A brief Report on the article "A two grid method for finite volume element approximations of second order non linear hyperbolic equations"

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Abstract: The authors consider a second order non linear hyperbolic equation. A semi discrete finite volume element method, based on the two–grid method, is suggested and analyzed. The idea of the two grid method is to reduce the non linear and non symmetric problem on a fine grid into a linear and symmetric problem on a coarse grid. The basic mechanisms are two quasi uniform triangulation of Ω, T_H and T_h, with two different sizes H and h ($H > h$), and the corresponding finite element spaces V_H and V_h which satisfy $V_H \subset V_h$.

An H^1 error estimate of order $h + H^3 \log |H|$ is proved. A Numerical test is presented to justify the efficiency of the method.

Key words and phrases: two grid method; second order non linear hyperbolic; finite volume element method; error estimates

1 Some Overall remarks

- The Bochner spaces are not defined more precise in the article: for instance in the page [CHE 10, Section 2, Page 2976], the space $L^2(H^2)$ it is defined by $\{u(\cdot, t) \in H^2(\Omega)\}$, but we see in this definition the spatial space H^2 and we do no see the temporal space L^2; see the exact defintion below.
- The authors mentioned that the data of the problem must satisfy some regularity and compatibility conditions in order that problem [1]–[4] has a unique solution; would be fine if the authors could refer to some reference on this subject.
- The authors mentioned the weak formulation [7]–[9] for problem [1]–[4]:
  - there is the spatial spaces $H_0^1(\Omega)$ but there is no temporal space; more precise which space belongs $u(t)$ as a function of t?
  - in which sense equalities [8] hold?

2 Problem to be solved

The authors consider the following equation:

$$u_{tt}(\mathbf{x},t) - \nabla\left(\mathcal{A}(\mathbf{x}), \nabla u(\mathbf{x})\right) = f(u(\mathbf{x})),\ (x,t) \in \Omega \times (0,T), \qquad [1]$$

with

$$u(\mathbf{x},t) = 0,\ (x,t) \in \partial\Omega \times (0,T), \qquad [2]$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \qquad [3]$$

$$u_t(\mathbf{x},0) = u_1(\mathbf{x}). \qquad [4]$$

The following assumption is needed

Assumption 2.1 The following assumptions are needed:

- The domain $\Omega \subset \mathbb{R}^2$ is an open convex polygonal subset of $\mathbb{R}^2$,
- The given source term f is a real valued function satisfying

$$|\,f'(\mathbf{x})| + |\,f''(\mathbf{x})| \leq M,\ \forall \mathbf{x} \in \mathbb{R}. \qquad [5]$$

- The matrix $\mathcal{A}(\mathbf{x}) = \left(\,a_{ij}(\mathbf{x})\right)_{i,j=1}^{2}$ is a symmetric and uniformly positive definite matrix in Ω.

In the article under consideration, it is said "It is assumed that the functions f, u_0, and u_1 have enough regularity and they satisfy appropriate compatibility conditions so that the boundary value problem [1]–[3] has unique solution satisfying the regularity results as demanded by our subsequent analysis". For moment, I have no knowledge, which compataibility conditions are required at least for existence and uniqueness; would be useful if the authors at least indicated as some reference on this subject!

3 Discretization method?

The finite volume element method (FVEM) is a type of important tool fr solving differential equations. It has been widely used in several engineering fields, such as Fluid Mechanics, Heat and Mass Transfer, and Petrolium Engineering. Perhaps, the important property of FVEM is that it can preserve the conservation laws (Mass, Momentum, and Heat Flux) on each computational cell. This important property, combined with adequate accuracy and ease of implementation, has attracted more people to work in this field. the theoretical framework and the basis tools for the analysis of FVEM have been developed in the last decades.

4 Two Grid Technique?

DEFINITION 4.1 Two grid method is a discretization technique for non linear equations based on two grids of different sizes. The main idea is to use a coarse grid space to produce a rough approximation of the solution of non linear problems, and then use it as the initial guess for one Newton like iteration on the fine grid. This method involves a non linear solve on the coarse grid with grid size H and a linear solve on the fine grid with grid size $h << H$.

5 Some weak formulation for the problem to be approximated

We quote here a weak formlation for problem [1]–[3]. To get more and precise information on the weak for such problems, it is useful to read Evan [EVA 98]. Some notations are often used in the study of evolutive problems. The Sobolev space $W^{s,p}(\Omega)$, with $1 \leq p \leq \infty$, denotes the space of functions that have generalized derivatives up to order s in the space $L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$\| u\|_{s,p,\Omega} = \| u\|_{s,p} = \left(\sum_{\alpha \leq s} \int_\Omega | D^\alpha u(\mathbf{x}) d\,\mathbf{x} \right)^{\frac{1}{p}}, \qquad [6]$$

with the standard modification when $p = \infty$. It is often used $H^s(\Omega)$ to denote $W^{s,2}(\Omega)$. The space $L^2(H^2) = L^2(0,T;H^2(\Omega))$ denotes the functions u such that $\int_0^T \| u(\cdot,t)\|_{2,2,\Omega}^2\, dt < \infty$. The space $L^2(W^{1,\infty}) = L^2(0,T;W^{1,\infty}(\Omega))$ denotes the functions u such that $\int_0^T \| u(\cdot,t)\|_{1,\infty,\Omega}\, dt < \infty$.
The following weak formulation is provided, in the article under consideration [CHE 10]: find $u(\cdot,t) \in H_0^1(\Omega)$, $0 < t < T$ such that

$$(u_{tt}, v) + a(u,v) = (f, v), \ \forall v \in H_0^1(\Omega), \qquad [7]$$

and

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \ u_t(\mathbf{x},0) = u_1(\mathbf{x}), \ \forall \mathbf{x} \in \Omega, \qquad [8]$$

where $(\cdot,\cdot)$ denotes the L^2–inner product and the bilinear $a(\cdot,\cdot)$ is defined by

$$a(u,v) = \int_\Omega a(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x}) d\mathbf{x}, \ \forall u,v \in H_0^1(\Omega). \qquad [9]$$

6 The finite volume element method

Let $\mathcal{T}_h$ be a quasi–uniform triangulation Ω with $h = \max h_K$, where h_K is the diameter of the triangle $K \in \mathcal{T}_h$. Based on this triangulation, let $\mathcal{V}_h$ be the standard conforming finite element space of piecewise linear functions.
In order to describe the finite volume element, we shall introduce a dual partition $\mathcal{T}_h^\star$ based upon

the primary partition $\mathcal{T}_h$ whose elements are called *control volumes*. We refer to the Handbook of Eymard et al. for a description of finite volume element methods.

7 Results of [CHE 10]

- The authors first mentioned the error estimate in the both norms of $L^\infty(H^1)$ and $L^\infty(L^2)$. I liked very much these estimates.
- they used these previous estimates with the two Grid methods to obtain estimates mentioned in the Abstract above.

References

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