AUTHOR QUERY FORM

ELSEVIER	**Journal:** YJDEQ **Article Number:** 7984	**Please e-mail your responses and any corrections to:** **E-mail:** corrections.esch@elsevier.vtex.lt

Dear Author,

Please check your proof carefully and mark all corrections at the appropriate place in the proof (e.g., by using on-screen annotation in the PDF file) or compile them in a separate list. Note: if you opt to annotate the file with software other than Adobe Reader then please also highlight the appropriate place in the PDF file. To ensure fast publication of your paper please return your corrections within 48 hours.

For correction or revision of any artwork, please consult *http://www.elsevier.com/artworkinstructions*

Any queries or remarks that have arisen during the processing of your manuscript are listed below and highlighted by flags in the proof. Click on the 'Q' link to go to the location in the proof.

Location in article	Query / Remark: click on the Q link to go Please insert your reply or correction at the corresponding line in the proof
Q1	Your article is registered as a regular item and is being processed for inclusion in a regular issue of the journal. If this is NOT correct and your article belongs to a Special Issue/Collection please contact <p.kanagaraj@elsevier.com> immediately prior to returning your corrections. (p. 1/ line 1)
Q2	Please confirm that given names and surnames have been identified correctly and are presented in the desired order. (p. 1/ line 12)
Q3	Please indicate which author(s) should be marked as 'Corresponding author'. (p. 1/ line 13)

Please check this box if you have no corrections to make to the PDF file ☐

Thank you for your assistance.

ARTICLE IN PRESS
JID:YJDEQ AID:7984 /FLA [m1+; v1.211; Prn:31/08/2015; 8:35] P.1 (1-44)
Available online at www.sciencedirect.com
ScienceDirect
Journal of Differential Equations
J. Differential Equations ••• (••••) •••–•••
www.elsevier.com/locate/jde

Stochastic variational inequalities on non-convex domains

Rainer Buckdahn [a], Lucian Maticiuc [b], Étienne Pardoux [c], Aurel Răşcanu [b,d]

[a] *Laboratoire de Mathématiques, CNRS-UMR 6205, Université de Bretagne Occidentale, 6, avenue Victor Le Gorgeu, 29238 Brest Cedex 3, France*
[b] *Faculty of Mathematics, "Alexandru Ioan Cuza" University, Carol I Blvd., no. 9, Iasi, 700506, Romania*
[c] *LATP, Université de Provence, CMI, 39, rue F. Joliot-Curie, 13453 Marseille Cedex 13, France*
[d] *"Octav Mayer" Mathematics Institute of the Romanian Academy, Iasi branch, Carol I Blvd., no. 8, Iasi, 700506, Romania*

Received 5 December 2013; revised 2 May 2015

Abstract

The objective of this work is to prove in a first step the existence and the uniqueness of a solution of the following multivalued deterministic differential equation:

$$\begin{cases} dx(t) + \partial^- \varphi(x(t))(dt) \ni dm(t), & t > 0, \\ x(0) = x_0 \,, \end{cases}$$

where $m : \mathbb{R}_+ \to \mathbb{R}^d$ is a continuous function and $\partial^- \varphi$ is the Fréchet subdifferential of a (ρ, γ)-semiconvex function φ; the domain of φ can be non-convex, but some regularities of the boundary are required.

The continuity of the map $m \mapsto x : C([0,T];\mathbb{R}^d) \to C([0,T];\mathbb{R}^d)$ associating to the input function m the solution x of the above equation, as well as tightness criteria allows to pass from the above deterministic case to the following stochastic variational inequality driven by a multi-dimensional Brownian motion:

$$\begin{cases} X_t + K_t = \xi + \int_0^t F(s, X_s)ds + \int_0^t G(s, X_s)dB_s, & t \geq 0, \\ dK_t(\omega) \in \partial^- \varphi(X_t(\omega))(dt). \end{cases}$$

E-mail addresses: rainer.buckdahn@univ-brest.fr (R. Buckdahn), lucian.maticiuc@uaic.ro (L. Maticiuc), pardoux@cmi.univ-mrs.fr (É. Pardoux), aurel.rascanu@uaic.ro (A. Răşcanu).

http://dx.doi.org/10.1016/j.jde.2015.08.023
0022-0396/

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

MSC: 60H10; 60J60

Keywords: Skorohod problem; Stochastic variational inequalities; Fréchet subdifferential

1. Introduction

Given a multi-dimensional Brownian motion $B = (B_t)$ and a proper (ρ, γ)-semiconvex function φ (for the definition the reader is referred to Definition 11 from the next section) defined over a possibly non-convex domain $\mathrm{Dom}(\varphi)$ and a random variable ξ independent of B, which takes its values in the closure of $\mathrm{Dom}(\varphi)$, we are interested in the following multi-valued stochastic differential equations (also called stochastic variational inequality) driven by the Fréchet subdifferential operator $\partial^- \varphi$:

$$\begin{cases} X_t + K_t = \xi + \int_0^t F(s, X_s)\, ds + \int_0^t G(s, X_s)\, dB_s, \ t \geq 0, \\ dK_t(\omega) \in \partial^- \varphi(X_t(\omega))(dt). \end{cases} \tag{1}$$

However, in order to study the above system, we shall first solve the following deterministic counterpart of the above equation. Given a continuous function $m : \mathbb{R}_+ \to \mathbb{R}^d$ and an initial value $x_0 \in \overline{\mathrm{Dom}(\varphi)}$ we look for a pair of continuous functions $(x, k) : \mathbb{R}_+ \to \mathbb{R}^{2d}$ such that

$$\begin{cases} x(t) + k(t) = x_0 + \int_0^t f(s, x(s))\, ds + m(t), \quad t \geq 0, \\ dk(t) \in \partial^- \varphi(x(t))(dt) \end{cases} \tag{2}$$

(for the notation $dk(t) \in \partial^- \varphi(x(t))(dt)$ see Definition 11 and Remark 15).

With the particular choice of $f \equiv 0$ and φ as convexity indicator function of a closed domain $E \subset \mathbb{R}^d$,

$$\varphi(x) = I_E(x) := \begin{cases} 0, & \text{if } x \in E, \\ +\infty, & \text{if } x \notin E, \end{cases}$$

equation (2) turns out to be just the Skorohod problem, i.e., a reflection problem, associated to the data x_0, m and E. For this reason we will refer to equation (1) as Skorohod equation.

The existence of solutions for both the Skorohod equation (2) and for the stochastic equation (1), has been well studied by different authors for the case, where φ is a convex function. In this case $\partial^- \varphi$ becomes a maximal monotone operator and the domain $\mathrm{Dom}(\varphi)$ in which the solution is kept is convex. By replacing $\partial^- \varphi$ by a general maximal monotone operator A, E. Cépa

generalized in [14] the above equation in the finite dimensional case, while A. Răşcanu [32] investigated the infinite dimensional case.

Deterministic variational inequalities with regular inputs, i.e., deterministic equations of type (2), with convex φ and $m = 0$ have been well studied and the corresponding results have by now become classical. As concerns the non-convex framework, the reader is referred to A. Marino, M. Tosques [26], M. Degiovanni, A. Marino, M. Tosques [19], A. Marino, C. Saccon, M. Tosques [25] or R. Rossi, G. Savaré [36,37]. They used the concept of ϕ-convexity (see, e.g., [26, Definition 4.1]) and they provide the existence, uniqueness and continuous dependence on the initial data of the solutions for the evolution equation of type (2) in the case $m = 0$, $f = 0$ (or $f(t, x(t)) = f(t)$ in [37]). We specify that our notion of (ρ, γ)-semiconvex function corresponds to the particular case of a ϕ-convex function of order $r = 1$ with $\phi(x, y, \varphi(x), \varphi(y), \alpha) := \rho + \gamma|\alpha|$. This particular form was required by the presence of the singular input dm/dt.

Related to our problem is the research on non-convex differential inclusions, see, e.g., F. Papalini [30], T. Cardinali, G. Colombo, F. Papalini, M. Tosques [10] and A. Cernea, V. Staicu [15]. In [15] it is proved the existence of a solution for the Cauchy problem

$$x' \in -\partial^- \varphi(x) + F(x) + f(t, x), \quad x(0) = x_0,$$

where φ is a ϕ-convex function of order $r = 2$, F is an upper semicontinuous and cyclically monotone multifunction and f is a Carathéodory function.

The particular case of a reflection problem, i.e., with $\varphi = I_E$, was extended to that of moving domains $E(t)$, $t \geq 0$, by considering the following problem (which solution is called *sweeping process*):

$$\begin{cases} x'(t) + N_{E(t)}(x(t)) \ni f(t, x(t)), & t \geq 0, \\ x(0) = x_0, \end{cases}$$

where $N_{E(t)}(x(t)) = \partial^- I_{E(t)}(x(t))$ is the external normal cone to $E(t)$ in $x(t)$.

This problem was introduced by J.J. Moreau (see [29]) for the case $f \equiv 0$ and convex sets $E(t)$, $t \geq 0$, and it has been intensively studied since then by several authors; see, e.g., C. Castaing [11], M.D.P. Monteiro Marques [28]. For the case of a sweeping process without the assumption of convexity on the sets $E(t)$, $t \geq 0$, we refer to [3,4,16,18]. The extension to the case $f \not\equiv 0$ was considered, e.g., in C. Castaing, M.D.P. Monteiro Marques [12] and J.F. Edmond, and L. Thibault [21] (see also the references therein). Another extension was made in [27] and [34] by considering the quasivariational sweeping process

$$\begin{cases} x'(t) + N_{E(t,x(t))}(x(t)) \ni 0, & t \geq 0, \\ x(0) = x_0. \end{cases}$$

The works mentioned above concern the case with vanishing driving force $m = 0$. Let us discuss now the case of equation (2) with singular input dm/dt. The associated reflection problem with singular input dm/dt has been investigated by A.V. Skorohod in [39] and [40] (for the particular case of $E = [0, \infty)$), by H. Tanaka in [41] (for a general convex domain E), and by P.-L. Lions and A.S. Sznitman in [24] and Y. Saisho in [38] for a non-convex domains. Generalizations from the reflection problem (with $\partial\varphi = \partial I_E$) to the case of $\partial\varphi$ for a general convex function φ, and

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

even to the case of a maximal monotone operator A, were discussed by A. Răşcanu in [32], V. Barbu and A. Răşcanu in [2] and by E. Cépa in [13] and [14].

Concerning the stochastic equations, we shall mention the papers [24] by P.-L. Lions and A.S. Sznitman and [38] by Y. Saisho, but also [20] by P. Dupuis and H. Ishii for stochastic differential equations with reflecting boundary conditions. On the other hand, A. Răşcanu [32], I. Asiminoaei and A. Răşcanu [1] as well as A. Bensoussan and A. Răşcanu [5] studied stochastic variational inequalities (1) in the convex case.

More recently, A.M. Gassous, A. Răşcanu and E. Rotenstein obtained in [22] existence and uniqueness results for stochastic variational inequalities with oblique subgradients. More precisely, in their equation the direction of reflection at the boundary of the convex domain differs from the normal direction, an effect which is caused by the presence of a multiplicative Lipschitz matrix acting on the subdifferential operator. In the authors' approach it turned out to be crucial to pass first by a study of the Skorohod problem with generalized reflection.

The objective of the present work is twofold: We generalize both the (non-)convex reflection problem as well as convex variational inequalities to non-convex variational inequalities. Some studies in this direction have been made already by A. Răşcanu and E. Rotenstein in [33]: a non-convex setup for multivalued (deterministic) differential equations driven by oblique subgradients has been established and the uniqueness and the local existence of the solution has been proven.

Our approach here in the present manuscript is heavily based on an a priori discussion of the generalized Skorohod problem (2) with $f \equiv 0$ and a (ρ, γ)-semiconvex φ. We give useful a priori estimates and prove the existence and the uniqueness of a solution (x, k) for the generalized Skorohod problem:

$$\begin{cases} x(t) + k(t) = x_0 + m(t), \quad t \geq 0, \\ dk(t) \in \partial^- \varphi(x(t))(dt), \end{cases} \tag{3}$$

where $x_0 \in \overline{\mathrm{Dom}(\varphi)}$, the input m is a continuous function starting from zero, and $\partial^- \varphi$ is the Fréchet subdifferential of a proper, lower semicontinuous and (ρ, γ)-semiconvex function φ. Here the set $\mathrm{Dom}(\varphi)$ is not necessarily convex, but however two assumptions are required:

1. $\mathrm{Dom}(\varphi)$ satisfies the *uniform exterior ball condition* (see Definition 1);
2. $\mathrm{Dom}(\varphi)$ satisfies the (γ, δ, σ)-*shifted uniform interior ball condition*, i.e.

there are some suitable constants $\gamma \geq 0$ and $\delta, \sigma > 0$ such that, for all $y \in \mathrm{Dom}(\varphi)$, there are some $\lambda_y \in (0, 1]$ and $v_y \in \mathbb{R}^d$, $|v_y| \leq 1$ with $\lambda_y - \left(|v_y| + \lambda_y\right)^2 \gamma \geq \sigma$ and

$$\overline{B}\left(x + v_y, \lambda_y\right) \subset \mathrm{Dom}(\varphi), \quad \text{for all } x \in \mathrm{Dom}(\varphi) \cap \overline{B}(y, \delta).$$

We observe that this condition is fulfilled if, in particular, the domain $\mathrm{Dom}(\varphi)$ satisfies the *uniform interior drop condition* (see Definition 5). It is worth pointing out that the shifted uniform interior ball condition is comparable with assumption (5) of P.-L. Lions, A.S. Sznitman [24] (or *Condition (B)* from [38]) (see the Remarks 20–21).

The application $m \mapsto x : C\left([0, T]; \mathbb{R}^d\right) \to C\left([0, T]; \mathbb{R}^d\right)$, which associates to the input function m the solution x of (3), will be proven to be continuous. This allows to derive the existence of a solution to the associated stochastic equation with additive noise $M = (M_t)$:

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

$$\begin{cases} X_t(\omega) + K_t(\omega) = \xi(\omega) + M_t(\omega), \ t \geq 0, \ \omega \in \Omega, \\ dK_t(\omega) \in \partial^- \varphi(X_t(\omega))(dt). \end{cases} \tag{4}$$

After having the existence, the uniqueness and properties for the equations (3) and (4), we will be able to extend the study to the more general equations (2) and (1), where f (respectively F) is continuous, satisfies a one-sided Lipschitz condition with respect to the second variable and a boundedness assumption.

The article is organized as it follows: The next section is devoted to a recall of such basic notions as those of as a semiconvex set, a (ρ, γ)-semiconvex function or a Fréchet subdifferential. Some notions, like for instance that of a (ρ, γ)-semiconvex function, are illustrated by an example. In Section 3 we give the definition of the solution to the generalized Skorohod problem (3), we prove the existence and the uniqueness of a solution (x, k) and we give some useful a priori estimations. Moreover, we extend equation (3) to the stochastic case (4). Section 4 is devoted to the proof of the both main results of Section 3. Finally, the Sections 5 and 6 study the extension of the results established in Section 3 to the equations (2) and (1). Appendix A is devoted to important auxiliary results such as applications of Fatou's Lemma, some complements concerning tightness in $C(\mathbb{R}_+; \mathbb{R}^d)$ or a very useful forward stochastic inequality, which are used in our approach.

2. Preliminaries

We introduce first some definitions and results concerning the notions of normal cone, uniform exterior ball conditions, semiconvex sets, (ρ, γ)-semiconvex functions and Fréchet subdifferential of a function.

Here and everywhere below E will be a nonempty closed subset of $\mathbb{R}^d$. Let $N_E(x)$ be the closed external normal cone of E at $x \in \mathrm{Bd}(E)$ i.e.

$$N_E(x) := \left\{ u \in \mathbb{R}^d : \lim_{\delta \searrow 0} \frac{\mathrm{d}_E(x + \delta u)}{\delta} = |u| \right\},$$

where $\mathrm{d}_E(z) := \inf\{|z - x| : x \in E\}$ is the distance of a point $z \in \mathbb{R}^d$ to E.

Definition 1. Let $r_0 > 0$. We say that E satisfies the r_0-uniform exterior ball condition (we write it r_0-*UEBC* for brevity) if

$$B(x + u, r_0) \cap E = \emptyset, \quad \text{where } B(x, r) \text{ denotes the ball from } \mathbb{R}^d \text{ of center } x \text{ and radius } r$$

or equivalently if

$$N_E(x) \neq \{0\} \quad \text{for all } x \in \mathrm{Bd}(E)$$

and

$$\text{for all } x \in \mathrm{Bd}(E) \text{ and } u \in N_E(x) \text{ such that } |u| = r_0 \text{ it holds that } \mathrm{d}_E(x + u) = r_0.$$

We remark that for all $v \in N_E(x)$ with $|v| \leq r_0$ we also have $\mathrm{d}_E(x + v) = |v|$.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Definition 2. Let $\gamma \geq 0$. A set E is γ-semiconvex if for all $x \in \mathrm{Bd}(E)$ there exists $\hat{x} \in \mathbb{R}^d \setminus \{0\}$ such that

$$\langle \hat{x}, y - x \rangle \leq \gamma \left| \hat{x} \right| |y - x|^2, \quad \forall y \in E.$$

We have the following equivalence:

Lemma 3. *(See [31, Lemma 6.47].) Let $r_0 > 0$. The set E satisfies the r_0-UEBC if and only if E is $\frac{1}{2r_0}$-semiconvex.*

For a given $z \in \mathbb{R}^d$ we denote by $\Pi_E(z)$ the set of elements $x \in E$ such that $\mathrm{d}_E(z) = |z - x|$. Obviously, $\Pi_E(z)$ is nonempty since E is nonempty and closed. Moreover, under the r_0-uniform exterior ball condition, it follows that the set $\Pi_E(z)$ is a singleton for all z such that $\mathrm{d}_E(z) < r_0$. In this case $\pi_E(z)$ will denote the unique element of $\Pi_E(z)$ and it is called the projection of z on E. We recall the following well-known property of the projection.

