Some highlights on the equivalent creteria in the convergence of series

Written by Prof. Bradji, Abdallah

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Provisional home page: `http://www.cmi.univ-mrs.fr/~bradji`

Aim of this note

It is known that if two sequences U_n and V_n have a constant sign for a sufficiently large n and $\lim_{n\to+\infty}\frac{U_n}{V_n} = L \neq 0$ then the series corresponding to the general terms U_n and V_n converge or diverge together. We will show, using a counter example, that when U_n (or V_n) has not the same sign, the above implication is not true.

1 Details

Assume that U_n and V_n are two sequences and they have a constant sign for a sufficiently large n and $\lim_{n\to+\infty}\frac{U_n}{V_n} = L \neq 0$. We assume without loss of generality that U_n and V_n are positive (elsewhere, we consider $-U_n$ and $-V_n$). There exists $N \in \mathbb{N}^\star$ such that for all $n \geq N$, we have

$$\frac{L}{2} \leq \frac{U_n}{V_n} \leq \frac{3L}{2}. \tag{1}$$

This gives that, for all $n \geq N$

$$\frac{L}{2}V_n \leq U_n \leq \frac{3L}{2}V_n. \tag{2}$$

Using the Comparison criteria, the series corresponding to the general terms U_n and V_n converge or diverge together.
If we assume only that $\lim_{n\to+\infty}\frac{U_n}{V_n} = L \neq 0$, i.e in the absence of the assumption that U_n and V_n have a constant sign, we can not ensure that the series corresponding to the general terms U_n and V_n converge or diverge together. Let us consider the example of two series with the following general terms:

$$U_n = \frac{(-1)^n}{\sqrt{n}}, \quad n \geq 2 \tag{3}$$

and

$$V_n = \ln(1 + U_n), \quad n \geq 2. \tag{4}$$

Since

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1 \tag{5}$$

and

$$\lim_{n\to+\infty} U_n = 0 \tag{6}$$

then

$$\lim_{n\to+\infty}\frac{\ln(1+U_n)}{U_n} = 1. \tag{7}$$

Which is

$$\lim_{n\to+\infty}\frac{V_n}{U_n} = 1. \tag{8}$$

The series $\sum_{n\geq 2} U_n$ of general term U_n converges thanks to alternating test since $\frac{1}{\sqrt{n}}$ decreases monotonically and $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$. However, we will show that the series $\sum_{n\geq 2} V_n$ of general term V_n diverges. Indeed, using a Taylor expansion yields

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}\frac{1}{(1+\theta x)^3}, \tag{9}$$

where $\theta \in (0,1)$.
Taking $x = U_n$ in (9) yields that

$$V_n - U_n = -\frac{U_n^2}{2} + \xi_n, \tag{10}$$

where

$$\xi_n = \frac{U_n^3}{3} \frac{1}{(1+\theta U_n)^3}. \tag{11}$$

We first remark that the series whose the general term is given by the first term in right hand side of (10) converges, i.e.

$$-\sum_{n\geq 2} \frac{U_n^2}{2} = -\sum_{n\geq 2} \frac{1}{2n} \quad \text{diverges}. \tag{12}$$

In addition to this, since $\lim_{n\to\infty} U_n = 0$, then $\lim_{n\to\infty} \frac{1}{(1+\theta U_n)^3} = 1$. Consequently the sequence $\frac{1}{(1+\theta U_n)^3}$ is bounded, i.e. there exists $M > 0$ such that

$$\left| \frac{1}{(1+\theta U_n)^3} \right| \leq M, \quad \forall n \geq 2. \tag{13}$$

This implies that

$$|\xi_n| \leq \frac{M}{3n^{\frac{3}{2}}}. \tag{14}$$

Consequently, the series $\sum_{n\geq 2} \xi_n$ is absolutely convergent, i.e.

$$\sum_{n\geq 2} |\xi_n| \quad \text{converges}. \tag{15}$$

Therefore

$$\sum_{n\geq 2} \xi_n \quad \text{converges}. \tag{16}$$

Using (10), (12), and (16), we remark that $\sum_{n\geq 2}(V_n - U_n)$ is the sum of two series: one is convergent and the other one is divergent. Consequently $\sum_{n\geq 2}(V_n - U_n)$ is divergent. Therefore, since $\sum_{n\geq 2} U_n$ is convergent, $\sum_{n\geq 2}(V_n - U_n) + \sum_{n\geq 2} U_n$ divergent. Which gives that

$$\sum_{n\geq 2} V_n \quad \text{diverges}. \tag{17}$$

References

[1] K. Allab, Elément d'Analyse: Fonction d'une Variable Réelle. OPU, 1990.

[2] W. F. Trench, Introduction to Real Analysis. ISBN 0-13-045786-8, Free Hyperlinked Edition 2.03, November 2012.