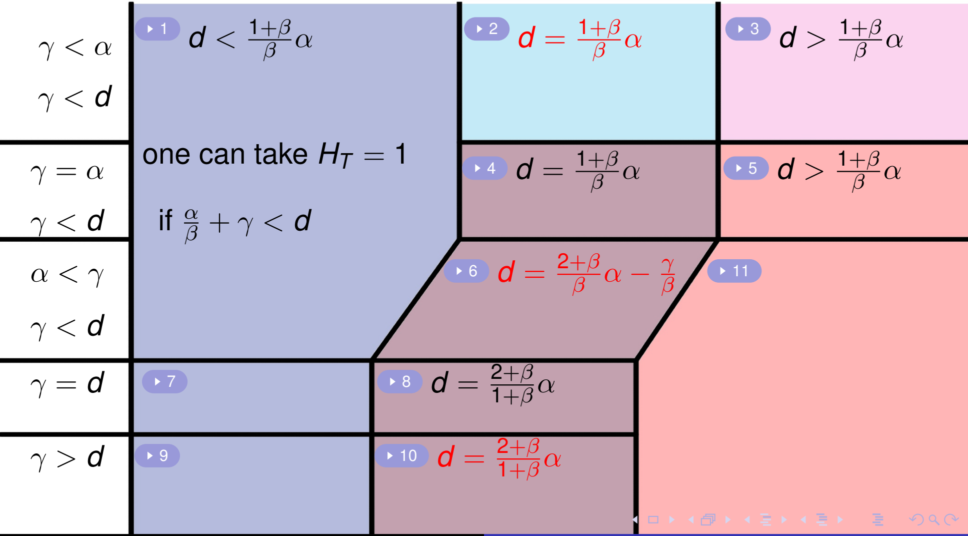
for low dimensions ξ has the form:
$\xi_t = \int_0^t \int_{\mathbb{R}^d} f(t,r,x)M(drdx)$

Critical dimension:

- Lebesgue measure case ($\gamma = 0$): $d = \frac{1+\beta}{\beta}\alpha$
- finite measure case (with high density) ($\gamma > d$): $d = \frac{2+\beta}{1+\beta}$
- $0 \leq \gamma \leq d$: $d = \frac{2+\beta}{\beta}\alpha - \frac{\gamma \vee \alpha}{\beta}$

🟪 🟦 🟫 of the form $K\lambda\xi$ 🟪 🟥 $\mathcal{S}'(\mathbb{R}^d)$ process

🟫 🟥 $X(t) = X(1), t > 0$ 🟪 lrd ▸ lrd



$\gamma < \alpha$ $\gamma < d$	▸ 1 $d < \frac{1+\beta}{\beta}\alpha$ one can take $H_T = 1$ if $\frac{\alpha}{\beta} + \gamma < d$	▸ 2 $d = \frac{1+\beta}{\beta}\alpha$	▸ 3 $d > \frac{1+\beta}{\beta}\alpha$
$\gamma = \alpha$ $\gamma < d$		▸ 4 $d = \frac{1+\beta}{\beta}\alpha$	▸ 5 $d > \frac{1+\beta}{\beta}\alpha$
$\alpha < \gamma$ $\gamma < d$		▸ 6 $d = \frac{2+\beta}{\beta}\alpha - \frac{\gamma}{\beta}$	▸ 11
$\gamma = d$	▸ 7	▸ 8 $d = \frac{2+\beta}{1+\beta}\alpha$	
$\gamma > d$	▸ 9	▸ 10 $d = \frac{2+\beta}{1+\beta}\alpha$	