Lemma 4. *Let the r_0-UEBC be satisfied, $\varepsilon \in (0, r_0)$ and $\overline{U}_\varepsilon(E) := \left\{ y \in \mathbb{R}^d : \mathrm{d}_E(y) \leq \varepsilon \right\}$ denoting the closed ε-neighborhood of E.*

Then:

- *the closed external normal cone of E at x is given by*

$$N_E(x) = \left\{ \hat{x} : \langle \hat{x}, y - x \rangle \leq \frac{1}{2r_0} \left| \hat{x} \right| |y - x|^2, \ \forall y \in E \right\};$$

- *the projection π_E restricted to $\overline{U}_\varepsilon(E)$ is Lipschitz with Lipschitz constant $L_\varepsilon = r_0/(r_0 - \varepsilon)$; and*
- *the function $\mathrm{d}_E^2(\cdot)$ is of class C^1 on $\overline{U}_\varepsilon(E)$ with*

$$\frac{1}{2} \nabla \mathrm{d}_E^2(z) = z - \pi_E(z) \quad \text{and} \quad z - \pi_E(z) \in N_E(\pi_E(z)), \ \forall z \in \overline{U}_\varepsilon(E).$$

Let us introduce now the notion of *drop of vertex x and running direction v*.
Let $x, v \in \mathbb{R}^d$, $r > 0$. The set

$$D_x(v, r) := \mathrm{conv}\left\{ x, \overline{B}(x + v, r) \right\} = \left\{ x + t(u - x) : u \in \overline{B}(x + v, r), \ t \in [0, 1] \right\}$$

is called (v, r)-drop of vertex x and running direction v. Remark that if $|v| \leq r$, then $D_x(v, r) = \overline{B}(x + v, r)$.

Definition 5. The set $E \subset \mathbb{R}^d$ satisfies the uniform interior drop condition if there exist $r_0, h_0 > 0$ such that for all $x \in E$ there exists $v_x \in \mathbb{R}^d$ with $|v_x| \leq h_0$ and

$$D_x(v_x, r_0) \subset E$$

(we also say that E satisfies the uniform interior (h_0, r_0)-drop condition).

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Remark 6. It is easy to see that if there exists $r_0 > 0$ such that E^c satisfies the r_0-*UEBC*, then E satisfies the uniform interior (h_0, r_0)-drop condition.

Indeed let $x \in Bd\,(E^c) = Bd\,(E)$ and $u_x \in N_{E^c}(x)$ with $|u_x| = r_0$.

Then

$$D_x(u_x, r_0/2) \subset D_x(u_x, r_0) = \overline{B}(x + u_x, r_0) \subset E.$$

It is easy to see that, for any $x \in int\,(E)$, there exists a direction $v_x \in \mathbb{R}^d$ such that $|v_x| \leq r_0$ and $D_x(v_x, r_0/2) \subset E$.

We state below that the drop condition implies a weaker condition, but is not equivalent with this (for the proof see Proposition 4.35 in [31]).

Proposition 7. *Let the set E be as above with* $\mathrm{Int}\,(E) \neq \emptyset$. *If set E satisfies the uniform interior (h_0, r_0)-drop condition then E satisfies the shifted uniform interior ball condition, which means that there exist $\gamma \geq 0$ and $\delta, \sigma > 0$, and for every $y \in E$ there exist $\lambda_y \in (0, 1]$ and $v_y \in \mathbb{R}^d$, $|v_y| \leq 1$ such that*

$$\begin{aligned} &(i) \quad \lambda_y - \left(|v_y| + \lambda_y\right)^2 \gamma \geq \sigma, \\ &(ii) \quad \overline{B}\left(x + v_y, \lambda_y\right) \subset E, \quad \forall\, x \in E \cap \overline{B}(y, \delta) \end{aligned} \tag{5}$$

(this condition will be called (γ, δ, σ)-SUIBC).

Example 8. Let E be a set for which there exists a function $\phi \in C_b^2\left(\mathbb{R}^d\right)$ such that

$$\begin{aligned} &(i) \quad E = \left\{x \in \mathbb{R}^d : \phi(x) \leq 0\right\}, \\ &(ii) \quad \mathrm{Int}\,(E) = \left\{x \in \mathbb{R}^d : \phi(x) < 0\right\}, \\ &(iii) \quad \mathrm{Bd}\,(E) = \left\{x \in \mathbb{R}^d : \phi(x) = 0\right\} \text{ and } |\nabla\phi(x)| = 1,\ \forall\, x \in \mathrm{Bd}\,(E). \end{aligned}$$

Then the set E satisfies the uniform exterior ball condition and the uniform interior drop condition.

Indeed, using the definition of E we see that, for $x \in \mathrm{Bd}\,(E)$, the gradient $\nabla\phi(x)$ is a unit normal vector to the boundary, pointing towards the exterior of E. Therefore, for any $x \in \mathrm{Bd}\,(E)$, the normal cone is given by $N_E(x) = \{c\nabla\phi(x) : c \geq 0\}$ and $N_{E^c}(x) = \{-c\nabla\phi(x) : c \geq 0\}$.

Since $\phi(y) \leq 0 = \phi(x)$, for all $y \in E$, $x \in \mathrm{Bd}\,(E)$,

$$\begin{aligned} \langle \nabla\phi(x), y - x\rangle &= \phi(y) - \phi(x) - \int_0^1 \langle \nabla\phi(x + \lambda(y - x)) - \nabla\phi(x), y - x\rangle\, d\lambda \\ &\leq M|y - x|^2 = M|\nabla\phi(x)|\,|y - x|^2, \end{aligned}$$

which means, using Definition 2 and Lemma 3, that E satisfies $\frac{1}{2M}$-*UEBC*.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Since $\phi(y) \geq 0 = \phi(x)$, for all $y \in E^c$, $x \in \mathrm{Bd}(E)$,

$$\langle -\nabla\phi(x), y-x\rangle = -\phi(y) + \int_0^1 \langle \nabla\phi(x+\lambda(y-x)) - \nabla\phi(x), y-x\rangle\, d\lambda$$
$$\leq M|y-x|^2 = M|-\nabla\phi(x)||y-x|^2,$$

which yields that E^c satisfies $\frac{1}{2M}$-*UEBC* and consequently (see Remark 6) E satisfies the uniform interior drop condition.

If E denotes a closed subset of $\mathbb{R}^d$ let E_ε be the ε-interior of E, i.e.

$$E_\varepsilon := \{x \in E : \mathrm{d}_{E^c}(x) \geq \varepsilon\}.$$

Example 9. Let $E \subset \mathbb{R}^d$ be a closed convex set with nonempty interior. If there exists $r_0 > 0$ such that (the r_0-interior of E) $E_{r_0} \neq \emptyset$ and $h_0 = \sup_{z\in E} \mathrm{d}_{E_{r_0}}(z) < \infty$ (in particular if E is a bounded closed convex set with nonempty interior), then E satisfies the uniform interior (h_0, r_0)-drop condition.

Moreover for every $0 < \delta \leq \frac{r_0}{2(1+h_0)} \wedge 1$, E satisfies (γ, δ, σ)-*SUIBC* with $\lambda_y = \sigma = \delta$.

For the proof, let $y \in E$, $\hat{y}$ the projection of y on the set E_{r_0} and $v_y = \frac{1}{1+h_0}(\hat{y} - y)$. Hence $|\hat{y} - y| \leq h_0$, $|v_y| \leq 1$ and for all $x \in E \cap \overline{B}(y, \delta)$

$$\overline{B}(x + v_y, \delta) \subset \overline{B}\Big(y + v_y, \frac{r_0}{1+h_0}\Big) \subset \mathrm{conv}\{y, \overline{B}(\hat{y}, r_0)\} = D_y(\hat{y} - y, r_0) \subset E.$$

Let $\varphi\colon \mathbb{R}^d \to (-\infty, +\infty]$ be a function with domain defined by

$$\mathrm{Dom}(\varphi) := \Big\{v \in \mathbb{R}^d : \varphi(v) < +\infty\Big\}.$$

We recall now the definition of the Fréchet subdifferential (for this kind of subdifferential operator see, e.g., [26] and the monograph [35, cap. VIII]):

Definition 10. The Fréchet subdifferential $\partial^-\varphi$ is defined by:

a_1) $\partial^-\varphi(x) = \emptyset$, if $x \notin \mathrm{Dom}(\varphi)$ and
a_2) for $x \in \mathrm{Dom}(\varphi)$,

$$\partial^-\varphi(x) = \{\hat{x} \in \mathbb{R}^d : \liminf_{y\to x} \frac{\varphi(y) - \varphi(x) - \langle \hat{x}, y-x\rangle}{|y-x|} \geq 0\}.$$

Taking into account this definition we will say that φ is proper if the domain $\mathrm{Dom}(\varphi) \neq \emptyset$ and has no isolated points.

We set

$$\mathrm{Dom}(\partial^-\varphi) = \{x \in \mathbb{R}^d : \partial^-\varphi(x) \neq \emptyset\},$$
$$\partial^-\varphi = \{(x, \hat{x}) : x \in \mathrm{Dom}(\partial^-\varphi),\ \hat{x} \in \partial^-\varphi(x)\}.$$

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

In the particular case of the indicator function of the closed set E,

$$\varphi(x) = I_E(x) := \begin{cases} 0, & \text{if } x \in E, \\ +\infty, & \text{if } x \notin E, \end{cases}$$

the function φ is lower semicontinuous and the Fréchet subdifferential becomes

$$\partial^- I_E(x) = \left\{ \hat{x} \in \mathbb{R}^d : \limsup_{y \to x,\ y \in E} \frac{\langle \hat{x}, y - x \rangle}{|y - x|} \leq 0 \right\}.$$

Moreover, in this particular case we deduce that, for any closed subset E of a Hilbert space,

$$\partial^- I_E(x) = N_E(x) \tag{6}$$

(for the proof see Colombo and Goncharov [17]).

Definition 11. Let $\rho, \gamma \geq 0$. The function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ is said to be (ρ, γ)-semiconvex function if

a_1) $\overline{\mathrm{Dom}(\varphi)}$ is γ-semiconvex,
a_2) $\mathrm{Dom}\left(\partial^- \varphi\right) \neq \emptyset$,
a_3) For all $\left(x, \hat{x}\right) \in \partial^- \varphi$ and $y \in \mathbb{R}^d$:

$$\langle \hat{x}, y - x \rangle + \varphi(x) \leq \varphi(y) + \left(\rho + \gamma \left|\hat{x}\right|\right) |y - x|^2.$$

Remark 12. Let E be a nonempty closed subset of $\mathbb{R}^d$. We have:

1. I_E is $(0, \gamma)$-semiconvex iff E is γ-semiconvex (see (6) and Definitions 2 and 11; we also mention that in the definition of γ-semiconvexity we can take $x \in E$, but in this case $\hat{x}$ should be taken 0).
2. I_E is $(0, \gamma)$-semiconvex (or, see [26, Definition 4.1], ϕ-convex of order $r = 1$ with $\phi(x, y, \varphi(x), \varphi(y), \alpha) = \gamma |\alpha|$) iff there exist $\delta, \mu > 0$ such that $x \mapsto d_E(x) + \mu |x|^2$ is convex on $B(y, \delta)$, for any $y \in E$ (see [31, Lemma 6.47]).
3. A convex function is also (ρ, γ)-semiconvex function, for any $\rho, \gamma \geq 0$ (see Definition 11 and the definition of the subdifferential of a convex function).
4. If E is convex, then E is γ-semiconvex for any $\gamma \geq 0$ (see the supporting hyperplane Theorem 4.1.6 from [7]).
5. If E has nonempty interior and is 0-semiconvex, then E is convex (see Definition 2 and [8, Exercise 2.27]).
6. If $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ is a (ρ, γ)-semiconvex function, then there exist $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}$ such that

$$\varphi(y) + a |y|^2 + b |y|^2 + c \geq 0, \quad \text{for all } y \in \mathbb{R}^d.$$

Indeed, by Definition 11, we have, for a fixed $\left(x_0, \hat{x}_0\right) \in \partial^- \varphi : a = \rho + \gamma \left|\hat{x}_0\right|$, $b = 2a |x_0| + \left|\hat{x}_0\right|$ and $c = a |x_0|^2 + \langle \hat{x}_0, x_0 \rangle - \varphi(x_0)$.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Example 13. If the bounded set E satisfies the r_0-*UEBC* and $g \in C^1\left(\mathbb{R}^d\right)$ (or $g : \mathbb{R}^d \to \mathbb{R}$ is a convex function), then $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ given by

$$\varphi(x) := I_E(x) + g(x)$$

is a lower semicontinuous and $\left(\frac{L}{2r_0}, \frac{1}{2r_0}\right)$-semiconvex function, where $|\nabla g(x)| \leq L$, for any $x \in E$ (or $|\partial g(x)| \leq L$, for any $x \in E$). Moreover

$$|\varphi(x) - \varphi(y)| \leq L|x - y|, \ \forall x, y \in \mathrm{Dom}(\varphi) = E.$$

In order to define the solution for the deterministic problem envisaged by our work it is necessary to introduce the bounded variation function space.

Let $T > 0$, $k : [0, T] \to \mathbb{R}^d$ and $\mathcal{D}$ be the set of the partitions of the interval $[0, T]$.

Set

$$S_\Delta(k) = \sum_{i=0}^{n-1} |k(t_{i+1}) - k(t_i)|$$

and

$$\updownarrow k \updownarrow_T := \sup_{\Delta \in \mathcal{D}} S_\Delta(k), \tag{7}$$

where $\Delta : 0 = t_0 < t_1 < \cdots < t_n = T$.

Write

$$BV([0, T]; \mathbb{R}^d) = \{k : [0, T] \to \mathbb{R}^d : \updownarrow k \updownarrow_T < \infty\}.$$

The space $BV([0, T]; \mathbb{R}^d)$ equipped with the norm $||k||_{BV([0,T];\mathbb{R}^d)} := |k(0)| + \updownarrow k \updownarrow_T$ is a Banach space.

Moreover, we have the duality

$$\left(C([0, T]; \mathbb{R}^d)\right)^* = \{k \in BV([0, T]; \mathbb{R}^d) : k(0) = 0\}$$

given by the Riemann–Stieltjes integral

$$(y, k) \mapsto \int_0^T \langle y(t), dk(t) \rangle.$$

We will say that a function $k \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^d)$ if, for every $T > 0$, $k \in BV([0, T]; \mathbb{R}^d)$.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

3. Generalized Skorohod problem

The aim of this section is to prove the existence and uniqueness result for the following deterministic Cauchy type differential equation:

$$\begin{cases} dx(t) + \partial^- \varphi(x(t))(dt) \ni dm(t), & t > 0, \\ x(0) = x_0, \end{cases} \tag{8}$$

where

$$\begin{aligned} &(i) \quad x_0 \in \overline{\mathrm{Dom}(\varphi)}, \\ &(ii) \quad m \in C\left(\mathbb{R}_+; \mathbb{R}^d\right),\ m(0) = 0, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\varphi : \mathbb{R}^d \to (-\infty, +\infty] \text{ is a proper lower semicontinuous} \\ &\qquad \text{and } (\rho, \gamma)\text{-semiconvex function.} \end{aligned} \tag{10}$$

Definition 14 *(Generalized Skorohod problem).* A pair (x, k) of continuous functions $x, k : \mathbb{R}_+ \to \mathbb{R}^d$, is a solution of equation (8) if

$$\begin{aligned} &(j) \quad x(t) \in \overline{\mathrm{Dom}(\varphi)},\ \forall t \geq 0 \text{ and } \varphi(x(\cdot)) \in L^1_{loc}(\mathbb{R}_+), \\ &(jj) \quad k \in BV_{loc}\left(\mathbb{R}_+; \mathbb{R}^d\right),\ k(0) = 0, \\ &(jjj) \quad x(t) + k(t) = x_0 + m(t),\ \forall t \geq 0, \\ &(jv) \quad \forall\, 0 \leq s \leq t,\ \forall y : \mathbb{R}_+ \to \mathbb{R}^d \text{ continuous:} \\ &\qquad \int_s^t \langle y(r) - x(r), dk(r) \rangle + \int_s^t \varphi(x(r))\, dr \leq \int_s^t \varphi(y(r))\, dr \\ &\qquad\qquad + \int_s^t |y(r) - x(r)|^2 (\rho dr + \gamma d \updownarrow k \updownarrow_r). \end{aligned} \tag{11}$$

In this case the pair (x, k) is said to be the solution of the generalized Skorohod problem $\left(\partial^- \varphi; x_0, m\right)$ (denoted by $(x, k) = \mathcal{SP}\left(\partial^- \varphi; x_0, m\right)$).

