A brief Report on the article [SHI 09]: "Nonconforming H^1–Galerkin Mixed FEM for Sobolev Equations on Anisotropic Meshes"

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Abstract

The authors consider a nonconforming mixed finite element approximation for the Sobolev equation on anisotropic meshes. Convergence order is provided in the both cases of semi and full discretization schemes. The error estimates are obtained without using Ritz–Volterra projection.

1 Introduction

It is well known that Sobolev equations have been found applications in many physical problems, such as the porous theories concerned with percolation into rocks with cracks, the transport problems of humidity in soil, the heat-conduction problems in different mediums and so on.

However, all of the above studies using finite element methods rely on the regularity assumption or the quasi-uniform assumption. i.e., $\frac{h_K}{\rho_K} \leq C_1$ and $\frac{h}{h_K} \leq C_2$, where K is an element, h_K, ρ_K denote the diameter of K and the biggest circle contained in K, respectively. C_1 and C_2 are two constants independent of $h = \max_K\{h_K\}$ and the function considered. But when the domain concerned is very narrow, if we employ the regular partition, the computing cost will be very high. The obvious idea to overcome this difficulty is to use the anisotropic meshes with fewer degrees of freedom.

2 Sobolev equations

Let Ω be an open polygonal subset of $\mathbb{R}^2$ which is composed by rectanglar subsets $\mathcal{T}_h$. The authors considered the following Sobolev equations:

$$u_t(x,t) - \nabla\cdot\left(a(x,t)\nabla u_t(x,t) + b(x,t)\nabla u(x,t)\right) = f(x,t),\ x\in\Omega,\ t\in(0,T], \tag{1}$$

where T is a given positive constant; the function a, b are continuous functions with bounded derivatives and

$$|\,b(x,t)| \le b_1,\ 0 < a_0 \le |\,b(x,t)| \le a_1,\ x\in\Omega,\ t\in(0,T], \tag{2}$$

$$u(x,0) = u_0,\ x\in\Omega \tag{3}$$

and

$$u(x,t) = 0,\ x\in\partial\Omega,\ t\in(0,T]. \tag{4}$$

Sobolev equations have been found applications in many physical problems, such as the porous theories concerned with percolation into rocks with cracks[4], the transport problems of humidity in soil, the heat-conduction problems[19] in different mediums and so on.

3 Formulation for problem [1]–[4]

Let us set $q = a\nabla u(x,t)$; so equation [1] becomes as:

$$u_t(x,t) - \nabla\cdot\left(q_t(x,t) + \beta(x,t)\,q(x,t)\right) = f(x,t),\ x\in\Omega,\ t\in(0,T], \tag{5}$$

where

$$\beta(x,t) = \frac{b(x,t) - a_t(x,t)}{a(x,t)}q(x,t)$$

To define a weak formulation for problem [1]–[4], we denote by $(\cdot,\cdot)$ the usual inner product, $H(\div;\Omega) = \{v\in\left(L^2(\Omega)\right)^2, \nabla\cdot v = 0\}$.

The following weak formulation is given in [SHI 09], where $\alpha = \frac{1}{a}$: find $\{u,q\} : [0,T]\to H_0^1(\Omega)\times H(div;\Omega)$ such that

$$(\nabla u, \nabla v) = (\alpha\,q, \nabla v),\ \forall v\in H_0^1(\Omega) \tag{6}$$

$$(\alpha_t\,q, w) + (\alpha\,q_t, w) + (\nabla\cdot(q_t(x,t) + \beta(x,t)\,q(x,t)), \nabla\cdot w) = -(f, \nabla\cdot w),\ \forall w\in H(div;\Omega) \tag{7}$$

and

$$u(x,0) = u_0(x),\ x\in\Omega. \tag{8}$$

4 Anisotropic finite element discretization

The finite element discretization is performed thanks to partition by rectangles $K \in \mathcal{T}_h$. For $K \in \mathcal{T}_h$, the lenght of edges parallel to x and y–axis are $2h_x$ and $2h_y$, and (x_K, y_K) is the center; the four vertices are then denoted by $a_1(x_K - h_x, y_K - h_y)$, $a_2(x_K + h_x, y_K - h_y)$, $a_3(x_K + h_x, y_K + h_y)$, and $a_4(x_K - h_x, y_K + h_y)$. The four edges then of K are denoted by $\overline{a_1a_2}$, $\overline{a_2a_3}$, $\overline{a_3a_4}$ and $\overline{a_4a_1}$. Let $\hat{K} = [-1,1] \times [-1,1]$ be the reference element; the four vertices of $\hat{K}$ are then denoted by $\hat{a_1} = (-1,-1)$, $\hat{a_2} = (1,-1)$, $\hat{a_3} = (1,1)$, and $\hat{a_4} = (-1,1)$. The four edges of $\hat{K}$ are $\bar{l}_1 = \overline{\hat{a_1}\hat{a_2}}$, $\bar{l}_2 = \overline{\hat{a_2}\hat{a_3}}$, $\bar{l}_3 = \overline{\hat{a_3}\hat{a_4}}$ and $\bar{l}_4 = \overline{\hat{a_4}\hat{a_1}}$.
The affine mapping from $\hat{K}$ to K can be defined as: $x = x_K + h_x\hat{x}$ and $y = y_K + h_y\hat{y}$ We introduce two finite element types:

- first type:

$$\hat{\mathcal{P}}^1 = \mathrm{span}\{1, \hat{\mathrm{x}}, \hat{\mathrm{y}}, \varphi(\hat{\mathrm{x}}), \varphi(\hat{\mathrm{y}})\}, \qquad [9]$$

  where $\varphi(t) = \frac{1}{2}(3t^2 - 1)$; but i do not what is the difference between $\mathrm{span}\{1, \hat{\mathrm{x}}, \hat{\mathrm{y}}, \varphi(\hat{\mathrm{x}}), \varphi(\hat{\mathrm{y}})\}$ and $\mathrm{span}\{1, \hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{x}}^2, \hat{\mathrm{y}}^2\}$

- second type:

$$\hat{\mathcal{P}}^2 = \mathrm{span}\{1, \hat{\mathrm{x}}, \hat{\mathrm{y}}\} \times \mathrm{span}\{1, \hat{\mathrm{x}}, \hat{\mathrm{y}}\} \qquad [10]$$

- first interpolation type:

$$\frac{1}{\hat{K}} \int_{\hat{K}} \left(\hat{v} - \hat{I}^1 \hat{v} \right) d\hat{x} d\hat{y} = 0 \text{ and } \frac{1}{\hat{l}} \int_{\hat{l}} \left(\hat{v} - \hat{I}^1 \hat{v} \right) ds = 0. \qquad [11]$$

- second interpolation type::

$$\frac{1}{\hat{l}} \int_{\hat{l}} \hat{I}^2 \hat{v}\, ds = \frac{1}{2} \left(\hat{v}(\overline{a}_i) + \hat{v}(\overline{l}_{i+1}) \right) \qquad [12]$$

To define the finite element spaces, we define the affine application defined on $\hat{K}$ into K

$$x = x_K + h_x\hat{x} \text{ and } y = y_K + h_y\hat{y}. \qquad [13]$$

4.1 Finite element spaces

- first finite element space:

$$V_h = \{v, \hat{x} \in \hat{\mathcal{P}}_1, \int_F [v] ds = 0,\ F \subset \partial K\}. \qquad [14]$$

- second interpolation type::

$$W_h = \{q = (q_1, q_2), (\hat{q}_1, \hat{q}_2) \in \hat{\mathcal{P}}_2 \times \hat{\mathcal{P}}_2, q(a) = 0,\ \text{for any node } a \in \partial\Omega\}. \qquad [15]$$

4.2 Discrete problem

Find $\{u_h, q_h\} : [0,T] \to V_h \times W_h$ such that

$$(\nabla u_h, \nabla v_h) = (\alpha\, q_h, \nabla v_h), \forall v_h \in V_h \qquad [16]$$

$$(\alpha_t\, q_h, w_h) + (\alpha\,(q_h)t, w_h) + (\nabla \cdot ((q_h)_t(x,t) + \beta(x,t)\, q_h(x,t)), \nabla \cdot w_h) = -(f, \nabla \cdot w_h), \forall w_h \in W_h \qquad [17]$$

and

$$u_h(x,0) = (\Pi_1 u_0)(x),\ x \in \Omega. \qquad [18]$$

5 Convergence result

THEOREM 5.1 Assume that $u, u_t \in H^2(\Omega)$ and $q, q_t \in (H^2(\Omega))^2$, then the following error estimates hold

- estimate for u_h

$$|\, u - u_h|_h \le C\, h(x,0) = \Pi_1(u_0)(x),\ x \in \Omega, \qquad [19]$$

  where C depends on t, u, u_t, q, and q_t

- estimate for q_h

$$\|\, q_h\|_{H(div;\Omega)} = \le C\, h, \qquad [20]$$

  where

$$|\, v_h|_h^2 = \sum_{K \in \mathcal{T}_h} |\, v_h|_{1,\Omega}^2, \qquad [21]$$

$$\|\, q_h\|_{H(div;\Omega)}^2 = \sum_{K \in \mathcal{T}_h} \left(|\, q_h|_{0,\Omega}^2 + |\, \nabla \cdot\, q_h|_{0,\Omega}^2 \right). \qquad [22]$$

References

[SHI 09] DONG-YANG SHI, HAI-HONG WANG: Nonconforming H^1–Galerkin mixed finite element method for Sobolev equations on anisotropic meshes. *Acta Mathematicae Applicatae Sinica*, Vol. 25, No. 2, 335–344, 2009.