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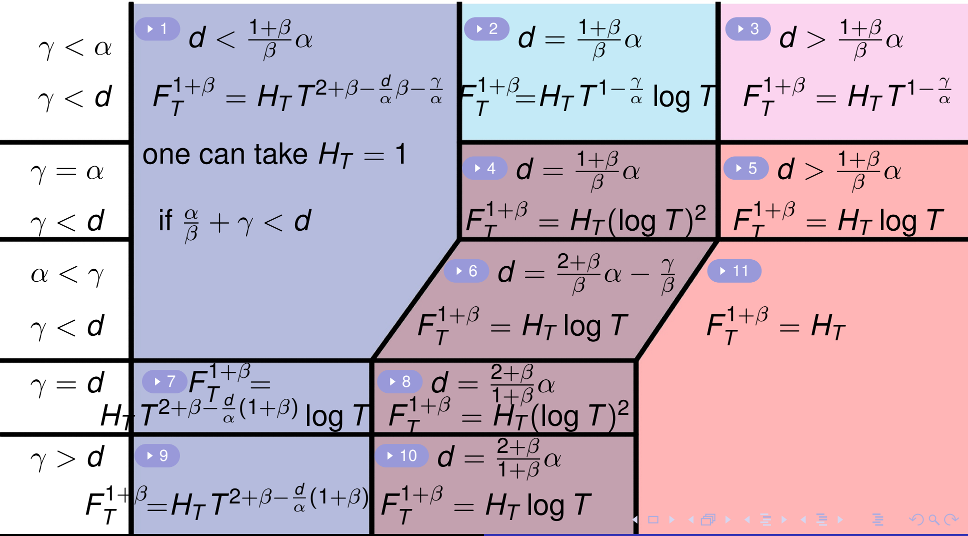

$\gamma < \alpha$, $\gamma < d$

▸ 1 $d < \frac{1+\beta}{\beta}\alpha$

$F_T^{1+\beta} = H_T T^{2+\beta-\frac{d}{\alpha}\beta-\frac{\gamma}{\alpha}}$

one can take $H_T = 1$

if $\frac{\alpha}{\beta} + \gamma < d$

▸ 2 $d = \frac{1+\beta}{\beta}\alpha$

$F_T^{1+\beta} = H_T T^{1-\frac{\gamma}{\alpha}} \log T$

▸ 3 $d > \frac{1+\beta}{\beta}\alpha$

$F_T^{1+\beta} = H_T T^{1-\frac{\gamma}{\alpha}}$

$\gamma = \alpha$, $\gamma < d$

▸ 4 $d = \frac{1+\beta}{\beta}\alpha$

$F_T^{1+\beta} = H_T (\log T)^2$

▸ 5 $d > \frac{1+\beta}{\beta}\alpha$

$F_T^{1+\beta} = H_T \log T$

$\alpha < \gamma$, $\gamma < d$

▸ 6 $d = \frac{2+\beta}{\beta}\alpha - \frac{\gamma}{\beta}$

$F_T^{1+\beta} = H_T \log T$

▸ 11

$F_T^{1+\beta} = H_T$

$\gamma = d$

▸ 7 $F_T^{1+\beta} = H_T T^{2+\beta-\frac{d}{\alpha}(1+\beta)} \log T$

▸ 8 $d = \frac{2+\beta}{1+\beta}\alpha$

$F_T^{1+\beta} = H_T (\log T)^2$

$\gamma > d$

▸ 9

$F_T^{1+\beta} = H_T T^{2+\beta-\frac{d}{\alpha}(1+\beta)}$

▸ 10 $d = \frac{2+\beta}{1+\beta}\alpha$

$F_T^{1+\beta} = H_T \log T$

Case 1: Let $\gamma < d$, $d < \frac{2+\beta}{\beta} - \frac{\gamma \vee \alpha}{\beta}$, $F_T^{1+\beta} = H_T T^{2+\beta-(d\beta+\gamma)/\alpha}$

$$\lim_{T\to\infty} T^{1-(d-\gamma)\beta/\alpha} H_T^{-\beta} = 0 \quad \text{(only needed if } d < \frac{\alpha}{\beta} + \gamma)$$

Then $X_T \underset{c}{\Rightarrow} K\lambda\xi$ where ξ

$$\xi_t = \int_{\mathbb{R}^{d+1}} \left(1\!\!1_{[0,t]}(r) \left(\int_{\mathbb{R}^d} p_r(x-y)|y|^{-\gamma} dy \right)^{1/(1+\beta)} \int_r^t p_{u-r}(x) du \right) M(drdx), \quad t \geq 0,$$

back to the general scheme

Local extinction

Theorem

If $\alpha = 2$ *and* $d < \frac{\alpha}{\beta} + \gamma$ *and the initial intensity measure is* μ_γ*, then the particle system suffers a.s. local extinction, i.e. for each bounded Borel set* A

$$P(\lim_{t\to\infty} N_t(A) = 0) = 1.$$

Hypothesis: the same is true for $\alpha < 2$

Theorem

Let $0 < \alpha \leq 2$*, initial intensity measure* $= \mu_\gamma$, $\gamma < \alpha$ *and* $d \geq \frac{\alpha}{\beta} + \gamma$*. Then there is no almost sure local extinction.*