If $\varphi = I_E$ then $\partial^- \varphi = N_E$ and we say that (x, k) is solution of the Skorohod problem $(E; x_0, m)$ and we write $(x, k) = \mathcal{SP}(E; x_0, m)$.

Remark 15. The notation

$$dk(t) \in \partial^- \varphi(x(t))(dt)$$

means that $x, k : \mathbb{R}_+ \to \mathbb{R}^d$ are continuous functions satisfying conditions (11-j, jj, jv).

The next result provides an equivalent condition for (11-jv) and will be used later in the proof of the continuity of the mapping $(x_0, m) \mapsto (x, k) = \mathcal{SP}\left(\partial^- \varphi; x_0, m\right)$ and for the main existence result in the stochastic case.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Lemma 16. *We suppose that φ satisfies assumption (10) and let $x, k : \mathbb{R}_+ \to \mathbb{R}^d$ be two continuous functions satisfying (11-j, jj). Then the pair (x, k) satisfies (11-jv) if and only if there exists a continuous increasing function $A : \mathbb{R}_+ \to \mathbb{R}_+$, such that*

$$(jv') \quad \forall\, 0 \le s \le t,\ \forall y : \mathbb{R}_+ \to \mathbb{R}^d \text{ continuous:}$$
$$\int_s^t \langle y(r) - x(r), dk(r)\rangle + \int_s^t \varphi(x(r))\, dr \le \int_s^t \varphi(y(r))\, dr + \int_s^t |y(r) - x(r)|^2\, dA_r\,. \tag{12}$$

Proof. We only need to prove that (12) $\Rightarrow$ (11-*jv*).

Denote

$$Q_r := r + \updownarrow k \updownarrow_r + A_r$$

and let $\lambda, \eta : \mathbb{R}_+ \to [0,1]$ and $\theta : \mathbb{R}_+ \to \mathbb{R}^d$ with $|\theta(r)| \le 1$, for any $r \in \mathbb{R}_+$, be some measurable functions (given by the Radon–Nikodym's theorem) such that

$$dk(r) = \theta(r)\, dQ_r\,, \quad dr = \lambda(r)\, dQ_r \quad \text{and} \quad dA_r = \eta(r)\, dQ_r\,.$$

Clearly $d \updownarrow k \updownarrow_r = |\theta(r)|\, dQ_r$. From (12) we deduce that, for all $t \in \mathbb{R}_+$, $\varepsilon > 0$ and $z \in \mathrm{Dom}(\varphi)$

$$\int_{t-\varepsilon}^{t+\varepsilon} \langle z - x(r), \theta(r)\rangle\, dQ_r + \int_{t-\varepsilon}^{t+\varepsilon} \varphi(x(r))\lambda(r)\, dQ_r \le \varphi(z) \int_{t-\varepsilon}^{t+\varepsilon} \lambda(r)\, dQ_r + \int_{t-\varepsilon}^{t+\varepsilon} |z - x(r)|^2 \eta(r)\, dQ_r$$

and therefore

$$\langle z, \int_{t-\varepsilon}^{t+\varepsilon} \theta(r)\, dQ_r\rangle - \int_{t-\varepsilon}^{t+\varepsilon} \langle x(r), \theta(r)\rangle\, dQ_r + \int_{t-\varepsilon}^{t+\varepsilon} \varphi(x(r))\lambda(r)\, dQ_r \le \varphi(z) \int_{t-\varepsilon}^{t+\varepsilon} \lambda(r)\, dQ_r$$
$$+ |z|^2 \int_{t-\varepsilon}^{t+\varepsilon} \eta(r)\, dQ_r - 2\langle z, \int_{t-\varepsilon}^{t+\varepsilon} x(r)\eta(r)\, dQ_r\rangle + \int_{t-\varepsilon}^{t+\varepsilon} |x(r)|^2 \eta(r)\, dQ_r\,. \tag{13}$$

Multiplying by $\dfrac{1}{Q([t-\varepsilon, t+\varepsilon])}$ and using the Lebesgue–Besicovitch differentiation theorem, we deduce, passing to the limit in the seven above integrals, that there exists $\Gamma_1 \subset \mathbb{R}_+$ with $\int_{\Gamma_1} dQ_r = 0$, such that for all $z \in \mathrm{Dom}(\varphi)$ and $r \in \mathbb{R}_+ \setminus \Gamma_1$

$$\langle z - x(r), \theta(r)\rangle + \varphi(x(r))\lambda(r) \le \varphi(z)\lambda(r) + |z - x(r)|^2 \eta(r)\,.$$

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Hence, from the definition of the Fréchet subdifferential we obtain

$$\theta(r) \in \partial^- I_{\overline{\mathrm{Dom}(\varphi)}}(x(r)), \quad \forall r \in \Gamma_2 \setminus \Gamma_1$$

and

$$\frac{\theta(r)}{\lambda(r)} \in \partial^- \varphi(x(r)), \quad \forall r \in (\mathbb{R}_+ \setminus \Gamma_2) \setminus \Gamma_1,$$

where $\Gamma_2 = \{r \geq 0 : \lambda(r) = 0\}$ with $\int_{\Gamma_2} dr = \int_{\Gamma_2} \lambda(r)\, dQ_r = 0$.

Since $I_{\overline{\mathrm{Dom}(\varphi)}}$ is $(0,\gamma)$-semiconvex,

$$\langle y(r) - x(r), \theta(r)\rangle \leq \gamma\, |\theta(r)|\, |y(r) - x(r)|^2, \quad \forall\, r \in \Gamma_2 \setminus \Gamma_1.$$

On the other hand, since φ is a (ρ,γ)-semiconvex function, we have for any continuous $y : \mathbb{R}_+ \to \mathbb{R}^d$,

$$\Big\langle y(r) - x(r), \frac{\theta(r)}{\lambda(r)}\Big\rangle + \varphi(x(r)) \leq \varphi(y(r)) + |y(r) - x(r)|^2 \Big(\rho + \gamma \Big|\frac{\theta(r)}{\lambda(r)}\Big|\Big), \quad \forall\, r \in (\mathbb{R}_+ \setminus \Gamma_2) \setminus \Gamma_1.$$

Therefore (with the convention $0 \cdot (+\infty) = 0$) we deduce that, for all $r \in \mathbb{R}_+ \setminus \Gamma_1$,

$$\langle y(r) - x(r), \theta(r)\rangle + \varphi(x(r))\,\lambda(r) \leq \varphi(y(r))\,\lambda(r) + |y(r) - x(r)|^2 (\rho\lambda(r) + \gamma\, |\theta(r)|).$$

Integrating on $[s,t]$ with respect to the measure dQ_r we infer that (11-*jv*) holds. □

Lemma 17. *If $dk(t) \in \partial^- \varphi(x(t))\,(dt)$ and $d\hat{k}(t) \in \partial^- \varphi\big(\hat{x}(t)\big)\,(dt)$, then for all $0 \leq s \leq t$:*

$$\int_s^t |x(r) - \hat{x}(r)|^2 \Big(2\rho dr + \gamma d\,\updownarrow k \updownarrow_r + \gamma d\,\updownarrow \hat{k} \updownarrow_r\Big) + \int_s^t \big\langle x(r) - \hat{x}(r), dk(r) - d\hat{k}(r)\big\rangle \geq 0. \tag{14}$$

Proof. The conclusion follows from (11-*jv*) written for (x,k) with $y = \hat{x}$ and for $(\hat{x},\hat{k})$ with $y = x$. □

Notation 18. *Let $\|x\|_{[s,t]} := \sup_{r\in[s,t]} |x_r|$ and $\|x\|_t := \|x\|_{[0,t]}$.*

Theorem 19 *(Uniqueness). Let assumptions (9) and (10) be satisfied. If $(x,k) = \mathcal{SP}\big(\partial^-\varphi; x_0, m\big)$ and $(\hat{x},\hat{k}) = \mathcal{SP}(\partial^-\varphi; \hat{x}_0, \hat{m})$ then for all $t \geq 0$:*

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

$$\left\|x-\hat{x}\right\|_t^2 \leq 2\Big(\left|x_0-\hat{x}_0\right|^2+\left\|m-\hat{m}\right\|_t^2+2\left\|m-\hat{m}\right\|_t\|k-\hat{k}\|_t\Big) \cdot \exp(8\rho t+4\gamma\,\updownarrow k\updownarrow_t+4\gamma\,\updownarrow\hat{k}\updownarrow_t)\,. \tag{15}$$

In particular the uniqueness of the problem $\mathcal{SP}\left(\partial^{-}\varphi; x_0, m\right)$ *follows.*

Proof. We clearly have

$$\left|x(t)-m(t)-\hat{x}(t)+\hat{m}(t)\right|^2=\left|x_0-\hat{x}_0\right|^2 + 2\int_0^t \langle m(r)-\hat{m}(r), dk(r)-d\hat{k}(r)\rangle - 2\int_0^t \langle x(r)-\hat{x}(r), dk(r)-d\hat{k}(r)\rangle.$$

Using (14) it follows that

$$\tfrac{1}{2}\left|x(t)-\hat{x}(t)\right|^2-\left|m(t)-\hat{m}(t)\right|^2 \leq \left|x(t)-m(t)-\hat{x}(t)+\hat{m}(t)\right|^2 \leq \left|x_0-\hat{x}_0\right|^2+2\left\|m-\hat{m}\right\|_t\updownarrow k-\hat{k}\updownarrow_t + 2\int_0^t \left|x(r)-\hat{x}(r)\right|^2\left(2\rho dr+\gamma d\,\updownarrow k\updownarrow_r+\gamma d\,\updownarrow\hat{k}\updownarrow_r\right),$$

which implies, via Gronwall's inequality, the desired conclusion. □

To derive the uniform boundedness and the continuity of the solution of the generalized Skorohod problem we need to introduce some additional assumptions:

$$|\varphi(x)-\varphi(y)| \leq L+L|x-y|\,,\ \forall\, x, y \in \mathrm{Dom}(\varphi) \tag{16}$$

and

$$\mathrm{Dom}(\varphi)\ \text{satisfies the}\ (\gamma,\delta,\sigma)\text{-shifted uniform interior ball condition} \tag{17}$$

(for the definition of (γ,δ,σ)-*SUIBC*, see definition (5)).

We mention that assumption (16) is obviously satisfied by the function φ given in the Example 13.

Using Proposition 7 we see that assumption (17) is fulfilled if we impose that

$$\mathrm{Dom}(\varphi)\ \text{satisfies the}\ (h_0,r_0)\text{-drop condition,} \tag{18}$$

condition which can be more easily visualized.

Note that the lower semicontinuity of φ and the assumption (16) clearly yield that the $\mathrm{Dom}(\varphi)$ is a closed set, and, from the assumption (17) it can be derived that

$$\mathrm{Int}(\mathrm{Dom}(\varphi)) \neq \emptyset.$$

Remark 20. Technical condition (5) from assumption (17) will provide an estimate for the total variation of k (see Lemma 29). On the other hand, assumption (5) from P.-L. Lions and A.S. Sznitman [24] (or [38, *Condition (B)*]) has the same role (see [24, Lemma 1.2]).

In the particular case $\varphi = I_E$, as in [24], it is essentially used in the representation of the bounded variation process k and in our case it is used in the subdifferential inequality (11-*jv*). Hence assumption (17) is required by the transition from the particular case of the indicator function to the case of a general convex l.s.c. function φ.

Remark 21. We notice that assumption (18) is similar to *Condition (B') from* [38] (the uniform interior cone condition). But the running direction from the drop condition (18) is not required to be uniform with respect to the vertex, like in [38, *Condition (B')*].

In order to prove some a priori estimates, let us introduce the following notation: for $y \in C\left([0,T];\mathbb{R}^d\right)$ and $\varepsilon > 0$ write

$$\boldsymbol{\mu}_y(\varepsilon) = \varepsilon + \mathbf{m}_y(\varepsilon),$$

where $\mathbf{m}_y(\varepsilon)$ is the modulus of continuity, given by

$$\mathbf{m}_y(\varepsilon) := \sup\left\{|y(t) - y(s)| : |t-s| \le \varepsilon,\ t,s \in [0,T]\right\}.$$

The function $\boldsymbol{\mu}_y : [0,T] \to [0, \boldsymbol{\mu}_y(T)]$ is a strictly increasing continuous function and therefore the inverse function $\boldsymbol{\mu}_y^{-1} : [0, \boldsymbol{\mu}_y(T)] \to [0,T]$ is well defined and is also a strictly increasing continuous function. Using this inverse function let C be a positive constant and

$$\begin{aligned} \Delta_m &:= 1/\boldsymbol{\mu}_m^{-1}\left(\delta^2 \exp[-C(1+T+\|m\|_T)]\right), \\ C_{T,m} &:= \exp\left[C(1+T+\|m\|_T+\Delta_m)\right]. \end{aligned} \tag{19}$$

Remark 22. It is easy to prove that, for any compact subset $\mathcal{M} \subset C\left([0,T];\mathbb{R}^d\right)$,

$$\Delta_{\mathcal{M}} := \sup_{m\in\mathcal{M}} \Delta_m < \infty \quad \text{and} \quad C_{T,\mathcal{M}} := \sup_{m\in\mathcal{M}} C_{T,m} < \infty. \tag{20}$$

The main results of this section are the following two theorems whose proofs will be given in the next section:

Theorem 23. *Let assumptions (9), (10), (16) and (17) be satisfied. Then there exists a constant C, depending only on the constants from the assumptions, such that if $(x,k) = \mathcal{SP}\left(\partial^-\varphi; x_0, m\right)$ then*

$$\begin{aligned} &(a) \quad \|k\|_{BV([0,T];\mathbb{R}^d)} = \updownarrow k \updownarrow_T \le C_{T,m}\ , \\ &(b) \quad \|x\|_T \le |x_0| + C_{T,m}\ , \\ &(c) \quad |x(t) - x(s)| + \updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le C_{T,m}\cdot\sqrt{\boldsymbol{\mu}_m(t-s)}, \quad \forall\, 0 \le s \le t \le T. \end{aligned} \tag{21}$$

If moreover $\hat{m} \in C\left([0,T];\mathbb{R}^d\right)$, $\hat{x}_0 \in \overline{\mathrm{Dom}(\varphi)}$ and $(\hat{x},\hat{k}) = \mathcal{SP}(\partial^-\varphi; \hat{x}_0, \hat{m})$, then

$$\|x-\hat{x}\|_T + \|k-\hat{k}\|_T \le A\left(C_{T,m}, C_{T,\hat{m}}\right)\cdot\left[\,|x_0-\hat{x}_0| + \sqrt{\|m-\hat{m}\|_T}\right], \tag{22}$$

where A is a continuous function.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

We can now derive the following continuity result of the mapping $(x_0, m) \mapsto (x, k) = \mathcal{SP}\left(\partial^- \varphi; x_0, m\right)$.

