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Exercise 1. *Let $P_1(t)$ and $P_3(t)$ be two mutually idependent Poisson processes, with rates resp. λ_1 and λ_2.*

What is the law of the process $P(t) = P_1(t) + P_2(t)$?

SOLUTION For any $k \geq 1$, any $0 = t_0 < t_1 < \cdots < t_k$, The two sequences $(P_1(t_1), P_1(t_2) - P_1(t_1), \ldots, P_1(t_k) - P_1(t_{k-1}))$ and $(P_2(t_1), P_2(t_2) - P_2(t_1), \ldots, P_2(t_k) - P_2(t_{k-1}))$ are mutually independent, and made of mutually independent r.v.'s, hence in particular the random vectors

$$\begin{pmatrix} P_1(t_j) - P_1(t_{j-1}) \\ P_2(t_j) - P_2(t_{j-1}) \end{pmatrix}, \ 1 \leq j \leq k$$

are mutually independent, and also $P_1(t_j) - P_1(t_{j-1})$ and $P_2(t_j) - P_2(t_{j-1})$ are independent, so that $P(t_j) - P(t_{j-1})$ is the sum of two independent Poisson r.v.'s, hence it is a Poisson r.v., with parameter $(\lambda_1 + \lambda_2)(t_j - t_{j-1})$, and the increments $\{P(t_j) - P(t_{j-1}),\ 1 \leq j \leq k\}$ are clearly independent. Hence we have characterized the law of $\{P(t),\ t \geq 0\}$ as that of a Poisson process with parameter $\lambda_1 + \lambda_2$.

Note that one could also consider the jumps times of $P(t)$. Let T_1^1 denote the first jump time of $P_1(t)$ and T_2^1 the first jump time of $P_2(t)$. Clearly the first jump time of $P(t)$ is $\inf(T_1^1, T_2^1)$. It is easy to see that

$$\begin{aligned} \mathbb{P}(\inf(T_1^1, T_2^1) > t) &= \mathbb{P}(T_1^1 > t, T_2^1 > t) \\ &= \mathbb{P}(T_1^1 > t)\mathbb{P}(T_2^1 > t) \\ &= e^{-\lambda_1 t} e^{-\lambda_2 t} = e^{-(\lambda_1 + \lambda_2)t}, \end{aligned}$$

where we have used the independence of T_1^1 and T_2^1, hence the first jump of $P(t)$ follows the exponential law with parameter $\lambda_1 + \lambda_2$. However, we should then prove that the further increments are mutually independent, with the correct distribution, which requires to use the strong Markov property, hence the other approach is simpler.

Exercise 2. *Let $P(t)$ be a rate λ Poisson process. What is the behaviour, as $N \to \infty$, of*

$$\frac{P(Nt) - \lambda Nt}{\sqrt{N}} \ ?$$

SOLUTION If we follow the first step of the proof of Lemma 30 in the Notes, we get that

$$X_N := \frac{P(Nt) - \lambda Nt}{\sqrt{N}} \Rightarrow \mathcal{N}(0, \lambda t).$$

Let us give an alternative proof. For any $r > 0$, we consider

$$\begin{aligned}
\mathbb{E}\left(e^{-rX_N}\right) &= e^{r\lambda\sqrt{N}t} \sum_{k=0}^{\infty} \frac{(\lambda Nt)^k}{k!} e^{-rk/\sqrt{N}} \\
&= e^{r\lambda\sqrt{N}t} e^{\lambda Nt\left(e^{-\frac{r}{\sqrt{N}}} - 1\right)} \\
&= e^{\frac{\lambda t}{2} r^2 + O\left(\frac{1}{\sqrt{N}}\right)} \\
&\to \mathbb{E}\left(e^{-rX}\right),
\end{aligned}$$

if $X \simeq \mathcal{N}(0, \lambda t)$.

Exercise 3. *Consider a SEIRS model with constant population size N, where the individuals jump from S to E upon contact with an I individual, from E to I at rate α, from I to R at rate β, from R to S at rate γ.*

a. Write the ODE and the SDE corresponding to this model.

b. Can this model have a stable endemic equilibrium ? Same question in case $\gamma = 0$ (model SEIR).

SOLUTION a. Let us first write the SDE for the numbers of individuals in the various compartments, namely $S(t), E(t), I(t), R(t)$. $P_1(t), P_2(t), P_3(t), P_4(t)$

being mutually independent standard Poisson processes,

$$
\begin{aligned}
S(t) &= S(0) - P_1\left(\frac{cp}{N}\int_0^t S(u)I(u)du\right) + P_4\left(\gamma\int_0^t R(u)du\right),\\
E(t) &= E(0) + P_1\left(\frac{cp}{N}\int_0^t S(u)I(u)du\right) - P_2\left(\alpha\int_0^t E(u)dru\right),\\
I(t) &= I(0) + P_2\left(\alpha\int_0^t E(u)du\right) - P_3\left(\beta\int_0^t I(u)du\right),\\
R(t) &= P_3\left(\beta\int_0^t I(u)du\right) - P_4\left(\gamma\int_0^t R(u)dru\right).
\end{aligned}
$$

The equations for the proportions read

$$
\begin{aligned}
s_N(t) &= s_N(0) - N^{-1}P_1\left(cpN\int_0^t s_N(u)i_N(u)du\right) + N^{-1}P_4\left(\gamma N\int_0^t r_N(u)du\right),\\
e_N(t) &= e_N(0) + N^{-1}P_1\left(cpN\int_0^t s_N(u)i_N(u)du\right) - N^{-1}P_2\left(\alpha N\int_0^t e_N(u)du\right),\\
i_N(t) &= i_N(0) + N^{-1}P_2\left(\alpha N\int_0^t e_N(u)du\right) - N^{-1}P_3\left(\beta N\int_0^t i_N(u)du\right),\\
r_N(t) &= r_N(0) + N^{-1}P_3\left(\beta N\int_0^t i_N(u)du\right) - N^{-1}P_4\left(\gamma N\int_0^t r_N(u)du\right).
\end{aligned}
$$

Finally the ODE reads

$$
\begin{aligned}
\dot{s}(t) &= -cpi(t) + \gamma r(t),\\
\dot{e}(t) &= cpi(t) - \alpha e(t),\\
\dot{i}(t) &= \alpha e(t) - \beta i(t),\\
\dot{r}(t) &= \beta i(t) - \gamma r(t)
\end{aligned}
$$

b If α and γ are very large, we are almost in the SIS model. Here $R_0 = cp/\beta$. If $R_0 > 1$, the SIS model has a stable endemic equilibrium, and our SEIRS model also has a stable endemic equilibrium, provided α and γ are large enough.

However, if $\gamma = 0$, we are in the SEIR model, all those who become infected become removed soon or later, and the worst thing that can happen is that all susceptibles become exposed, but then there will be no more susceptible and the epidemy will die out. There can't be an endemic equilibrium in that case.