Note: there is always extinction in probability for $\gamma > 0$ since

$$E\left\langle N_T, \varphi\right\rangle \sim \begin{cases} T^{-\frac{\gamma}{\alpha}} & \text{if} \quad \gamma < d \\ T^{-\frac{d}{\alpha}} \log T & \quad \gamma = d \\ T^{-\frac{d}{\alpha}} & \quad \gamma > d \end{cases}$$

Case 3: Assume $\gamma < d, \gamma < \alpha$, $d > \alpha\frac{1+\beta}{\beta}$, and

$$F_T^{1+\beta} = H_T T^{1-\gamma/\alpha},$$

with $H_T \geq 1$. Then $X_T \Rightarrow_f X$, where X is an $\mathcal{S}'(\mathbb{R}^d)$-valued $(1+\beta)$-stable process with independent increments

- if $\beta < 1$:

$$Ee^{i\langle X(t)-X(s),\varphi\rangle} = \exp\left\{\left(t^{1-\frac{\gamma}{\alpha}} - s^{1-\frac{\gamma}{\alpha}}\right)\right.$$
$$\left.\int_{\mathbb{R}^d} \frac{V}{2}\left|G\varphi(x)\right|^{1+\beta}\left(1 - i(\text{sgn}\, G\varphi(x))\tan\frac{\pi}{2}(1+\beta)\right)dx\right\}$$

- if $\beta = 1$, additional term with $\varphi(x)G\varphi(x)$.

back to the general scheme

Case 2: Assume $\gamma < d, \gamma < \alpha$, $d = \alpha\frac{1+\beta}{\beta}$ and

$$F_T^{1+\beta} = H_T T^{1-\gamma/\alpha} \log T,$$

with $H_T \geq 1$. Then $X_T \Rightarrow_f K\lambda\eta$, where η is a $(1+\beta)$-stable process with independent, non-stationary increments (for $\gamma > 0$) whose laws are determined by

$$Ee^{iz(\eta_t - \eta_s)} = \exp\left\{\left(t^{1-\frac{\gamma}{\alpha}} - s^{1-\frac{\gamma}{\alpha}}\right)|z|^{1+\beta} \left(1 - i(\operatorname{sgn} z)\tan\frac{\pi}{2}(1+\beta)\right)\right\}$$

$z \in \mathbb{R}, t \geq s \geq 0$.

back to the general scheme

Long range dependence

Case 1: $\gamma < d$ and $d < \frac{\alpha(2+\beta)}{\beta} - \frac{\gamma \vee \alpha}{\beta}$.

- $\beta = 1$ Gaussian case.
  Let $0 \leq u < v < s < t$. Then

$$Cov(\xi_v - \xi_u, \xi_{t+T} - \xi_{t+T}) \sim CT^{-\frac{d}{\alpha}}.$$

- $\beta < 1$ We introduce dependence exponent:
  For $0 \leq u < v < s < t$, $T > 0$, $z_1, z_2 \in \mathbb{R}$ let

$$\begin{aligned} & D_T(z_1, z_2; u, v, s, t) \\ = \quad & \big| \log Ee^{i(z_1(\xi_v - \xi_u) + z_2(\xi_{T+t} - \xi_{T+s}))} \\ & - \log Ee^{iz_1(\xi_v - \xi_u)} - \log Ee^{iz_2(\xi_{T+t} - \xi_{T+s})} \big|. \end{aligned}$$

dependence exponent κ is defined as

$$\kappa = \inf_{z_1,z_2\in\mathbb{R}} \inf_{0\leq u<v<s<t} \sup\{\gamma > 0 : D_T(z_1,z_2;u,v,s,t) = o(T^{-\gamma}) \text{ as } T\to\infty\}.$$

For the process ξ in 1

$$\xi_t = \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p_r(x-y)\,|y|^{-\gamma}\,dy\right)^{\frac{1}{1+\beta}} \int_r^t p_{u-r}(x)duM(drdx),$$

we have

$$\kappa = \begin{cases} \dfrac{d}{\alpha} & \text{if} \quad \alpha = 2 \text{ or } \beta > \dfrac{d-\gamma}{d+\alpha}, \\ \dfrac{d}{\alpha}\left(1+\beta-\dfrac{d-\gamma}{\alpha+d}\right) & \text{if} \quad \alpha < 2 \text{ and } \beta \leq \dfrac{d-\gamma}{d+\alpha}. \end{cases}$$