Corollary 24. *Let assumptions (9), (10), (16) and (17) be satisfied. If* $x_{0n}, x_0 \in \overline{\mathrm{Dom}(\varphi)}$, $m_n, m \in C\left(\mathbb{R}_+; \mathbb{R}^d\right)$, $m_n(0) = 0$ *and*

$$\begin{aligned} &i) \quad (x_n, k_n) = \mathcal{SP}\left(\partial^- \varphi; x_{0n}, m_n\right), \\ &ii) \quad x_{0n} \to x_0, \\ &iii) \quad m_n \to m \text{ in } C\left([0,T]; \mathbb{R}^d\right), \ \forall T \geq 0, \end{aligned}$$

then

$$\sup_{n \in \mathbb{N}^*} \updownarrow k_n \updownarrow_T < \infty, \ \forall T \geq 0,$$

and there exist $x, k \in C\left(\mathbb{R}_+; \mathbb{R}^d\right)$ *such that, for any* $T \geq 0$,

$$\begin{aligned} &(a) \quad \|x_n - x\|_T + \|k_n - k\|_T \to 0, \\ &(b) \quad (x, k) = \mathcal{SP}\left(\partial^- \varphi; x_0, m\right). \end{aligned}$$

Proof. Let us fix arbitrary $T > 0$. The set $\mathcal{M} = \{m, m_n : n \in \mathbb{N}^*\}$ is a compact subset of $C\left([0,T]; \mathbb{R}^d\right)$. If $C_{T,m}$ is the constant defined by (19), then, using (20), it follows that

$$C_{T,\mathcal{M}} := \sup_{\nu \in \mathcal{M}} C_{T,\nu} < \infty.$$

Also

$$\boldsymbol{\mu}_{\mathcal{M}}(\varepsilon) := \sup_{\nu \in \mathcal{M}} \boldsymbol{\mu}_\nu(\varepsilon) \searrow 0, \text{ as } \varepsilon \searrow 0.$$

Let $a > 0$ be such that $|x_{0n}| \leq a$. By the estimates established in Theorem 23 we obtain: for all $n, l \in \mathbb{N}^*$ and for all $s, t \in [0,T]$, $s \leq t$,

$$\begin{aligned} &\|x_n\|_T + \updownarrow k_n \updownarrow_T \leq a + C_{T,\mathcal{M}}, \\ &|x_n(t) - x_n(s)| + \updownarrow k_n \updownarrow_t - \updownarrow k_n \updownarrow_s \leq C_{T,\mathcal{M}} \cdot \sqrt{\boldsymbol{\mu}_{\mathcal{M}}(t-s)} \end{aligned}$$

and

$$\|x_n - x_l\|_T + \|k_n - k_l\|_T \leq C_{T,\mathcal{M}} \cdot \left[|x_{0n} - x_{0l}| + \sqrt{||m_n - m_l||_T} \right].$$

Therefore there exist $x, k, A \in C\left(\mathbb{R}_+; \mathbb{R}^d\right)$ such that

$$x_n \to x, \quad k_n \to k, \text{ in } C([0,T]; \mathbb{R}^d), \text{ as } n \to \infty,$$

and, by Arzelà–Ascoli's theorem, on a subsequence denoted also with $\updownarrow k_n \updownarrow$,

$$\updownarrow k_n \updownarrow \to A, \text{ in } C([0,T]; \mathbb{R}^d), \text{ as } n \to \infty,$$

where A is an increasing function starting from zero.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Clearly, the pair (x,k) satisfies (11-j,jj,jjj) and (12), which means, using Lemma 16, that $(x,k)=\mathcal{SP}\left(\partial^{-}\varphi;x_0,m\right)$. □

Theorem 25. *Let assumptions (9), (10), (16) and (17) be satisfied.*
Then the generalized Skorohod problem

$$\begin{cases} x(t)+k(t)=x_0+m(t), & t\geq 0,\\ dk(t)\in\partial^{-}\varphi(x(t))\,(dt) \end{cases}$$

has a unique solution (x,k), in the sense of Definition 14 (and we write $(x,k)=\mathcal{SP}\left(\partial^{-}\varphi;x_0,m\right)$).

Before giving the proof of the main results, Theorems 23 and 25, let us examine the particular case of the indicator function of the closed set E (which yields the classical Skorohod problem).

If E satisfies the r_0-*UEBC*, then, by Lemmas 3 and 4 and Definition 11, the set E is $\frac{1}{2r_0}$-semiconvex and the indicator function $I_E(x)$ is a $(0,\frac{1}{2r_0})$-semiconvex function. Hence assumptions (10) and (16) are satisfied.

We write the definition of the solution of the Skorohod problem in the case of the indicator function.

Definition 26 *(Skorohod problem).* Let $E\subset\mathbb{R}^d$ be a set satisfying the r_0-*UEBC*. A pair (x,k) is a solution of the Skorohod problem if $x,k:\mathbb{R}_+\to\mathbb{R}^d$ are continuous functions and

$$\begin{aligned} &(j)\quad x(t)\in E,\\ &(jj)\quad k\in BV_{loc}\left(\mathbb{R}_+;\mathbb{R}^d\right),\ k(0)=0,\\ &(jjj)\quad x(t)+k(t)=x_0+m(t),\\ &(jv)\quad \forall\, 0\leq s\leq t\leq T,\ \forall y\in C(\mathbb{R}_+;E)\\ &\qquad \int_s^t\langle y(r)-x(r),dk(r)\rangle\leq\frac{1}{2r_0}\int_s^t|y(r)-x(r)|^2\,d\,\updownarrow k\updownarrow_r\ . \end{aligned}\tag{23}$$

The following theorem is a consequence of the main existence Theorem 25.

Theorem 27. *Let $x_0\in E$ and $m:\mathbb{R}_+\to\mathbb{R}^d$ be a continuous function such that $m(0)=0$. If E satisfies the r_0-UEBC, for some $r_0>0$, and the shifted uniform interior ball condition, then there exists a unique solution of the Skorohod problem, in the sense of Definition 26.*

Moreover, each one of the following conditions is equivalent with (23-jv):

$$(jv')\begin{cases} \updownarrow k\updownarrow_t=\int_0^t\mathbf{1}_{x(s)\in\mathrm{Bd}(E)}d\,\updownarrow k\updownarrow_s\ ,\\ k(t)=\int_0^t n_{x(s)}d\,\updownarrow k\updownarrow_s\,,\\ \qquad \text{where } n_{x(s)}\in N_E(x(s)) \text{ and } |n_{x(s)}|=1,\ d\,\updownarrow k\updownarrow_s\text{-a.e.}, \end{cases}\tag{24}$$

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

and

$$(jv'') \quad \exists\, \beta > 0 \text{ such that } \forall\, y : \mathbb{R}_+ \to E \text{ continuous,}$$
$$\int_s^t \langle y(r) - x(r), dk(r) \rangle \leq \beta \int_s^t |y(r) - x(r)|^2 \, d\updownarrow k \updownarrow_r . \tag{25}$$

Proof. The uniqueness is ensured by Theorem 25. For a direct proof it is sufficient to use inequality (15) and the next one: if $(x,k) = \mathcal{SP}(E; x_0, m)$ and $(\hat{x}, \hat{k}) = \mathcal{SP}\left(E; \hat{x}_0, \hat{m}\right)$, then from (23) we get (see also Lemma 17)

$$\langle x(t) - \hat{x}(t), dk(t) - d\hat{k}(t) \rangle + \frac{1}{2r_0} \left| x(t) - \hat{x}(t) \right|^2 (d\updownarrow k \updownarrow_t + d\updownarrow \hat{k} \updownarrow_t) \geq 0.$$

The existence is due to Theorem 25 (but, for a direct proof of the existence, with condition (23-*jv*) replaced by (24), we refer the reader to [24]).

Proof of (24) $\Longrightarrow$ (23-*jv*): using Lemma 3 we see

$$\int_s^t \langle y(r) - x(r), dk(r) \rangle = \int_s^t \left\langle y(r) - x(r), n_{x(r)} d\updownarrow k \updownarrow_r \right\rangle$$
$$= \int_s^t \left\langle y(r) - x(r), n_{x(r)} \mathbf{1}_{x(r) \in \mathrm{Bd}(E)} d\updownarrow k \updownarrow_r \right\rangle$$
$$\leq \frac{1}{2r_0} \int_s^t \left| n_{x(r)} \right| |y(r) - x(r)|^2 \mathbf{1}_{x(s) \in \mathrm{Bd}(E)} d\updownarrow k \updownarrow_r$$
$$\leq \frac{1}{2r_0} \int_s^t |y(r) - x(r)|^2 \, d\updownarrow k \updownarrow_r .$$

Clearly (23-*jv*) $\Longrightarrow$ (25).

Proof of (25) $\Longrightarrow$ (24): let $[s,t]$ be an interval such that $x(r) \in \mathrm{Int}(E)$ for all $r \in [s,t]$. Then there exists $\delta = \delta_{s,t} > 0$ such that

$$\inf_{r \in [s,t]} \mathrm{d}_{\mathrm{Bd}(E)}\left(x(r)\right) \geq \delta.$$

Let $\lambda \in [0, \delta]$ and $\alpha \in C\left([0,T]; \mathbb{R}^d\right)$ such that $\|\alpha\|_T \leq 1$. Setting $y(r) = x(r) + \lambda\alpha(r)$ in (25) we obtain

$$\int_s^t \langle \alpha(r), dk(r) \rangle \leq \beta\lambda \int_s^t d\updownarrow k \updownarrow_r .$$

Hence, passing to the limit, for $\lambda \to 0$, and taking $\sup_{\|\alpha\|_T \leq 1}$, we deduce the implication:

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

$$x(r) \in \mathrm{Int}(E),\ \forall r \in [s,t] \Longrightarrow \updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s = 0. \tag{26}$$

Let $\ell(r)$ be a measurable function such that $|\ell(r)| = 1$, $d\updownarrow k\updownarrow_r$-a.e. and

$$k(t) = \int_0^t \ell(r)\, d\updownarrow k\updownarrow_r\ .$$

Since (25) holds for all $0 \le s \le t$, we deduce, using the Lebesgue–Besicovitch theorem, that

$$\beta |y(r) - x(r)|^2 - \langle \ell(r), y(r) - x(r)\rangle \ge 0, \quad d\updownarrow k\updownarrow_r\text{-a.e.},$$

for any $y \in C([0,T];E)$.

Therefore, from Lemma 4, we infer that

$$\ell(r) \in N_E(x(r)),\ d\updownarrow k\updownarrow_r\text{-a.e.} \tag{27}$$

We have thus proved inequality (24), since we have (26) and (27). □

4. Generalized Skorohod problem: proofs

In order to prove Theorem 25 let us first prove some auxiliary results.

Let $(x,k) = \mathcal{SP}\left(\partial^- \varphi; x_0, m\right)$ and $y \in C(\mathbb{R}_+; E)$, where $E = \mathrm{Dom}(\varphi)$. From (16) and (11–*jv*) we have, for all $0 \le s \le t$,

$$\int_s^t \langle y(r) - x(r), dk(r)\rangle \le L(t-s) + L\int_s^t |y(r) - x(r)|\, dr + \int_s^t |y(r) - x(r)|^2 (\rho dr + \gamma d\updownarrow k\updownarrow_r). \tag{28}$$

Suppose that $x(r) \in \mathrm{Int}(\mathrm{Dom}(\varphi))$ for all $r \in [s,t]$, and let

$$0 < b \le \inf_{r\in[s,t]} \mathrm{d}_{\mathrm{Bd}(E)}(x(r)).$$

Write $y(r) = x(r) + \lambda b\alpha(r)$ with $\alpha \in C\left(\mathbb{R}_+; \mathbb{R}^d\right)$, $\|\alpha\|_{[s,t]} \le 1$ and $0 < \lambda < 1$. Hence the above inequality becomes, for $\lambda = \left[(1+\gamma)(1+b)^2\right]^{-1}$

$$\lambda b \int_s^t \langle \alpha(r), dk(r)\rangle \le (L + Lb)(t-s) + \lambda^2 b^2 \left[\rho(t-s) + \gamma(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s)\right]$$

$$\le \left(L + Lb + \lambda^2 b^2 \rho\right)(t-s) + \frac{\lambda b}{1+b}(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s).$$

Taking the supremum over all α such that $\|\alpha\|_{[s,t]} \le 1$ we see that

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

$$\frac{\lambda b^2}{1+b}\left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s\right) \leq \left(L + Lb + \lambda^2 b^2 \rho\right)(t-s).$$

Consequently, the following result is proved:

Lemma 28. *Let φ such that assumption (16) is satisfied and $(x,k) = \mathcal{SP}\left(\partial^- \varphi; x_0, m\right)$. If $x(r) \in \mathrm{Int}(\mathrm{Dom}(\varphi))$, for all $r \in [s,t]$, then there exists a positive constant $C = C(L, \rho, \gamma, b)$ such that*

$$\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \leq C(t-s)$$

where

$$0 < b \leq \inf_{r\in[s,t]} \mathrm{d}_{\mathrm{Bd}(E)}(x(r)).$$

More generally we have

Lemma 29. *Let φ be a (ρ,γ)-semiconvex function and $(x,k) = \mathcal{SP}\left(\partial^- \varphi; x_0, m\right)$. Assume that φ satisfies assumption (16) and set $\mathrm{Dom}(\varphi)$ satisfies assumption (17). If $0 \leq s \leq t$ and*

$$\sup_{r\in[s,t]} |x(r) - x(s)| \leq \delta,$$

then there exists $\sigma > 0$ such that

$$\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \leq \frac{1}{\sigma}|k(t) - k(s)| + \frac{3L+4\rho}{\sigma}(t-s). \tag{29}$$

Proof. Let us fix arbitrarily $\alpha \in C\left(\mathbb{R}_+; \mathbb{R}^d\right)$ such that $\|\alpha\|_{[s,t]} \leq 1$. From assumptions (16)–(17), if

$$y(r) := x(r) + v_{x(s)} + \lambda_{x(s)}\alpha(r), \quad r \in [s,t],$$

then $y(r) \in E$.

Moreover

$$|y(r) - x(r)| \leq \left|v_{x(s)}\right| + \lambda_{x(s)} \leq 2$$

and

$$|\varphi(y(r)) - \varphi(x(r))| \leq 3L.$$

From (28) we deduce that

$$\lambda_{x(s)} \int_s^t \langle \alpha(r), dk(r)\rangle \leq -\int_s^t \left\langle v_{x(s)}, dk(r)\right\rangle + (3L+4\rho)(t-s)$$
$$+ \gamma \int_s^t \left(\left|v_{x(s)}\right| + \lambda_{x(s)}\right)^2 d\updownarrow k\updownarrow_r .$$

Taking the supremum over all α such that $\|\alpha\|_{[s,t]} \leq 1$ we see, using also (5), that

$$\sigma\left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s\right) \leq |k(t) - k(s)| + (3L + 4\rho)(t - s)$$

and the lemma follows. □

Proof of Theorem 23. We denote by C, C', C'' generic constants independent of x_0, $\hat{x}_0$, m, $\hat{m}$ and T, but possibly depending on constants L, δ, σ, ρ, γ provided by the assumptions.

Step 1. *Some estimates of the modulus of continuity of the function x.*

Let $0 \leq s \leq t \leq T$.
Since

$$|x(t) - x(s) - m(t) + m(s)| = |k(t) - k(s)| \leq \updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s,$$

it follows that

$$|x(t) - x(s)| \leq |m(t) - m(s)| + \updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s.$$

We clearly have

$$|x(t) - x(s) - m(t) + m(s)|^2 = 2\int_s^t \langle m(r) - m(s), dk(r)\rangle + 2\int_s^t \langle x(s) - x(r), dk(r)\rangle.$$

From (11-*jv*) written for $y(r) \equiv x(s)$ and (16) we have

$$\int_s^t \langle x(s) - x(r), dk(r)\rangle \leq L(t - s) + L\int_s^t |x(s) - x(r)|\, dr + \int_s^t |x(s) - x(r)|^2 (\rho dr + \gamma d\updownarrow k\updownarrow_r).$$

Thus, using also the inequality $\frac{1}{2}|\alpha|^2 \leq |\alpha - \beta|^2 + |\beta|^2$, we get

$$\frac{1}{2}|x(t) - x(s)|^2 \leq |m(t) - m(s)|^2 + 2\mathbf{m}_m(t - s)\left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s\right) + C(t - s) + C\int_s^t |x(r) - x(s)|^2 (dr + d\updownarrow k\updownarrow_r),$$

and, by Gronwall's inequality,

$$|x(t)-x(s)|^2 \le \left[\mathbf{m}_m^2(t-s)+\mathbf{m}_m(t-s)\left(\updownarrow k\updownarrow_t-\updownarrow k\updownarrow_s\right)+(t-s)\right] \cdot \exp\left[C\left(1+t-s+\updownarrow k\updownarrow_t-\updownarrow k\updownarrow_s\right)\right], \tag{30}$$

for all $0 \le s \le t \le T$.

Step 2. *Estimates of the differences* $|x(t)-x(s)|$ *and* $\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s$ *under the assumption* $|x(t)-x(s)| \le \delta$.

Let $0 \le s \le r \le t \le T$ such that $|x(t)-x(s)| \le \delta$. From (29) we have

$$\begin{aligned}
&\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le C\,|k(t)-k(s)| + C(t-s)\\
&= C\,|x(t)-x(s)-m(t)+m(s)| + C(t-s)\\
&\le C\,|x(t)-x(s)| + C\mathbf{m}_m(t-s) + C\ (t-s) \le C\delta + C\boldsymbol{\mu}_m(t-s).
\end{aligned}$$

Using the estimate (30), it clearly follows that

$$|x(t)-x(s)|^2 \le \boldsymbol{\mu}_m(t-s)\exp\left[C\left(1+T+\|m\|_T\right)\right], \text{ for all } 0 \le s \le t \le T,$$

since $(t-s)+\mathbf{m}_m(t-s) = \boldsymbol{\mu}_m(t-s) \le T + 2\|m\|_T$.

