A brief Report on the article “Taylor’s Decomposition on Two Points for One–Dimensional Bratu Problem”

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Abstract: The authors present a Taylor’s decomposition method on two points to approximate the one dimensional nonlinear Bratu problem. They first derived an equivalent first order differential equation system. A Taylor’s decomposition method on two points is applied to approximate this first order differential equation system. The computation of the eigenvalues of the problem is given. The application and error analysis of the method for the nonlinear initial value problem corresponding to the Bratu problem are discussed. Numerival tests justifying the efficiency of the numerical method are presented.

Key words and phrases: Bratu problem; Taylor’s decomposition method on two points; One dimensional problem; eigenvalue

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1 Motivation and some useul information

The one dimensional Bratu problem is the following second order nonlinear boundary value problem:

$$y''(x) + \lambda \exp\left(y(x)\right) = 0,\ x \in (0,1), \quad [1]$$

with

$$y(0) = y(1) = 0. \quad [2]$$

So the one dimensional Bratu problem is a nonlinear eigenvalue value problem with λ is the eigenvalue.

The two dimensional Bratu problem refer to the following problem defined on $(0,1)^2$

$$\Delta\, u(\mathbf{x}) + \lambda \exp\left(u(\mathbf{x})\right) = 0,\ \mathbf{x} = (x,y) \in \Omega = (0,1)^2, \quad [3]$$

with

$$u(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\,\Omega. \quad [4]$$

The authors stated the following property of the existence of solutions to [1]–[2]: there exists a value $\lambda_c > 0$ such that

1. there are two known solutions for $\lambda < \lambda_c$,
2. there is no solutions for $\lambda > \lambda_c$,
3. there exists a unique solution for $\lambda = \lambda_c$.

Bratu problem appears for instance in

1. fuel ignition (what is ignition!!) model found in *Combustion theory*,
2. Chandrasekhar model for the expansion of the universe.

Several numerical methods to devoted to approximate the Bratu problem, e.g. Mickens finite difference scheme, weighed residual method, Adomian decomposition method, Laplace transform decomposition algorithms.
The exact solution for problem [1]–[2] is:

$$y(x) = -2\log\{\frac{\cosh(x-\frac{1}{2})\frac{\theta}{2}}{\cosh\frac{\theta}{2}}\}, \qquad [5]$$

where θ solves

$$\theta = \sqrt{2\lambda}\cosh\frac{\theta}{2}. \qquad [6]$$

There are two solutions to [6] for values $0 < \lambda < \lambda_c$. For $\lambda > \lambda_c$, there is no solution. The solution [6] is unique only for the critical value $\lambda = \lambda_c$ which solves

$$1 = \sqrt{2\lambda_c}\sinh\frac{\theta}{2}\frac{1}{4}. \qquad [7]$$

The critical θ_c is given by

$$\theta_c = 4.78971456. \qquad [8]$$

The exact value of θ_c can be therefore used in [7] to obtain

$$\lambda_c = \frac{8}{\sinh^2\frac{\theta_c}{4}} = 3.513830719. \qquad [9]$$

2 What's Taylor's decomposition method?

Consider the following linear first order differential equation:

$$y'(t) + a(t)y(t) = f(t),\ 0 < t < T, \qquad [10]$$

with

$$y(0) = y_0. \qquad [11]$$

Taylor's decomposition method on two points in the construction of single–step difference scheme of order $p+q$ is based on the following Theorem:

THEOREM 2.1 Let v a smooth function on interval $[0,T]$ and consider the uniform discrtization of $[0,T]$ $t_k = kh$, $k = 0,\dots,n$. Then the following relation holds:

$$v(t_k)-v(t_{k-1})+\sum_{j=1}^{p}\alpha_j v^{(j)}(t_k)h^j-\sum_{j=1}^{p}\beta_j v^{(j)}(t_{k-1})h^j+\frac{(-1)^p}{(p+q)!}\int_{t_{k-1}}^{t_k}(t_k-s)^q(s-t_{k-1})^p v^{(p+q)}(s)ds, \quad [12]$$

where

$$\alpha_j = \frac{(p+q-j)!)p!(-1)^j}{(p+q)!)j!(p-j)!}, \quad [13]$$

$$\beta_j = \frac{(p+q-j)!)q!}{(p+q)!)j!(q-j)!}. \quad [14]$$

Negleting the r.h.s. of the [12] and using equation [10] (this means that the derivatives of y can be expressed using y) to get a two points finite difference scheme of order $p+q$.

3 How the authors got higher order scheme for one dimensional Bratu problem?

Problem [1]–[2] can be written as, by setting $y'(x) = z(x)$:

$$Y'(x) = F(Y(x)), \quad [15]$$

$$A_0Y(0) + A_1Y(1) = 0, \quad [16]$$

where

$$Y(x) = \begin{pmatrix} y(x) \\ z(x) \end{pmatrix} \quad [17]$$

$$F(Y(x)) = \begin{pmatrix} z(x) \\ -\lambda \exp y(x) \end{pmatrix} \quad [18]$$

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad [19]$$

$$A_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad [20]$$

So, we apply results of the previous subsection on the equation [15].

References

[EVR 10] M. EVRENOSOGLU AND S. SOMALI: Taylor's Decomposition on Two Points for One–Dimensional Bratu Problem. *Numerical Methods Partial Differential Equations*, **26**, 412–425, 2010.