A brief Report on the article "Global error bounds for the Petrov–Galerkin discretization of the neutron transport equation "

Britton Chang, Anne Greenbaum, and Eric Machorro

Numer. Linear Algebra Appl. 2011; 18:141–154.

Report done by Professor Bradji, Abdallah

Provisional home page: http://www.latp.univ-mrs.fr/~bradji

Last update: Sunday 24th July, 2011.

Abstract: The Neutron transport equation is considered and a bilinear finite element scheme is suggested. A second convergence order, in the quadtatic norm, is proved using a uniform bound for the 2–norm of the inverse of the system matrix. This proves that the global error is of order two.

Key words and phrases: Neutron transport equation; Global error bounds; Petrov Galerkin discretization; 2–norm of the system matrix

Subject Classification : 65N30; 65N15; 65F05; 65F35

1 Introduction: what importance of bounds of the matrices...

Let us consider a finite difference scheme which can be written as

$$\mathbb{L}^h u^h = f^h, \text{ on } \Omega^h \tag{1}$$

where $\mathbb{L}$ is a square matrix and h is a postive parameter is expected to tend to zero.
Equation [1] is expected to approximate the following differential equation

$$\mathbb{L}u =, \text{ on } \Omega. \tag{2}$$

Two notions are known in the context of finite difference methods: stability and consistency. These two concepts are useful to prove the convergence of finite difference schemes.

1. stability states that the matrix is bounded independently of h in some reasonable norms, see for instance [GOD 77].

2. consistency states that the trunction error tends to zero in the folowing sense: by acting $\mathbb{L}^h$ on the exact solution u (in some sense) of [2], we should find

$$\mathbb{L}^h u - f \to 0, \text{ as } h \to 0. \tag{3}$$

Of course the previous convergence should be measured in some reasonable norms, see for instance [GOD 77].

Under properties of stability and consistency, the solution u^h of finite difference scheme [1] converges towards the exact solution u of [2]

By this way, we understand that the bound of the matrix $\mathbb{L}^h$ is interesting to get the convergence of the solution of finite difference scheme [1] towards the exact solution of [2].

2 some of physics on the Neutron transport equation

From Wikipedia, http://en.wikipedia.org/wiki/, I learned the following information "Neutron transport is the study of the motions and interactions of neutrons with materials. Nuclear scientists and engineers often need to know where neutrons are in an apparatus, what direction they are going, and how quickly they are moving. It is commonly used to determine the behavior of nuclear reactor cores and experimental or industrial neutron beams. Neutron transport is a type of radiative transport. "

3 what is the Neutron transport equation?

Let $\mathcal{D}$ be the domain $(0, a_1) \times (0, a_2)$ and $\mathbf{n}$ be a unit vector in $\mathbb{R}^2$. Let $\sigma \geq 0$ be the absorption cross section (for simplicty it is assumed constant), and $q(x) \geq\geq 0$ be an external source of neutrons (I do not know why the source term is assumed to be positive). The Neutron transport equation is

$$\mathbf{n} \cdot \nabla \psi(x) + \sigma \psi(x) = q(x), \ x \in \mathcal{D}, \qquad [4]$$

with a Dirichlet boundary conditions.

4 scheme?

A bilinear finite element scheme is suggested. The scheme can be written as

$$\begin{aligned} \frac{\mathbf{n}_1}{h}\left(\psi_{i,j} - \psi_{i-1,j}\right) + \frac{\mathbf{n}_2}{k}\left(\psi_{i,j} - \psi_{i,j-1}\right) \quad &+ \quad \frac{\sigma}{4}\left(\psi_{i,j} + \psi_{i-1,j} + \psi_{i,j-1} + \psi_{i-1,j-1}\right) \\ &= \quad q_{i,j}, \ x \in \mathcal{D}^{h,k}, \qquad [5] \end{aligned}$$

where $q_{i,j}$ is the mean value of q on $(x_{i-1}, x_i) \times (y_{j-1}, y_j)$. The values $\psi_{i,j}$ are known on the boundary using the stated Dirichlet boundary conditions.

5 some basic ideas from the article...

1. local truncation error: it is defined by the quantity $\tau_{i,j}$ given by

$$\tau_{ij} = (\mathbb{L}\psi)_{ij} - q(x_i, y_j) = \mathbf{n} \cdot \nabla\psi(x_i, y_j) + \sigma\psi(x_i, y_j) - q(x_i, y_j), \qquad [6]$$

where

$$\begin{aligned}(\mathbb{L}\psi)_{ij} &= \frac{\mathbf{n}_1}{h}\left(\psi(x_i, y_j) - \psi(x_{i-1}, j)\right) + \frac{\mathbf{n}_2}{k}\left(\psi(x_i, y_j) - \psi(x_i, y_{j-1})\right)\\ &+ \frac{\sigma}{4}\left(\psi(x_i, y_j) + \psi(x_{i-1}, y_j) + \psi(x_i, y_{j-1}) + \psi(x_{i-1}, y_{j-1})\right). \qquad [7]\end{aligned}$$

2. order of local truncation error:

   (a) first approach: Taylor expansion it is proved in [GRE 86] that τ_{ij} is of order $(\max(h, k))^2$.

   (b) second approach: Bramble–Hilbert lemma convenient linear form is chosen and the we apply Bramble–Hilbert lemma ([BRA 70])

   (c) third approach: using a bound of matrix: we recommand the information provided in http://en.wikipedia.org/wiki/Matrix_norm. The following useful inequality in the matrices is used

$$\|A\|_2^2 \leq \|A\|_1 \cdot \|A\|_\infty. \qquad [8]$$

References

[BRA 70] Bramble JH, Hilbert SR.: Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolations. SIAM Journal on Numerical Analysis, **7**, 113–124 (1970).

[GOD 77] Godounov, S, et Riabenki: Schémas aux Différences. *Traduction franÃ§aise Edition MIR. Moscou*, 1977.

[GOL 91] Golub GH, Van Loan CF.: Matrix Computations, *John Hopkins: Baltimore*, 1991.

[GRE 86] Greenbaum A., Ferguson J.M.: A Petrov–Galerkin finite element for solving the neutron transport equation.*Journal of Computational Physics*, **64**, 97–111 (1986).