Hence if $0 \le s \le t \le T$ and $|x(t)-x(s)| \le \delta$ then

$$|x(t)-x(s)| + \updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le \sqrt{\boldsymbol{\mu}_m(t-s)}\cdot\exp\left[C\left(1+T+\|m\|_T\right)\right]. \tag{31}$$

Step 3. *Adapted time partition and local estimates.*

Let the sequence given by (the definition is suggested by [24])

$$\begin{aligned}
t_0 &= T_0 = 0,\\
T_1 &= \inf\left\{t \in [t_0, T] : \mathrm{d}_{\mathrm{Bd}(E)}(x(t)) \le \delta/4\right\},\\
t_1 &= \inf\left\{t \in [T_1, T] : |x(t)-x(T_1)| > \delta/2\right\},\\
T_2 &= \inf\left\{t \in [t_1, T] : \mathrm{d}_{\mathrm{Bd}(E)}(x(t)) \le \delta/4\right\},\\
&\cdots\cdots\cdots\\
t_i &= \inf\left\{t \in [T_i, T] : |x(t)-x(T_i)| > \delta/2\right\}\\
T_{i+1} &= \inf\left\{t \in [t_i, T] : \mathrm{d}_{\mathrm{Bd}(E)}(x(t)) \le \delta/4\right\}\\
&\cdots\cdots\cdots
\end{aligned}$$

Clearly

$$0 = T_0 = t_0 \le T_1 < t_1 \le T_2 < \cdots < t_i \le T_{i+1} < t_{i+1} \le \cdots \le T.$$

Let $t_i \le r \le T_{i+1}$. Then $x(r) \in \mathrm{Int}(E)$ and $\mathrm{d}_{\mathrm{Bd}(E)}(x(r)) \ge \delta/4$. By Lemma 28 we get

$$|k(t)-k(s)| \le \updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le C(t-s) \text{ for } t_i \le s \le t \le T_{i+1}.$$

Also for $t_i \le s \le t \le T_{i+1}$:

$$|x(t) - x(s)| \le |k(t) - k(s)| + |m(t) - m(s)| \le C(t-s) + |m(t) - m(s)|$$
$$\le C\boldsymbol{\mu}_m(t-s)$$

and then

$$|x(t) - x(s)| + \updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s \le C\boldsymbol{\mu}_m(t-s).$$

On each of the intervals $[T_i, t_i]$, we have

$$|x(t) - x(s)| \le \delta, \quad \text{for all } T_i \le s \le t \le t_i,$$

and consequently, for all $T_i \le s \le t \le t_i$, inequality (31) holds.

If $T_i \le s \le t_i \le t \le T_{i+1}$ then

$$\begin{aligned}
&|x(t) - x(s)| + \updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s \\
&\le |x(t) - x(t_i)| + \updownarrow k \updownarrow_t - \updownarrow k \updownarrow_{t_i} + |x(t_i) - x(s)| + \updownarrow k \updownarrow_{t_i} - \updownarrow k \updownarrow_s \\
&\le C\boldsymbol{\mu}_m(t - t_i) + \sqrt{\boldsymbol{\mu}_m(t_i - s)} \cdot \exp\left[C(1 + T + \|m\|_T)\right] \\
&\le \sqrt{\boldsymbol{\mu}_m(t-s)} \times \exp\left[C'(1 + T + \|m\|_T)\right].
\end{aligned}$$

Consequently for all $i \in \mathbb{N}$ and $T_i \le s \le t \le T_{i+1}$, inequality (31) holds.

Step 4. *Getting inequalities (21).*

Since $\boldsymbol{\mu}_m^{-1} : \left[0, \boldsymbol{\mu}_m(T)\right] \to [0, T]$ is well defined and is a strictly increasing continuous function, from

$$\begin{aligned}
&\tfrac{\delta}{2} \le |x(t_i) - x(T_i)| \le \sqrt{\boldsymbol{\mu}_m(t_i - T_i)} \times \exp\left[C(1 + T + \|m\|_T)\right] \\
&\le \sqrt{\boldsymbol{\mu}_m(T_{i+1} - T_i)} \cdot \exp\left[C(1 + T + \|m\|_T)\right],
\end{aligned}$$

we deduce that

$$\begin{aligned}
T_{i+1} - T_i &\ge \boldsymbol{\mu}_m^{-1}\left(\frac{\delta^2}{4}\exp\left[-2C(1 + T + \|m\|_T)\right]\right) \\
&\ge \boldsymbol{\mu}_m^{-1}\left(\delta^2 \exp\left[-2C'(1 + T + \|m\|_T)\right]\right) > 0.
\end{aligned}$$

Hence the bounded increasing sequence $(T_i)_{i\ge 0}$ has a finite number of terms, therefore there exists $j \in \mathbb{N}^*$ such that $T = T_j$. Then

$$T = T_j = \sum_{i=1}^{j} (T_i - T_{i-1}) \ge j\Delta_m^{-1},$$

where $\Delta_m := 1/\boldsymbol{\mu}_m^{-1}\left(\delta^2 \exp\left[-C'(1 + T + \|m\|_T)\right]\right)$ (see definition (19)).

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Let $0 \leq s \leq t \leq T$. We have

$$\begin{aligned}
\updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s &= \sum_{i=1}^{j} \left(\updownarrow k \updownarrow_{(t \wedge T_i) \vee s} - \updownarrow k \updownarrow_{(t \wedge T_{i-1}) \vee s} \right) \\
&\leq \sum_{i=1}^{j} \sqrt{\boldsymbol{\mu}_m \left((t \wedge T_i) \vee s - (t \wedge T_{i-1}) \vee s \right)} \cdot \exp\left[C \left(1 + T + \|m\|_T \right) \right] \\
&\leq j \sqrt{\boldsymbol{\mu}_m (t-s)} \cdot \exp\left[C \left(1 + T + \|m\|_T \right) \right] \\
&\leq T \Delta_m \sqrt{\boldsymbol{\mu}_m (t-s)} \cdot \exp\left[C \left(1 + T + \|m\|_T \right) \right]
\end{aligned}$$

and consequently

$$\updownarrow k \updownarrow_T \leq T \Delta_m \sqrt{\boldsymbol{\mu}_m (T)} \cdot \exp\left[C \left(1 + T + \|m\|_T \right) \right] \leq \exp\left[C' \left(1 + T + \|m\|_T + \Delta_m \right) \right]$$

and

$$|x(t)| = |x_0 + m(t) - k(t)| \leq |x_0| + \|m\|_t + \updownarrow k \updownarrow_t \leq |x_0| + \|m\|_T + \updownarrow k \updownarrow_T .$$

Hence there exists a positive constant $C = C(L, \delta, \sigma, \rho, \gamma)$ such that, under notations (19),

$$\updownarrow k \updownarrow_T \leq C_{T,m}, \quad \text{and} \quad \|x\|_T \leq |x_0| + C_{T,m},$$

which is part of conclusion (21).

In order to end the proof of (21) it is sufficient to remark that, for any $0 \leq s \leq t \leq T$,

$$\begin{aligned}
|x(t) - x(s)|^2 &\leq \left[\mathbf{m}_m^2 (t-s) + \mathbf{m}_m (t-s) C_{T,m} + (t-s) \right] \cdot \exp\left[C \left(1 + C_{T,m} \right) \right] \\
&\leq C_{T,m} \, \boldsymbol{\mu}_m (t-s) .
\end{aligned}$$

Step 5. *Getting inequality (22).*

Since $\updownarrow k \updownarrow_T + \updownarrow \hat{k} \updownarrow_T \leq C_{T,m} + C_{T,\hat{m}}$, from inequality (15) we have

$$\begin{aligned}
\left\| x - \hat{x} \right\|_T^2 &\leq 2 \Big[\left| x_0 - \hat{x}_0 \right|^2 + \left\| m - \hat{m} \right\|_T^2 + 2 \left\| m - \hat{m} \right\|_T \|k - \hat{k}\|_T \Big] \\
&\qquad \cdot \exp\left[4\gamma (2t + \updownarrow k \updownarrow_T + \updownarrow \hat{k} \updownarrow_T) \right] \\
&\leq A^2 \left(C_{T,m}, C_{T,\hat{m}} \right) \left[\left| x_0 - \hat{x}_0 \right|^2 + \left\| m - \hat{m} \right\|_T \right],
\end{aligned}$$

(where A is a continuous function) and the conclusion follows since $k - \hat{k} = x_0 - \hat{x}_0 + m - \hat{m} - \left(x - \hat{x} \right)$. □

Proof of the Theorem 25. Uniqueness was proved in Theorem 19. To prove existence, let $m_n \in C^1\left(\mathbb{R}_+; \mathbb{R}^d \right)$ with $m_n(0) = 0$ be such that $\|m_n - m\|_T \to 0$ for all $T \geq 0$. Since $m_n \in C^1\left(\mathbb{R}_+; \mathbb{R}^d \right)$, we deduce, using the results from the papers [19] or [37], that there exists a unique solution (x_n, k_n) of the $\mathcal{SP}\left(\partial^- \varphi; x_0, m_n \right)$, and by Corollary 24 we see that there exist $x, k \in C\left(\mathbb{R}_+; \mathbb{R}^d \right)$ such that for all $T \geq 0$

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

$$\|x_n - x\|_T + \|k_n - k\|_T \to 0, \text{ as } n \to \infty, \text{ and}$$
$$(x, k) = \mathcal{SP}\left(\partial^- \varphi; x_0, m\right),$$

which completes the proof. □

5. Generalized Skorohod equations

Consider the next (non-convex) variational inequality with singular input (which will be called generalized Skorohod differential equation):

$$\begin{cases} x(t) + k(t) = x_0 + \int_0^t f(s, x(s))\, ds + m(t), \quad t \geq 0, \\ dk(t) \in \partial^- \varphi(x(t))\,(dt) \end{cases} \tag{32}$$

(for the notation $dk(t) \in \partial^- \varphi(x(t))\,(dt)$ we recall Remark 15).

We introduce the following supplementary assumptions:

$$f(\cdot, x) : \mathbb{R}_+ \to \mathbb{R}^d \text{ is measurable, } \forall x \in \mathbb{R}^d, \tag{33}$$

and there exists $\mu \in L^1_{loc}(\mathbb{R}_+)$, such that a.e. $t \geq 0$:

$$\begin{aligned} &(i) \quad x \mapsto f(t, x) : \mathbb{R}^d \to \mathbb{R}^d \text{ is continuous,} \\ &(ii) \quad \langle x - y, f(t, x) - f(t, y)\rangle \leq \mu(t)\,|x - y|^2, \ \forall x, y \in \mathbb{R}^d, \\ &(iii) \quad \int_0^T f^{\#}(s)\, ds < \infty, \ \forall T \geq 0, \end{aligned} \tag{34}$$

where

$$f^{\#}(t) := \sup\left\{|f(t, x)| : x \in \overline{\mathrm{Dom}(\varphi)}\right\}. \tag{35}$$

Clearly, assumption (34-*iii*) is satisfied if, as example, $f(t, x) = f(t)$ or if $\mathrm{Dom}(\varphi)$ is bounded.

Proposition 30 *(Generalized Skorohod equation). Let $\varphi : \mathbb{R}^d \to (-\infty, \infty]$ and $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be such that assumptions (9), (10), (16), (17) and (33), (34) are satisfied.*

Then the generalized Skorohod equation (32) has a unique solution.

Proof. Let (x, k) and $(\hat{x}, \hat{k})$ be two solutions. Then

$$|x(t) - \hat{x}(t)|^2 + 2\int_0^t \langle x(r) - \hat{x}(r), dk(r) - d\hat{k}(r)\rangle$$
$$= 2\int_0^t \langle x(r) - \hat{x}(r), f(r, x(r)) - f\left(r, \hat{x}(r)\right)\rangle dr \leq 2\int_0^t \mu^+(r)\,|x(r) - \hat{x}(r)|^2 dr,$$

and using Lemma 17 it follows that

$$\left|x\left(t\right)-\hat{x}\left(t\right)\right|^{2}\leq 2\int_{0}^{t}\left|x\left(r\right)-\hat{x}\left(r\right)\right|^{2}dA_{r},$$

where

$$A_{t}=2\rho t+\gamma\,\updownarrow k\updownarrow_{t}+\gamma\,\updownarrow\hat{k}\updownarrow_{t}+\int_{0}^{t}\mu^{+}\left(r\right)dr.$$

Applying a Gronwall's type inequality, we see that $x=\hat{x}$.

Concerning the existence, we shall obtain the solution (x,k) as the limit in $C\left([0,T];\mathbb{R}^{d}\right)\times C\left([0,T];\mathbb{R}^{d}\right)$ of the sequence $(x_{n},k_{n})_{n\in\mathbb{N}^{*}}$ defined by an approximate Skorohod equation

$$\begin{cases} x_{n}\left(t\right)=x_{0}\ ,\ \text{for } t<0, \\ x_{n}\left(t\right)+k_{n}\left(t\right)=x_{0}+\int_{0}^{t}f\left(s,x_{n}(s-1/n)\right)ds+m\left(t\right), \quad \text{for } t\geq 0, \\ dk_{n}\left(t\right)\in\partial^{-}\varphi\left(x_{n}\left(t\right)\right)\left(dt\right). \end{cases} \tag{36}$$

For any $i\in\mathbb{N}$, for $t\in\left[\frac{i}{n},\frac{i+1}{n}\right]$, we can write

$$x_{n}\left(t\right)+\left[k_{n}\left(t\right)-k_{n}\left(i/n\right)\right]=x_{n}\left(i/n\right)+\int_{i/n}^{t}f\left(s,x_{n}(s-1/n)\right)ds+m\left(t\right)-m\left(i/n\right),$$

therefore by iteration over the intervals $\left[\frac{i}{n},\frac{i+1}{n}\right]$ there exists (via Theorem 25) a unique pair $(x_{n},k_{n})=\mathcal{SP}\left(\partial^{-}\varphi;x_{0},m_{n}\right)$, with

$$m_{n}\left(t\right)=\int_{0}^{t}f\left(s,x_{n}(s-1/n)\right)ds+m\left(t\right).$$

Let $T>0$. If $\mathcal{M}$ denotes the set

$$\mathcal{M}:=\left\{m_{n}:n\in\mathbb{N}^{*}\right\},$$

then $\mathcal{M}$ is a relatively compact subset of $C\left([0,T];\mathbb{R}^{d}\right)$ since it is a bounded and equicontinuous subset of $C\left([0,T];\mathbb{R}^{d}\right)$.

Indeed

$$\left\|m_{n}\right\|_{T}\leq\int_{0}^{T}f^{\#}\left(s\right)ds+\left\|m\right\|_{T}$$

and, for $s < t$,

$$|m_n(t) - m_n(s)| \le \int_s^t f^{\#}(r)\, dr + |m(t) - m(s)|.$$

Then by Theorem 23 and Remark 22,

$$\|x_n\|_T + \updownarrow k_n \updownarrow_T \le |x_0| + C_{T,\mathcal{M}} \tag{37}$$

and for all $0 \le s \le t$:

$$|x_n(t) - x_n(s)| + \updownarrow k_n \updownarrow_t - \updownarrow k_n \updownarrow_s \le C_{T,\mathcal{M}} \sqrt{\boldsymbol{\mu}_{\mathcal{M}}(t-s)},$$

where $\boldsymbol{\mu}_{\mathcal{M}}(\varepsilon) := \varepsilon + \sup_{m \in \mathcal{M}} \mathbf{m}_m$.

Hence, by Arzelà–Ascoli's theorem, the set $\{x_n : n \in \mathbb{N}^*\}$ is a relatively compact subset of $C([0,T];\mathbb{R}^d)$.

Let $x \in C([0,T];\mathbb{R}^d)$ be such that, along a sequence still denoted by $\{x_n : n \in \mathbb{N}^*\}$,

$$\|x_n - x\|_T \to 0, \text{ as } n \to \infty.$$

Then, uniformly with respect to $t \in [0,T]$,

$$m_n(t) \to \int_0^t f(s, x(s))\, ds + m(t), \text{ as } n \to \infty,$$

and

$$k_n(t) \to k(t) = x_0 + \int_0^t f(s, x(s))\, ds + m(t) - x(t), \text{ as } n \to \infty.$$

Using Corollary 24 we infer that

$$(x,k) = \mathcal{SP}\left(\partial^- \varphi; x_0, \int_0^{\cdot} f(s, x(s))\, ds + m\right)$$

i.e. (x,k) is a solution of problem (32). The uniqueness of the solution implies that the whole sequence (x_n, k_n) is convergent to that solution (x,k). The proof is completed now. □

Remark 31. As it can be seen in the above proof, assumption (34-*iii*) is essential in order to obtain inequality (37), i.e. the boundedness of the sequence $(x_n, k_n)_n$ defined by (36).

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Remark 32. If we replace assumptions (16), (17) and (34-*iii*) by the hypotheses

$$\text{Int}\,(\text{Dom}\,(\varphi)) \neq \emptyset$$

and

$$\int_0^T f_R^{\#}(s)\,ds < \infty, \quad \forall R, T > 0,$$

where

$$f_R^{\#}(t) := \sup\left\{|f(t,x)| : |x| \leq R\right\},$$

then, following the calculus from the convex case (see, e.g., Remark 4.15 and Proposition 4.16 from [31]), we deduce

$$\begin{aligned}
&\frac{1}{2}|x(t)-u_0|^2 + \frac{r_0}{2}\updownarrow k\updownarrow_t + \frac{r_0}{2}\int_0^t |f(r,x(r))|\,dr \\
&\leq \frac{1}{4}\|x-u_0\|_t^2 + C_0 + C_0\|m\|_T \\
&+ 2\int_0^t \mu^+(r)\|x-u_0\|_r^2\,dr + 2\int_0^t \mu^+(r)\|x-u_0\|_r^2\,(\rho dr + \gamma d\updownarrow k\updownarrow_r),
\end{aligned} \tag{38}$$

where $u_0 \in \mathbb{R}^d$ and $r_0 \in [0,1)$ are such that $\overline{B}(u_0, r_0) \subset \text{Int}\,(\text{Dom}\,(\varphi))$.