Cases 7 and 9: $\gamma \geq d$, $d < \alpha\frac{2+\beta}{1+\beta}$ general scheme

The limit process is of the form $K\lambda\zeta$, where

$$\zeta_t = \int_{\mathbb{R}^{d+1}} \left(1\!\!1_{[0,t]}(r) p_r^{1/(1+\beta)}(x) \int_r^t p_{u-r}(x)du \right) M(drdx), \quad t \geq 0.$$

Dependence exponent of ζ is

$$\kappa = \frac{d}{\alpha}$$

Other comments

- The same results hold for superprocesses (some differences if $\beta = 1$ in “large” dimensions)
- Questions
  - probabilistic interpretation of the two long range dependence regimes for $\gamma < d$ in “low” dimensions
  - interpretation of the “critical” dimension

Idea of the proof of convergence

1. Let $\left(\tilde{X}(t)\right)_{t\in\mathbb{R}_+}$ be an $\mathcal{S}'(\mathbb{R}^d)$-valued process. Define $\mathcal{S}'(\mathbb{R}^{d+1})$-valued random variable $\tilde{X}$ by:

$$\left\langle \tilde{X}, \Phi \right\rangle = \int_0^1 \langle X(t), \Phi(\cdot, t)\rangle \, dt, \qquad \Phi \in \mathcal{S}(\mathbb{R}^{d+1})$$

In $C([0,1], \mathcal{S}'(\mathbb{R}^d))$:
convergence $\left\langle \tilde{X}_T, \Phi \right\rangle \Rightarrow \left\langle \tilde{X}, \Phi \right\rangle$, in law $\forall \Phi \in \mathcal{S}(R^{d+1})$
+ tightness of $\langle X_T, \varphi\rangle_{T\geq 2}$ in $C([0,1], \mathbb{R})$, $\forall \varphi \in \mathcal{S}(\mathbb{R}^d)$
$\Longrightarrow$ convergence of $X_T \Rightarrow X$ in law in w $C([0,1], \mathcal{S}'(\mathbb{R}^d))$.
(Remark: in general convergence of finite dimensional disributions of $X_T \not\Leftrightarrow$ convergence in law of $\tilde{X}_T$)

2. Convergence of $\left\langle \tilde{X}, \Phi \right\rangle$ – is reduced to te case $\Phi \geq 0$.
3. The limit process is totally skewed to the right $\Rightarrow$ one can use Laplace transform.
   We show

$$Ee^{-\langle \tilde{X}_T, \Phi \rangle} = \exp\left\{ -\int_{\mathbb{R}^d} v_T(x, T)\nu_T(dx) + \frac{1}{F_T} E \int_0^T \langle N_s, \Phi \rangle \, ds \right\},$$

   where ν_T – initial intensity measure, v_T satisfies a certain integral equation.
   (Remark: a similar formula holds for finite dimensional distributions: $(\langle X_T(t_1), \varphi_1 \rangle, \ldots, \langle X_T(t_n), \varphi_n \rangle)$
   – by approximating $\sum_{j=1}^n \delta_{t_j} \varphi_j$ by $\Phi_m \in \mathcal{S}(\mathbb{R}^{d+1})$).

4.
$$Ee^{-\langle \tilde{X}_T, \Phi \rangle} \to Ee^{-\langle \tilde{X}, \Phi \rangle}, \qquad \Phi \geq 0.$$

5. Tightness (if the limit process is continuous)
   - $\beta = 1$ one can study
   $$E\left|\langle X_T(t_2) - X_T(t_1), \varphi \rangle\right|^k$$
   - $\beta \leq 1$. We show: there exist $h, \varepsilon, r > 0$ such that
   $$\begin{aligned} &P\left(\left|\langle X_T(t_2) - X_T(t_1), \varphi \rangle\right| \geq \delta\right) \\ &\leq C\delta \int_0^{\frac{1}{\delta}} \left(1 - \mathrm{Re}\, Ee^{-i\theta \langle X_T(t_2) - X_T(t_1), \varphi \rangle}\right) d\theta \\ &\leq \frac{C(\varphi)}{\delta^r} \left(t_2^h - t_1^h\right)^{1+\varepsilon} \end{aligned}$$
   for all $0 \leq t_1 \leq t_2$, $\delta > 0$.