In the convex case, namely $\gamma = 0$, this inequality yields the boundedness (21-*a*, *b*), but in the non-convex case ($\gamma \neq 0$) we cannot obtain (21-*a*, *b*).

Of course, if there exists $R > 0$ such that $\text{Dom}\,(\varphi) \subset B(0,R)$, then $\|x\|_T \leq R$ and from (38) we obtain $\updownarrow k\updownarrow_T \leq C$, if γ is such that

$$0 \leq \gamma < \frac{r_0}{2(r_0 + R + |u_0|)}.$$

If in the above proposition we take $\varphi = I_E$ we get, via Theorem 27,

Corollary 33 *(Skorohod equation). Let $x_0 \in E$ and $m : \mathbb{R}_+ \to \mathbb{R}^d$ be a continuous function such that $m(0) = 0$. If f satisfies assumptions (33)–(34) and E satisfies the r_0-UEBC (for some $r_0 > 0$) and the shifted uniform interior ball condition, then there exists a unique pair (x,k) such that:*

$$\begin{aligned}
&(j) \quad x, k \in C(\mathbb{R}_+; E), \quad k(0) = 0, \\
&(jj) \quad k \in BV_{loc}\left(\mathbb{R}_+; \mathbb{R}^d\right), \\
&(jjj) \quad x(t) + k(t) = x_0 + \textstyle\int_0^t f(s, x(s))\,ds + m(t), \\
&(jv) \quad \updownarrow k\updownarrow_t = \textstyle\int_0^t \mathbf{1}_{x(s)\in \text{Bd}(E)} d\updownarrow k\updownarrow_s, \\
&(v) \quad k(t) = \textstyle\int_0^t n_{x(s)}\, d\updownarrow k\updownarrow_s, \text{ where } n_{x(s)} \in N_E(x(s)) \quad \text{and} \\
&\qquad\qquad \left|n_{x(s)}\right| = 1,\ d\updownarrow k\updownarrow_s\text{-a.e.}
\end{aligned} \tag{39}$$

6. Non-convex stochastic variational inequalities

In the last section of the paper we will study the following multivalued stochastic differential equation (also called *stochastic variational inequality*) considered on a non-convex domain:

$$\begin{cases} X_t + K_t = \xi + \int_0^t F(s, X_s)\, ds + \int_0^t G(s, X_s)\, dB_s, \quad t \geq 0, \\ dK_t(\omega) \in \partial^- \varphi(X_t(\omega))(dt), \end{cases} \tag{40}$$

where φ is (ρ, γ)-semiconvex function and $\{B_t : t \geq 0\}$ is an $\mathbb{R}^k$-valued Brownian motion with respect to a stochastic basis (which is supposed to be complete and right-continuous) $\big(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}\big)$.

First we derive directly from Theorem 25 the existence result in the additive noise case.

Corollary 34. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t)_{t\geq 0}$ be given. If*

$$\xi \in L^0\big(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{\mathrm{Dom}(\varphi)}\big)$$

and M is a progressively measurable and continuous stochastic process (p.m.c.s.p. for short) with $M_0 = 0$, then there exists a unique pair (X, K) of p.m.c.s.p., solution of the problem

$$\begin{cases} X_t(\omega) + K_t(\omega) = \xi(\omega) + M_t(\omega), \ t \geq 0, \ \omega \in \Omega, \\ dK_t(\omega) \in \partial^- \varphi(X_t(\omega))(dt) \end{cases}$$

(in this case we shall write $(X_\cdot(\omega), K_\cdot(\omega)) = \mathcal{SP}\big(\partial^- \varphi; \xi(\omega), M_\cdot(\omega)\big)$, $\mathbb{P}$-a.s.).

Proof. Let ω be arbitrary but fixed. By Theorem 25, the Skorohod problem

$$(X_\cdot(\omega), K_\cdot(\omega)) = \mathcal{SP}\big(\partial^- \varphi; \xi(\omega), M_\cdot(\omega)\big)$$

has a unique solution

$$(X_\cdot(\omega), K_\cdot(\omega)) \in C(\mathbb{R}_+; \mathbb{R}^d) \times C(\mathbb{R}_+; \mathbb{R}^d).$$

Since $(\omega, t) \mapsto M_t(\omega)$ is progressively measurable and the mapping

$$(\xi, M) \mapsto X : \overline{\mathrm{Dom}(\varphi)} \times C([0,t]; \mathbb{R}^d) \to C([0,t]; \mathbb{R}^d)$$

is continuous for each $0 \leq t \leq T$, the stochastic process X is progressively measurable. Hence the conclusion follows. □

The next assumptions will be needed throughout this section:

(A$_1$) *(Carathéodory conditions.) The functions $F(\cdot,\cdot,\cdot) : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $G(\cdot,\cdot,\cdot) : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are $\big(\mathcal{P}, \mathbb{R}^d\big)$-Carathéodory functions, i.e.*

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

$$F(\cdot,\cdot,x) \text{ and } G(\cdot,\cdot,x) \text{ are p.m.s.p., } \forall\, x \in \mathbb{R}^d,$$
$$F(\omega,t,\cdot) \text{ and } G(\omega,t,\cdot) \text{ are continuous function } d\mathbb{P}\otimes dt\text{-a.e.} \tag{41}$$

(A$_2$) *(Boundedness conditions.) For all* $T \geq 0$:

$$\int_0^T F^{\#}(s)\,ds < \infty \quad \text{and} \quad \int_0^T |G^{\#}(s)|^2 ds < \infty,\ \mathbb{P}\text{-a.s.,} \tag{42}$$

where

$$F^{\#}(t) := \sup\left\{|F(t,x)| : x \in \overline{\mathrm{Dom}(\varphi)}\right\},$$
$$G^{\#}(t) := \sup\left\{|G(t,x)| : x \in \overline{\mathrm{Dom}(\varphi)}\right\}$$

(A$_3$) *(Monotonicity and Lipschitz conditions.) There exist* $\mu \in L^1_{loc}(\mathbb{R}_+)$ *and* $\ell \in L^2_{loc}(\mathbb{R}_+)$ with $\ell \geq 0$, such that $d\mathbb{P}\otimes dt$-a.e.

$$\begin{aligned} &(i) \quad \langle x-y, F(t,x)-F(t,y)\rangle \leq \mu(t)|x-y|^2, \quad \forall\, x,y \in \mathbb{R}^d,\\ &(ii) \quad |G(t,x)-G(t,y)| \leq \ell(t)|x-y|, \quad \forall\, x,y \in \mathbb{R}^d. \end{aligned} \tag{43}$$

We define now the notions of strong and weak solutions for the stochastic Skorohod equation (40).

Definition 35. Let $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t,B_t)_{t\geq 0}$ be given. A pair $(X,K):\Omega\times\mathbb{R}_+\to\mathbb{R}^d\times\mathbb{R}^d$ of continuous $\mathcal{F}_t$-p.m.c.s.p. is a strong solution of the stochastic Skorohod equation (40) if $\mathbb{P}$-a.s.,

$$\begin{aligned} &(j) \quad X_t \in \overline{\mathrm{Dom}(\varphi)},\ \forall\, t\geq 0,\ \varphi(X_\cdot) \in L^1_{loc}(\mathbb{R}_+),\\ &(jj) \quad K_\cdot \in BV_{loc}\left(\mathbb{R}_+;\mathbb{R}^d\right), \quad K_0 = 0,\\ &(jjj) \quad X_t + K_t = \xi + \int_0^t F(s,X_s)\,ds + \int_0^t G(s,X_s)\,dB_s,\ \forall\, t\geq 0,\\ &(jv) \quad \forall\, 0\leq s\leq t,\ \forall\, y:\mathbb{R}_+\to\mathbb{R}^d \text{ continuous}\\ &\qquad \int_s^t \langle y(r)-X_r, dK_r\rangle + \int_s^t \varphi(X_r)\,dr \leq \int_s^t \varphi(y(r))\,dr\\ &\qquad\qquad + \int_s^t |y(r)-X_r|^2\left(\rho\,dr + \gamma\, d\,\updownarrow K\updownarrow_r\right), \end{aligned} \tag{44}$$

which means that

$$(X_\cdot(\omega), K_\cdot(\omega)) = \mathcal{SP}\left(\partial^- \varphi; \xi(\omega), M_\cdot(\omega)\right), \quad \mathbb{P}\text{-a.s.,}$$

where

$$M_t = \int_0^t F(s,X_s)\,ds + \int_0^t G(s,X_s)\,dB_s\,.$$

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Definition 36. Let $F(\omega, t, x) := f(t, x)$, $G(\omega, t, x) := g(t, x)$ and $\xi(\omega) := x_0$ (be independent of ω). If there exist a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)_{t\geq 0}$, an $\mathbb{R}^k$-valued $\mathcal{F}_t$-Brownian motion $\{B_t : t \geq 0\}$ and a pair $(X., K.) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d \times \mathbb{R}^d$ of p.m.c.s.p. such that

$$(X.(\omega), K.(\omega)) = \mathcal{SP}\left(\partial^- \varphi; x_0, M.(\omega)\right), \quad \mathbb{P}\text{-}a.s.,$$

where

$$M_t = \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dB_s,$$

then the collection $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, X_t, K_t)_{t\geq 0}$ is called a weak solution of the stochastic Skorohod equation (40).

Since the stochastic process K is uniquely determined from (X, B) through equation (44-*jjj*), we can also say that X is a strong solution (and respectively $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, X_t)_{t\geq 0}$ is a weak solution).

We first give a uniqueness result for strong solutions.

Proposition 37 *(Pathwise uniqueness). Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t)_{t\geq 0}$ be given and assumptions (9), (10) and (16) be satisfied. The functions F and G are such that assumptions (A1)–(A3) are satisfied. Then the stochastic Skorohod equation (40) has at most one strong solution.*

Proof. Let (X, K) and $(\hat{X}, \hat{K})$ be two solutions corresponding to ξ and respectively $\hat{\xi}$. Since

$$dK_t \in \partial^- \varphi(X_t)(dt) \quad \text{and} \quad d\hat{K}_t \in \partial^- \varphi(\hat{X}_t)(dt),$$

by Lemma 17, for $p \geq 1$ and $\lambda > 0$

$$\begin{aligned}&\left\langle X_t - \hat{X}_t, (F(t, X_t)\, dt - dK_t) - (F(t, \hat{X}_t)dt - d\hat{K}_t)\right\rangle \\ &+ \left(\frac{1}{2}m_p + 9p\lambda\right)|G(t, X_t) - G(t, \hat{X}_t)|^2 dt \leq |X_t - \hat{X}_t|^2 dV_t,\end{aligned}$$

where

$$V_t = \int_0^t \left[\mu(s)\, ds + \left(\frac{1}{2}m_p + 9p\lambda\right)\ell^2(s)\, ds + 2\rho ds + \gamma d\,\updownarrow K\updownarrow_s + \gamma d\,\updownarrow \hat{K}\updownarrow_s\right].$$

Therefore, by Corollary 51 (from Appendix A), we get

$$\mathbb{E}\left[1 \wedge \left\|e^{-V}(X - \hat{X})\right\|_T^p\right] \leq C_{p,\lambda}\, \mathbb{E}\left[1 \wedge |\xi - \hat{\xi}|^p\right],$$

and the uniqueness follows. □

Remark also that in the case of additive noise (i.e. G does not depend upon X) we have existence of a strong solution.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Lemma 38. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t)_{t\geq 0}$ be given and assumptions (9), (10), (16) and (17) be satisfied. The function F is such that assumptions (A$_1$)–(A$_3$) are satisfied.*

If

$$\xi \in L^0\left(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{\mathrm{Dom}\,(\varphi)}\right)$$

and M is a p.m.c.s.p. with $M_0 = 0$, then there exists a unique pair (X, K) of p.m.c.s.p., solution of the problem

$$(X.(\omega), K.(\omega)) = \mathcal{SP}\left(\partial^-\varphi; \xi(\omega), M.(\omega)\right), \quad \mathbb{P}\text{-a.s.},$$

i.e.

$$\begin{cases} X_t(\omega) + K_t(\omega) = \xi(\omega) + \displaystyle\int_0^t F(\omega, s, X_s(\omega))\, ds + M_t(\omega), \quad t \geq 0, \\ dK_t(\omega) \in \partial^-\varphi(X_t(\omega))(dt), \end{cases}$$

$\mathbb{P}$*-a.s.*

Proof. Applying Corollary 34 to the approximating problem

$$\begin{cases} X_t^n(\omega) + K_t^n(\omega) = \xi(\omega) + \displaystyle\int_0^t F(\omega, s, X_{s-1/n}^n(\omega))ds + M_t(\omega), \quad t \geq 0, \\ dK_t^n(\omega) \in \partial^-\varphi\left(X_t^n(\omega)\right)(dt), \end{cases}$$

we conclude that there exists a unique solution (X^n, K^n) of p.m.c.s.p. The solution (X, K) is obtained as the limit of the sequence (X^n, K^n), exactly as in the proof of Proposition 30. □

In order to study the general stochastic Skorohod equation (40) we shall consider only the case when F, G and ξ are independent of ω and, to highlight this, the coefficients will be denoted by f and g respectively.

Let us consider equation

$$\begin{cases} X_t + K_t = x_0 + \displaystyle\int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dB_s, \quad t \geq 0, \\ dK_t(\omega) \in \partial^-\varphi(X_t(\omega))(dt), \end{cases} \tag{45}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$.

We recall the definition of $f^\#$, $g^\#$ given as in (35).

Theorem 39. *Let assumptions (9), (10), (16) and (17) be satisfied. The functions f and g are suppose to be $\left(\mathcal{B}_1, \mathbb{R}^d\right)$-Carathéodory functions satisfying moreover the boundedness conditions*

$$\int_0^T \left[|f^{\#}(s)|^2 + |g^{\#}(s)|^4 \right] ds < \infty, \ \forall T \geq 0.$$

If $x_0 \in \overline{\mathrm{Dom}(\varphi)}$ *then equation (45) has a weak solution* $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, X_t, K_t, B_t)_{t\geq 0}$.

Remark 40. Usually, when G is Lipschitz, a fixed point argument is used (based on Banach contraction theorem). But, in our case this argument doesn't work even for the drift part $F \equiv 0$, as it can be seen from inequality (22), we have different order of the estimates in the left and in the right side of the inequality.

Proof.

Step 1. *Approximating sequence.*

Let $\left(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t^B, B_t\right)_{t\geq 0}$ be a stochastic basis. Applying Lemma 38, we deduce that there exists a unique pair $(X^n, K^n) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d \times \mathbb{R}^d$ of $\mathcal{F}_t^B$-p.m.c.s.p. such that

$$\begin{cases} X_t^n + K_t^n = x_0 + \int_0^t f(s, X_{s-1/n}^n) ds + \int_0^t g(s, X_{s-1/n}^n) dB_s, \quad t \geq 0, \\ dK_t^n(\omega) \in \partial^- \varphi\left(X_t^n(\omega)\right)(dt). \end{cases} \tag{46}$$

Denote

$$M_t^n := \int_0^t f\left(s, X_{s-1/n}^n\right) ds + \int_0^t g\left(s, X_{s-1/n}^n\right) dB_s .$$

Since

$$\mathbb{E}\left[\sup_{0\leq\theta\leq\varepsilon} \left|M_{t+\theta}^n - M_t^n\right|^4\right] \leq C\left[\left(\int_t^{t+\varepsilon} f^{\#}(s)\, ds\right)^4 + \left(\int_t^{t+\varepsilon} |g^{\#}(s)|^2 ds\right)^2\right]$$
$$\leq \varepsilon C\left[\sup_{t\in[0,T]}\left(\int_t^{t+\varepsilon} |f^{\#}(s)|^2 ds\right)^2 + \sup_{t\in[0,T]} \int_t^{t+\varepsilon} |g^{\#}(s)|^4 ds\right]$$

using Proposition 46, we deduce that the family of laws of $\{M^n : n \geq 1\}$ is tight on $C\left(\mathbb{R}_+; \mathbb{R}^d\right)$.

Therefore by Theorem 45 for all $T \geq 0$

$$\lim_{N\nearrow\infty}\left[\sup_{n\geq 1} \mathbb{P}\left(\|M^n\|_T \geq N\right)\right] = 0,$$

and, for all $a > 0$ and $T > 0$,

$$\lim_{\varepsilon \searrow 0} \left[\sup_{n \geq 1} \mathbb{P}\left(\{ \mathbf{m}_{M^n}(\varepsilon) \geq a \} \right) \right] = 0. \tag{47}$$

Recalling the definition

$$\boldsymbol{\mu}_{M^n} = \varepsilon + \mathbf{m}_{M^n}(\varepsilon),$$

where $\mathbf{m}$ is the modulus of continuity, we see that we can replace in (47) $\mathbf{m}_{M^n}$ by $\boldsymbol{\mu}_{M^n}$.

Step 2. *Tightness.*

Let $T \geq 0$ be arbitrary. We now show that the family of laws of the random variables $U^n = (X^n, K^n, \updownarrow K^n \updownarrow)$ is tight on $C\left([0,T]; \mathbb{R}^{2d+1}\right)$.

From (21-*c*) we deduce that

$$\mathbf{m}_{U^n}(\varepsilon) \leq G\left(M^n\right) \sqrt{\boldsymbol{\mu}_{M^n}(\varepsilon)}, \text{ a.s.},$$

where $G : C\left([0,T]; \mathbb{R}^d\right) \to \mathbb{R}_+$ and

$$G(x) := C_{T,x} = \exp\left[C\left(1 + T + \|x\|_T + B_x\right)\right],$$
$$B_x := 1/\boldsymbol{\mu}_x^{-1}\left(\delta^2 e^{-C(1+T+\|x\|_T)}\right).$$

From (20) we see that G is bounded on compact subset of $C\left([0,T]; \mathbb{R}^d\right)$ and therefore by Proposition 47, $\{U^n; n \in \mathbb{N}^*\}$ is tight on $C\left([0,T]; \mathbb{R}^d\right)$.

Using the Prohorov theorem we see that there exists a subsequence (still denoted with n) such that

$$\left(X^n, K^n, \updownarrow K^n \updownarrow, B\right) \to (X, K, V, B) \quad \text{in law, as } n \to \infty$$

on $C\left([0,T]; \mathbb{R}^{2d+1+k}\right)$ and applying the Skorohod theorem, we can choose a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some random quadruples $(\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n)$, $(\bar{X}, \bar{K}, \bar{V}, \bar{B})$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathcal{L}(\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n) = \mathcal{L}\left(X^n, K^n, \updownarrow K^n \updownarrow, B^n\right)$$
$$\mathcal{L}(\bar{X}, \bar{K}, \bar{V}, \bar{B}) = \mathcal{L}(X, K, V, B)$$

and

$$(\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n) \xrightarrow{\mathbb{P}\text{-}a.s.} (\bar{X}, \bar{K}, \bar{V}, \bar{B}), \text{ as } n \to \infty, \text{ in } C([0,T]; \mathbb{R}^{2d+1+k}).$$

From Proposition 49 we deduce that $\left(\bar{B}^n, \{\mathcal{F}_t^{\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n}\}\right)$, $n \geq 1$, and $\left(\bar{B}, \{\mathcal{F}_t^{\bar{X}, \bar{K}, \bar{V}, \bar{B}}\}\right)$ are Brownian motions.

Step 3. *Passing to the limit.*

Since we also have $(X^n, K^n, \updownarrow K^n \updownarrow, B) \to (\bar{X}, \bar{K}, \bar{V}, \bar{B})$, in law, then by Corollary 43 we deduce that for all $0 \le s \le t$, $\mathbb{P}$-a.s.

$$\begin{aligned} &\bar{X}_0 = x_0, \quad \bar{K}_0 = 0, \quad \bar{X}_t \in \overline{\mathrm{Dom}(\varphi)}, \\ &\updownarrow \bar{K} \updownarrow_t - \updownarrow \bar{K} \updownarrow_s \le \bar{V}_t - \bar{V}_s \quad \text{and} \quad 0 = \bar{V}_0 \le \bar{V}_s \le \bar{V}_t \end{aligned} \tag{48}$$

Moreover, since for all $0 \le s < t$, $n \in \mathbb{N}^*$

$$\begin{aligned} \int_s^t \varphi\left(X_r^n\right) dr \le \int_s^t \varphi\left(y\left(r\right)\right) dr - \int_s^t \left\langle y\left(r\right) - X_r^n, dK_r^n \right\rangle \\ + \int_s^t \left| y\left(r\right) - X_r^n \right|^2 \left(\rho dr + \gamma d \updownarrow K^n \updownarrow_r\right), \text{ a.s.,} \end{aligned}$$

then by Corollary 44 we infer that

$$\begin{aligned} \int_s^t \varphi\left(\bar{X}_r\right) dr \le \int_s^t \varphi\left(y\left(r\right)\right) dr - \int_s^t \left\langle y\left(r\right) - \bar{X}_r, d\bar{K}_r \right\rangle \\ + \int_s^t \left| y\left(r\right) - \bar{X}_r \right|^2 \left(\rho dr + \gamma d\bar{V}_r\right). \end{aligned} \tag{49}$$

Hence, based on (48) and (52) and Lemma 16 we have

$$d\bar{K}_r \in \partial^- \varphi\left(\bar{X}_r\right)(dr).$$

Let

$$S_t\left(Y, B\right) := x_0 + \int_0^t f\left(s, Y_s\right) ds + \int_0^t g\left(s, Y_s\right) dB_s, \quad t \ge 0.$$

By Proposition 48 it follows

$$\mathcal{L}(\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n, S_t(\bar{X}^n, \bar{B}^n)) = \mathcal{L}\left(X^n, K^n, \updownarrow K^n \updownarrow, B^n, S_t(X^n, B^n)\right)$$

Since for every $t \ge 0$,

$$X_t^n + K_t^n - S_t(X^n, B^n) = 0, \ a.s.,$$

then by Corollary 43 we have

$$\bar{X}_t^n + \bar{K}_t^n - S_t(\bar{X}^n, \bar{B}^n) = 0, \ a.s.,$$

and consequently, letting $n \to \infty$,

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

$$\bar{X}_t + \bar{K}_t - S_t(\bar{X}, \bar{B}) = 0, \ a.s.$$

Hence we obtain that, $\mathbb{P}$-a.s.,

$$\bar{X}_t + \bar{K}_t = x_0 + \int_0^t f\left(s, \bar{X}_s\right) ds + \int_0^t g\left(s, \bar{X}_s\right) d\bar{B}_s, \ \forall\, t \in [0, T],$$

and consequently $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \mathcal{F}_t^{\bar{B},\bar{X}}, \bar{X}_t, \bar{K}_t, \bar{B}_t\right)_{t\geq 0}$ is a weak solution. □

Since the stochastic process K is uniquely determined by (X, B) via equation (45), then a weak solution for the stochastic differential equation is a sextuplet $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, X, B)$. We know that weak existence and pathwise uniqueness implies strong existence (see Theorem 3.55 in [31] or Theorem 1.1 in [23]). Hence we deduce from Theorem 39 and Proposition 37:

Theorem 41. *Let assumptions (9), (10), (16) and (17) be satisfied. The functions f and g are suppose to be $\left(\mathcal{B}_1, \mathbb{R}^d\right)$-Carathéodory functions satisfying moreover assumption* (A_3) *and boundedness conditions*

$$\int_0^T \left[|f^{\#}(s)|^2 + |g^{\#}(s)|^4\right] ds < \infty, \ \forall T \geq 0.$$

If $x_0 \in \overline{\mathrm{Dom}(\varphi)}$ then equation (45) has a unique strong solution $(X_t, K_t)_{t\geq 0}$.

Acknowledgments

The work of A.R. and R.B. was supported by grant "Deterministic and stochastic systems with state constraints", code 241/05.10.2011 and the work of L.M. by grant POSDRU/89/1.5/S/49944.

Appendix A

A.1. Applications of Fatou's Lemma

The next result is a well known consequence of weak convergence of probability measures (for its proof see, e.g. [31, Proposition 1.22]).

Proposition 42. *Let $(\mathbb{X}, \rho)$ be a separable metric space. Let $\varphi : \mathbb{X} \to (-\infty, \infty]$ be a lower semicontinuous function. If X and X_n are $\mathbb{X}$-valued random variables, for $n \in \mathbb{N}^*$, such that*

$$(i) \quad X_n \xrightarrow{law} X, \ as\ n \to \infty,$$

and there exists a continuous function $\alpha : \mathbb{X} \to \mathbb{R}$ such that

$$(ii) \quad \alpha(x) \leq \varphi(x), \quad \forall\, x \in \mathbb{X},$$
$$(iii) \quad \{\alpha(X_n) : n \in \mathbb{N}^*\} \ \textit{is a uniformly integrable family,}$$

then the expectations $\mathbb{E}\varphi(X)$ *and* $\mathbb{E}\varphi(X_n)$ *exist for all* $n \in \mathbb{N}$, *and*

$$-\infty < \mathbb{E}\varphi(X) \leq \liminf_{n\to+\infty} \mathbb{E}\varphi(X_n) \ .$$

For $0 \leq s < t \leq T$, we denote by $\updownarrow X \updownarrow_{[s,t]}$ (similar to (7)) the total variation of $X_\cdot$ on $[s,t]$, that is

$$\updownarrow X \updownarrow_{[s,t]}(\omega) = \sup\left\{\sum_{i=0}^{n-1} \left|X_{t_{i+1}}(\omega) - X_{t_i}(\omega)\right| : n \in \mathbb{N}^*, s = t_0 < t_1 < \cdots < t_n = t\right\}$$

We also use $\updownarrow X \updownarrow_T := \updownarrow X \updownarrow_{[0,T]}$.

Applying successively Proposition 42 for

$$\varphi(x) = \mathrm{d}_F(x(t)),$$

$$\varphi(x,y) = \left(\sum_{i=0}^{N-1} |x(t_{i+1}) - x(t_i)| - g(y)\right)^+,$$

where $s = t_0 < t_1 < \ldots < t_N = t$ is an arbitrary partition of $[s.t]$, and respectively

$$\varphi(x) = (x(s) - x(t))^+,$$

we get the next result:

Corollary 43. *Let* s, t *be arbitrary fixed such that* $0 \leq s \leq t \leq T$. *If* $g : C\left([0,T];\mathbb{R}^d\right) \to \mathbb{R}_+$ *is a continuous function and* X, V, X^n, V^n, $n \in \mathbb{N}^*$, *are random variables with values in* $C\left([0,T];\mathbb{R}^d\right)$, *such that*

$$\left(X^n, V^n\right) \xrightarrow{law} (X, V), \ as\ n \to \infty,$$

then the following implications hold true:

(*a*) *If* $X_t^n \in F$ *a.s., then* $X_t \in F$, *a.s., whenever* F *is closed subset of* $\mathbb{R}^d$;
(*b*) *If* $\updownarrow X^n \updownarrow_{[s,t]} \leq g(V^n)$ *a.s., then* $\updownarrow X \updownarrow_{[s,t]} \leq g(V)$, *a.s.;*
(*c*) *If* $d = 1$ *and* $X_s^n \leq X_t^n$ *a.s., then* $X_s \leq X_t$, *a.s.*

Now let us consider the partition

$$\Delta_N : s = r_0 < r_1 < \ldots < r_N = t, \ \ r_{i+1} - r_i = \frac{t-s}{N}$$

and the function $g : C\left([0,T];\mathbb{R}^d\right) \to [0,1]$, defined by

$$g(x,k,v) := \left(\sum_{i=0}^{N-1} |k(r_{i+1}) - k(r_i)| - v(t) + v(s)\right)^+ \wedge 1$$
$$+ \left[\int_s^t \varphi(x(r))\,dr - \sum_{i=0}^{N-1} \langle x(r_i), k(r_{i+1}) - k(r_i)\rangle - \mathbf{m}_x(1/N)(v(t) - v(s))\right]^+ \wedge 1.$$

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Applying again the generalization of the Fatou's Lemma (Proposition 42), it can be proved:

Corollary 44. *Let* (X, K, V), (X^n, K^n, V^n), $n \in \mathbb{N}$, *be* $C\left([0,T];\mathbb{R}^d\right)^2 \times C([0,T];\mathbb{R})$*-valued random variables, such that*

$$\left(X^n, K^n, V^n\right) \xrightarrow[n\to\infty]{law} (X, K, V)$$

and for all $0 \le s < t$, *and* $n \in \mathbb{N}^*$,

$$\updownarrow K^n \updownarrow_t - \updownarrow K^n \updownarrow_s \le V_t^n - V_s^n \quad a.s.$$

If $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ *is a lower semicontinuous function and*

$$\int_s^t \varphi\left(X_r^n\right) dr \le \int_s^t \left\langle X_r^n, dK_r^n \right\rangle, \text{ a.s. for all } n \in \mathbb{N}^*,$$

then

$$\updownarrow K \updownarrow_t - \updownarrow K \updownarrow_s \le V_t - V_s, \text{ a.s.}$$

and

$$\int_s^t \varphi\left(X_r\right) dr \le \int_s^t \left\langle X_r\ , dK_r \right\rangle, \text{ a.s.}$$

A.2. Complements on tightness

If $\left\{X_t^n : t \ge 0\right\}$, $n \in \mathbb{N}^*$, is a family of continuous stochastic processes then the following result is a consequence of the Arzelà–Ascoli theorem (see, e.g., Theorem 7.3 in [6]).

We recall the notations:

$$\|X^n\|_T := \sup\left\{\left|X_t^n\right| : t \in [0,T]\right\},$$
$$\mathbf{m}_{X^n}(\varepsilon; [0,T]) := \sup\left\{\left|X_t^n - X_s^n\right| : t, s \in [0,T],\ |t-s| \le \varepsilon\right\}.$$

Theorem 45. *The family* $\{X^n : n \in \mathbb{N}^*\}$ *is tight in* $C(\mathbb{R}_+;\mathbb{R}^d)$ *if and only if, for every* $T \ge 0$,

$$(j) \quad \lim_{N\nearrow\infty}\left[\sup_{n\ge1}\mathbb{P}^{(n)}\left(\left|X_0^n\right| \ge N\right)\right] = 0,$$

$$(jj) \quad \lim_{\varepsilon\searrow0}\left[\sup_{n\ge1}\mathbb{P}^{(n)}\left(\mathbf{m}_{X^n}(\varepsilon;[0,T]) \ge a\right)\right] = 0, \quad \forall a > 0.$$

Moreover, tightness yields that for all $T > 0$

$$\lim_{N\nearrow\infty}\left[\sup_{n\ge1}\mathbb{P}^{(n)}\left(\left\|X^n\right\|_T \ge N\right)\right] = 0.$$

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

Without using the above theorem, it can be proved the following criterion for tightness which is well adapted to our needs. The proof can be found in E. Pardoux and A. Răşcanu [31] (Proposition 1.53) and we will give the sketch of the proof.

Proposition 46. *Let $\{X_t^n : t \geq 0\}$, $n \in \mathbb{N}^*$, be a family of $\mathbb{R}^d$-valued continuous stochastic processes defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that for every $T \geq 0$, there exist $\alpha > 0$ and $b \in C(\mathbb{R}_+)$ with $b(0) = 0$, such that*

$$(j) \quad \lim_{N\to\infty} \left[\sup_{n\in\mathbb{N}^*} \mathbb{P}(\{|X_0^n| \geq N\})\right] = 0,$$

$$(jj) \quad \mathbb{E}\left[1 \wedge \sup_{0\leq s\leq\varepsilon} |X_{t+s}^n - X_t^n|^\alpha\right] \leq \varepsilon \cdot b(\varepsilon), \ \forall \varepsilon > 0, \ n \geq 1, \ t \in [0, T].$$

Then the family $\{X^n : n \in \mathbb{N}^*\}$ is tight in $C(\mathbb{R}_+; \mathbb{R}^d)$.

Proof. We fix $\varepsilon, T > 0$. From (j), there exists $M = M_\varepsilon \geq 1$ such that

$$\sup_{n\in\mathbb{N}^*} \mathbb{P}(\{|X_0^n| \geq M\}) < \frac{\varepsilon}{2}.$$

Let $\gamma_k = \dfrac{1}{2^{(k-1)/\alpha}}$ and $\varepsilon_k \searrow 0$ be such that $b(\varepsilon_k) \leq \dfrac{\varepsilon}{4^k\, T}$. Let $N_k = \left[\dfrac{T}{\varepsilon_k}\right]$ and $t_i = \dfrac{(i-1)T}{N_k}$.

Applying Theorem Arzelá–Ascoli we see that the set

$$\mathcal{K}_\varepsilon = \Big\{z \in C([0, T]; \mathbb{R}^d) : |z(0)| \leq M, \sup_{1\leq i\leq N_k} \sup_{0<s\leq\varepsilon_k} |z(t_i + s) - z(t_i)| \leq \gamma_k, \forall k \in \mathbb{N}^*\Big\}$$

is compact in $C([0, T]; \mathbb{R}^d)$.