Ad 3

Let

$$\Psi_T(x,s) = \frac{1}{F_T}\int_{\frac{s}{T}}^{1} \Phi(x,t)dt$$

$(\eta_s)_{s\geq 0}$ – standard α-stable process
g – probability generating function of the branching mechanism
N^x – empirical process for a tree starting from one particle at site x
Since N_0 is a Poisson random measure with intensity ν_T, we get

$$Ee^{-\langle \tilde{X}_T,\Phi\rangle} = \exp\left\{\int_{\mathbb{R}^d}\left(Ee^{-\int_0^T \langle N_s^x,\Psi_T(\cdot,s)\rangle ds} - 1\right)\nu_T(dx)\right\}$$
$$\times \exp\left\{\int_{\mathbb{R}^d}\int_0^T \mathcal{T}_s\Psi_T(\cdot,s)(x)ds\nu_T(dx)\right\} \qquad \text{(L)}$$

Let

$$w_T(x, r, t) := Ee^{-\int_0^T \langle N_s^x, \Psi_T(\cdot, r+s)\rangle ds}.$$

Conditioning with respect to the first branching

$$w_T(x, r, t) = e^{-Vt}\underbrace{Ee^{-\int_0^t \Psi_T(\eta_s^x, r+s)ds}}_{=: h_T(x, r, t)} + V\int_0^t e^{-V(t-s)}\underbrace{Ee^{-\int\limits_0^{t-s} \Psi_T(\eta_u^x, r+u)du} g(w_T(\eta_{t-s}^x, r+t-s, s))}_{=: k_T(x, r, s, t-s)} ds \tag{1}$$

By the Feynman Kac Formula:

$$\begin{cases} \frac{\partial}{\partial t} h_T(x,r,t) = \left(\Delta_\alpha + \frac{\partial}{\partial r} - \Psi_T(x,r)\right) h_T(x,r,t) \\ h_T(x,r,0) = 1 \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial \sigma} k_T(x,r,s,\sigma) = \left(\Delta_\alpha + \frac{\partial}{\partial r} - \Psi_T(x,r)\right) k_T(x,r,t) \\ k_T(x,r,s,0) = g(w_T(x,r,s)) \end{cases}$$

This and (1) gives

$$\begin{cases} \frac{\partial}{\partial t} w_T(x,r,t) = \left(\Delta_\alpha + \frac{\partial}{\partial r} - \Psi_T(x,r)\right) w_T(x,r,t) \\ \qquad\qquad\qquad\qquad + \frac{V}{1+\beta}\left(1 - w_T(x,r,t)\right)^{1+\beta} \\ w_T(x,r,0) = 1 \end{cases}$$

Writing the equation in the mild form and substituting $v_T(x,t) = 1 - w_T(x, T-t, t)$, $(0 \leq t \leq T)$ we obtain

$$v_T(x,t) = \int_0^t \mathcal{T}_{t-s}\left(\Psi_T(\cdot, T-t)(1 - v_T(\cdot, s)) - \frac{V}{1+\beta}(v_T(\cdot, t))^{1+\beta}\right)(x)ds,$$

From (L) we obtain

$$Ee^{-\langle \tilde{X}_T, \Phi \rangle}$$

$$= \exp\left\{\int_{\mathbb{R}^d}\left[-v_T(x,T) + \int_0^T \mathcal{T}_{T-s}\Psi_T(\cdot, T-s)(x)ds\right]\nu_T(dx)\right\}$$

$$= \exp\left\{\int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\left[\Psi_T(\cdot, T-s)v_T(\cdot, s) + (v_T(\cdot, s))^{1+\beta}\right](x)ds\nu_T(dx)\right\}$$

Usually (always for $\beta < 1$, and if $\beta = 1$ – in case of low dimensions)

$$\lim_{T\to\infty} Ee^{-\langle \tilde{X}_T, \Phi \rangle} =$$

$$\lim_{T\to\infty} \exp\left\{ \frac{V}{1+\beta} \int_0^T \mathcal{T}_{T-s} \left(\int_0^s \mathcal{T}_{s-u} \Psi_T(\cdot, T-u) \right)^{1+\beta} (x) \nu_T(dx) \right\}$$