From Markov's inequality and (jj)

$$\mathbb{P}\left(X^n \notin \mathcal{K}_\varepsilon\right) \leq \mathbb{P}(\{|X_0^n| > M\}) + \sum_{k\in\mathbb{N}^*} \sum_{i=1}^{N_k} \mathbb{P}(\{\sup_{0\leq s\leq\varepsilon_k} |X_{t_i+s}^n - X_{t_i}^n| > \gamma_k\})$$

$$< \frac{\varepsilon}{2} + \sum_{k\in\mathbb{N}^*} \sum_{i=1}^{N_k} \frac{\varepsilon_k \times b(\varepsilon_k)}{\gamma_k^\alpha} = \frac{\varepsilon}{2} + \sum_{k\in\mathbb{N}^*} \frac{N_k \times \varepsilon_k \times b(\varepsilon_k)}{\gamma_k^\alpha} \leq \varepsilon.$$

The proof is complete now. □

Proposition 47. *Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function satisfying $g(0) = 0$ and $G : C\left(\mathbb{R}_+; \mathbb{R}^d\right) \to \mathbb{R}_+$ be a mapping which is bounded on compact subsets of $C\left(\mathbb{R}_+; \mathbb{R}^d\right)$. Let X^n, Y^n, $n \in \mathbb{N}^*$, be random variables with values in $C\left(\mathbb{R}_+; \mathbb{R}^d\right)$. If $\{Y^n : n \in \mathbb{N}^*\}$ is tight and for all $n \in \mathbb{N}^*$*

$$(i) \quad \left|X_0^n\right| \leq G\left(Y^n\right), \ a.s.$$
$$(ii) \quad \mathbf{m}_{X^n}\left(\varepsilon;[0,T]\right) \leq G\left(Y^n\right) g\left(\mathbf{m}_{Y^n}\left(\varepsilon;[0,T]\right)\right), \ a.s., \ \forall\, \varepsilon, T>0,$$

then $\{X^n : n \in \mathbb{N}^*\}$ *is tight.*

Proof. Let $\delta > 0$ be arbitrary. Then there exists a compact set $K_\delta \subset C\left([0,\infty[\,;\mathbb{R}^d\right)$ such that for all $n \in \mathbb{N}^*$

$$\mathbb{P}\left(Y^n \notin K_\delta\right) < \delta\ .$$

Define $N_\delta = \sup_{x\in K_\delta} G(x)$. Then

$$\mathbb{P}\left(\left|X_0^n\right| > N_\delta\right) < \delta.$$

Let $a > 0$ be arbitrary. There exists $\varepsilon_0 > 0$ such that

$$\sup_{x\in K_\delta}\left[g\left(\mathbf{m}_x\left(\varepsilon;[0,T]\right)\right)\right] < \frac{a}{N_\delta}, \ \forall\, 0<\varepsilon<\varepsilon_0\ .$$

Consequently for all $n \in \mathbb{N}^*$,

$$\mathbb{P}\left(\mathbf{m}_{X^n}\left(\varepsilon;[0,T]\right) \geq a\right) \leq \mathbb{P}\left[g\left(\mathbf{m}_{Y^n}\left(\varepsilon;[0,T]\right)\right) \geq \tfrac{a}{N_\delta}, \ Y^n \in K_\delta\right] + \mathbb{P}\left(Y^n \notin K_\delta\right) \leq \delta$$

and the result follows. □

A.3. Itô's stochastic integral

In this subsection we consider $\{B_t : t \geq 0\}$ to be a k-dimensional Brownian motion on a stochastic basis (which is supposed to be complete and right-continuous) $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}\right)$.

Let $S_d[0,T]$ be the space of p.m.c.s.p. $X : \Omega \times [0,T] \to \mathbb{R}^d$ and $\Lambda_d(0,T)$ the space of p.m.c.s.p. $X : \Omega \times [0,T] \to \mathbb{R}^d$ such that

$$\int_0^T |X_t|^2\, dt < \infty, \ \mathbb{P}\text{-a.s.}$$

Write S_d (and Λ_d) for space of p.m.c.s.p. $X : \Omega \times [0,T] \to \mathbb{R}^d$ such that the restriction of X to $[0,T]$ belongs to S_d (respectively to Λ_d).

If $X \in S_{d\times k}$ and B is an $\mathbb{R}^k$-Brownian motion, then the stochastic process $\{(X_t, B_t) : t \geq 0\}$ can be seen as a random variable with values in the space $C(\mathbb{R}_+, \mathbb{R}^{d\times k}) \times C(\mathbb{R}_+, \mathbb{R}^k)$. The law of this random variable will be denoted $\mathcal{L}(X, B)$.

Proposition 48. *(See Corollary 2.13 in [31].) Let* $X, \hat{X} \in S_d[0,T]$, B, $\hat{B}$ *be two* $\mathbb{R}^k$*-Brownian motions and* $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ *be a function such that*

$$g(\cdot, y) \text{ is measurable } \forall\, y \in \mathbb{R}^d,$$
$$y \mapsto g(t, y) \text{ is continuous } dt\text{-a.e.}$$

If

$$\mathcal{L}(X, B) = \mathcal{L}(\hat{X}, \hat{B}) \text{ on } C(\mathbb{R}_+, \mathbb{R}^{d+k}),$$

then

$$\mathcal{L}\Big(X, B, \int_0^{\cdot} g(s, X_s)\, dB_s\Big) = \mathcal{L}\Big(\hat{X}, \hat{B}, \int_0^{\cdot} g(s, \hat{X}_s) d\hat{B}_s\Big) \text{ on } C(\mathbb{R}_+, \mathbb{R}^{d+k+d}).$$

We present now a continuity property of the mapping

$$(X, B) \longrightarrow \int_0^T X_s dB_s\ .$$

Given $B : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^k$ and $X : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^{d\times k}$ be two stochastic processes, let $\mathcal{F}_t^{B,X}$ be the natural filtration generated jointly by B and X.

For the proof of the next proposition see Proposition 2.4 in [9] or Proposition 2.14 in [31].

Proposition 49. *Let $B, B^n, \tilde{B}^n : \Omega \times \mathbb{R}_+ \to \mathbb{R}^k$ and $X, X^n, \tilde{X}^n : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{d\times k}$ be continuous stochastic processes such that*

(i) $\tilde{B}^n$ *is* $\mathcal{F}_t^{\tilde{B}^n, \tilde{X}^n}$*-Brownian motion* $\forall\, n \geq 1$*;*
(ii) $\mathcal{L}(\tilde{B}^n, \tilde{X}^n) = \mathcal{L}(B^n, X^n)$ *on* $C(\mathbb{R}_+, \mathbb{R}^k \times \mathbb{R}^{d\times k})$*, for all* $n \geq 1$*;*
(iii) $\int_0^T \left|X_s^n - X_s\right|^2 ds + \sup_{t\in[0,T]} \left|B_t^n - B_t\right| \to 0$*, in probability, as* $n \to \infty$*, for all* $T > 0$*.*

Then $\big(B^n, \{\mathcal{F}_t^{B^n, X^n}\}\big)$, $n \geq 1$, *and* $\big(B, \{\mathcal{F}_t^{B,X}\}\big)$ *are Brownian motions and as* $n \to \infty$

$$\sup_{t\in[0,T]} \left| \int_0^t X_s^n dB_s^n \longrightarrow \int_0^t X_s dB_s \right| \longrightarrow 0 \quad \text{in probability.} \tag{50}$$

A.4. A forward stochastic inequality

Let $X, \hat{X} \in S_d$ be two semimartingales defined by

$$X_t = X_0 + K_t + \int_0^t G_s dB_s\ ,\ t \geq 0, \quad \hat{X}_t = \hat{X}_0 + \hat{K}_t + \int_0^t \hat{G}_s dB_s\ ,\ t \geq 0, \tag{51}$$

where

- $K, \hat{K} \in S_d$;
- $K_{\cdot}(\omega), \hat{K}_{\cdot}(\omega) \in BV_{loc}\left(\mathbb{R}_+; \mathbb{R}^d\right)$, $K_0(\omega) = \hat{K}_0(\omega) = 0$, $\mathbb{P}$-a.s.;
- $G, \hat{G} \in \Lambda_{d\times k}$.

Assume that there exist $p \geq 1$ and $\lambda \geq 0$ and V a bounded variation p.m.c.s.p., with $V_0 = 0$, such that, as measures on $\mathbb{R}_+$,

$$\left\langle X_t - \hat{X}_t, dK_t - d\hat{K}_t \right\rangle + \left(\frac{1}{2}m_p + 9p\lambda\right)|G_t - \hat{G}_t|^2 dt \leq |X_t - \hat{X}_t|^2 dV_t. \tag{52}$$

Theorem 50. *(See Corollary 6.74 in [31].) Let $p \geq 1$. If the assumption (52) is satisfied with $\lambda > 1$, then there exists a positive constant $C_{p,\lambda}$ such that for all $\delta \geq 0$, $0 \leq t \leq s$:*

$$\mathbb{E}^{\mathcal{F}_t} \frac{||e^{-V}(X - \hat{X})||^p_{[t,s]}}{\left(1 + \delta||e^{-V}(X - \hat{X})||^2_{[t,s]}\right)^{p/2}} \leq C_{p,\lambda} \frac{e^{-pV_t}|X_t - \hat{X}_t|^p}{\left(1 + \delta e^{-2V_t}|X_t - \hat{X}_t|^2\right)^{p/2}}, \quad \mathbb{P}\text{-a.s.}$$

In particular for $\delta = 0$

$$\mathbb{E}^{\mathcal{F}_t}||e^{-V}(X - \hat{X})||^p_{[t,s]} \leq C_{p,\lambda}\, e^{-pV_t}|X_t - \hat{X}_t|^p, \quad \mathbb{P}\text{-a.s.},$$

for all $0 \leq t \leq s$.

As a consequence of the above theorem, since

$$\frac{1}{2}(1 \wedge r) \leq \frac{r}{\left(1 + r^2\right)^{1/2}} \leq 1 \wedge r, \quad \forall r \geq 0,$$

we obtain:

Corollary 51. *If assumption (52) is satisfied with $\lambda > 1$ and $p \geq 1$, then there exists a positive constant $C_{p,\lambda}$ depending only on (p, λ) such that $\mathbb{P}$-a.s.*

$$\mathbb{E}^{\mathcal{F}_t}\left[1 \wedge ||e^{-V}(X - \hat{X})||^p_{[t,s]}\right] \leq C_{p,\lambda}\left[1 \wedge |e^{-V_t}(X_t - \hat{X}_t)|^p\right],$$

for all $0 \leq t \leq s$.

References

[1] Ioan Asiminoaei, Aurel Răşcanu, Approximation and simulation of stochastic variational inequalities – splitting up method, Numer. Funct. Anal. Optim. 18 (3–4) (1997) 251–282.
[2] Viorel Barbu, Aurel Răşcanu, Parabolic variational inequalities with singular inputs, Differential Integral Equations 10 (1) (1997) 67–83.
[3] Houcine Benabdellah, Existence of solutions to the nonconvex sweeping process, J. Differential Equations 164 (2000) 286–295.
[4] Houcine Benabdellah, Charles Castaing, Anna Salvadori, Aicha Syam, Nonconvex sweeping processes, J. Appl. Anal. 2 (1996) 217–240.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

[5] Alain Bensoussan, Aurel Răşcanu, Stochastic variational inequalities in infinite-dimensional spaces, Numer. Funct. Anal. Optim. 18 (1–2) (1997) 19–54.
[6] Patrick Billingsley, Convergence of Probability Measures, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1999.
[7] Jonathan M. Borwein, Adrian S. Lewis, Convex Analysis and Nonlinear Optimization. Theory and Examples, second edition, CMS Books in Mathematics, Springer, New York, 2006.
[8] Stephen Boyd, Lieven Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
[9] Rainer Buckdahn, Aurel Răşcanu, On the existence of stochastic optimal control of distributed state system, Nonlinear Anal. 52 (4) (2003) 1153–1184.
[10] Tiziana Cardinali, Giovanni Colombo, Francesca Papalini, Mario Tosques, On a class of evolution equations without convexity, Nonlinear Anal. 28 (2) (1997) 217–234.
[11] Charles Castaing, Sur une nouvelle classe d'équation d'évolution dans les espaces de Hilbert, Sém. d'Anal. Convexe Montpellier 13 (10) (1983), 28 pp.
[12] Charles Castaing, Manuel D.P. Monteiro Marques, Evolution problems associated with nonconvex closed moving sets with bounded variation, Port. Math. 53 (1996) 73–87.
[13] Emmanuel Cépa, Équations différentielles stochastiques multivoques, in: Séminaire de Probabilités, XXIX, in: Lecture Notes in Math., vol. 1613, Springer, Berlin, 1995, pp. 86–107.
[14] Emmanuel Cépa, Problème de Skorohod multivoque, Ann. Probab. 26 (2) (1998) 500–532.
[15] Aurelian Cernea, Vasile Staicu, Existence of solutions to a class of evolution inclusions, Nonlinear Anal. 50 (2002) 1025–1034.
[16] Giovanni Colombo, Vladimir V. Goncharov, The sweeping processes without convexity, Set-Valued Var. Anal. 7 (1999) 357–374.
[17] Giovanni Colombo, Vladimir Goncharov, Variational inequalities and regularity properties of closed sets in Hilbert spaces, J. Convex Anal. 8 (1) (2001) 197–221.
[18] Giovanni Colombo, Manuel D.P. Monteiro Marques, Sweeping by a continuous prox-regular set, J. Differential Equations 187 (2003) 46–62.
[19] Marco Degiovanni, Antonio Marino, Mario Tosques, Evolution equations with lack of convexity, Nonlinear Anal. 9 (2) (1985) 1401–1443.
[20] Paul Dupuis, Hitoshi Ishii, SDEs with oblique reflection on nonsmooth domains, Ann. Probab. 21 (1) (1993) 554–580.
[21] Jean Fenel Edmond, Lionel Thibault, BV solutions of nonconvex sweeping process differential inclusion with perturbation, J. Differential Equations 226 (2006) 135–179.
[22] Anouar M. Gassous, Aurel Răşcanu, Eduard Rotenstein, Stochastic variational inequalities with oblique subgradients, Stochastic Process. Appl. 122 (2012) 2668–2700.
[23] Nobuyuki Ikeda, Shinzo Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, Amsterdam, 1981.
[24] Pierre-Louis Lions, Alain-Sol Sznitman, Stochastic differential equations with reflecting boundary conditions, Comm. Pure Appl. Math. 37 (4) (1984) 511–537.
[25] Antonio Marino, Claudio Saccon, Mario Tosques, Curves of maximal slope and parabolic variational inequalities on nonconvex constraints, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 16 (2) (1989) 281–330.
[26] Antonio Marino, Mario Tosques, Some variational problems with lack of convexity and some partial differential inequalities, in: A. Cellina (Ed.), Methods of Nonconvex Analysis, Proceedings from CIME, Varenna 1989, Springer, Berlin, 1990.
[27] Alexander Mielke, Riccarda Rossi, Existence and uniqueness results for a class of rate-independent hysteresis problems, Math. Models Methods Appl. Sci. 17 (2007) 81–123.
[28] Manuel D.P. Monteiro Marques, Differential Inclusions in Nonsmooth Mechanical Problems—Shocks and Dry Friction, Birkhäuser, Boston, 1993.
[29] Jean Jacques Moreau, Evolution problem associated with a moving convex set in a Hilbert space, J. Differential Equations 26 (1977) 347–374.
[30] Francesca Papalini, Nonlinear evolution equations without convexity, J. Math. Anal. Appl. 193 (1995) 406–418.
[31] Etienne Pardoux, Aurel Răşcanu, Stochastic Differential Equations, Backward SDEs, Partial Differential Equations, Stochastic Modelling and Applied Probability, vol. 69, Springer, 2014.
[32] Aurel Răşcanu, Deterministic and stochastic differential equations in Hilbert spaces involving multivalued maximal monotone operators, Panamer. Math. J. 6 (3) (1996) 83–119.
[33] Aurel Răşcanu, Eduard Rotenstein, A non-convex setup for multivalued differential equations driven by oblique subgradients, Nonlinear Anal. 111 (2014) 82–104.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023

[34] Thomas Roche, Uniqueness of a quasivariational sweeping process on functions of bounded variation, Ann. Sc. Norm. Super. Pisa Cl. Sci. XI (2) (2012) 363–394.
[35] R. Tyrrell Rockafellar, Roger J.-B. Wets, Variational Analysis, Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, 1997.
[36] Riccarda Rossi, Giuseppe Savaré, Existence and approximation results for gradient flows, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. IX Ser. Mat. Appl. 15 (3–4) (2004) 183–196.
[37] Riccarda Rossi, Giuseppe Savaré, Gradient flows of non convex functionals in Hilbert spaces and applications, ESAIM Control Optim. Calc. Var. 12 (3) (2006) 564–614.
[38] Yasumasa Saisho, Stochastic differential equations for multidimensional domain with reflecting boundary, Probab. Theory Related Fields 74 (3) (1987) 455–477.
[39] Anatoliy V. Skorokhod, Stochastic equations for diffusion processes in a bounded region, Theory Probab. Appl. 6 (3) (1961) 264–274.
[40] Anatoliy V. Skorokhod, Stochastic equations for diffusion processes in a bounded region. II, Theory Probab. Appl. 7 (1) (1962) 3–23.
[41] Hiroshi Tanaka, Stochastic differential equations with reflecting boundary conditions in convex regions, Hiroshima Math. J. 9 (1) (1979) 163–177.

Please cite this article in press as: R. Buckdahn et al., Stochastic variational inequalities on non-convex domains, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.023