A brief Report on the article [FEN6 09]“Mixed finite element methods for the fully nonlinear Monge–Ampère equation based on the vanishing moment method”

X. Feng and M. Neilan

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Report done by Professor Bradji, Abdallah

Provisional home page: http://www.cmi.univ-mrs.fr/~bradji

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Abstract: The aim of the article is to provide a mixed finite element approximation for Monge–Ampère problem. They suggest a vanishing moment method which consist of a quasilinear fourth order problem with a singularly perturbed parameter $\varepsilon > 0$. It is first justified that for each $\varepsilon > 0$, the stated quasilinear fourth order problem has a unique solution u^ε. It is proven that u^ε converges uniformly towards the exact viscosity solution of Monge–Ampère problem, as $\varepsilon \to 0$. The quasilinear fourth order problem is equivalent to a nonlinear system of second–order equations. Using this system, the authors derive a weak mixed formulation to the quasilinear fourth order problem in which the solution is denoted by $(u^\varepsilon, \sigma^\varepsilon)$. Thanks to the use of Hermann–Miyoshi mixed elements, the authors suggest a finite element scheme in which the solution is denoted by $(u_h^\varepsilon, \sigma_h^\varepsilon)$. The existence, uniqueness, and the convergence $(u_h^\varepsilon, \sigma_h^\varepsilon)$ towards $(u^\varepsilon, \sigma^\varepsilon)$ is proved, using fixed point technique (since the discrete problem is nonlinear), under the assumption that the mesh parameter h is small enought and under a regularity assumption on $(u_h^\varepsilon, \sigma_h^\varepsilon)$. Finally, the authors present numerical tests showing the error estimates when the mesh paremeter h is a power of ε.

Key words and phrases: fully nonlinear second order partial differential equation, Monge–Ampère equation, viscosity solution, vanishing moment method, moment solution, mixed finite element method, Hermann–Miyoshi mixed element, linearized problem

Subject Classification (to be checked because these subjects have ben taken from the article and I do not know if they are subject classification 2010): 65N30, 65M60, 35J60, 53C45

1 Some final remark

1.1 What I learned from this nice article!

1. what about the finite volume approximation of $\left(\partial^2 z/\partial x^2\right)\left(\partial^2 z/\partial y^2\right) - \left(\left(\partial^2 z/\partial x \partial y\right)\right)^2 = f$ (would say $\det\left(D^2 z\right) = f$ when the dimension $n = 2$?)

1.2 Some questions!!

1. first question

$$\int_\Omega \mathrm{div}(\sigma^\varepsilon)(x) \cdot \nabla v(x) dx = \int_\Omega \nabla \left(\mathrm{tr}(\sigma^\varepsilon)\right)(x) \cdot \nabla v(x) dx \quad ? \qquad [1]$$

   So, I have not understood yet the steps [23]–[25]?

2. what is the origin of terms "viscosity" and "moment"?

3. I have not checked how it obtained [27] from [17]; formulation [27] is new for me!!

2 Motivation and some useful information

2.1 What is Monge–Ampère equation?

- What is Monge–Ampère equation? It is a prototype of the fully nonlinear second–order PDEs

$$F(D^2u, Du, u, x) = 0, \; x \in \Omega \qquad [2]$$

  when F is defined by

$$F(D^2u, Du, u, x) = \det(D^2u) - f, \qquad [3]$$

  where Ω is a convex domain with smooth boundary $\partial\,\Omega$, $D^2u(x)$ and $det(D^2u(x))$ denote respectively the Hessian matrix of u at $x \in \Omega$ and the determinant of $D^2u(x)$.

- Monge–Ampère equation arises, see [GUT 01], naturally from differential geometry and from application such as mass transportation, meteorology, and geostrophic fluid dynamics.

2.2 Aims and definition of viscosity solutions?

The aim of [FEN6 09], which is the article under consideration, is to provide finite element approximations of viscosity solutions of the following Dirichlet problem for the fully nonlinear Monge–Ampère equation:

$$\det(D^2u^0(x)) = f(x), \; x \in \Omega \subset \mathbb{R}^n, \qquad [4]$$

$$u^0(x) = g(x), \; x \in \partial\Omega. \qquad [5]$$

So, the Monge–Ampère equation is a particular case of the fully nonlinear second order PDEs

$$F(D^2u, Du, u, x) = 0, \; x \in \Omega, \qquad [6]$$

in which $F(D^2u, Du, u, x) = \det(D^2u) - f$.

- According to [GIL 01], problem [4]–[5] has no solution, when the domain Ω is not strictly convex, even the data f, g and $\partial\Omega$ are smooth enough.
- A classical result for Aleksandrov states that problem [4]–[5] has unique generalized solution in the space of convex functions provided that $f > 0$.
- A *viscosity solution* is first introduced by Crandall and Lions [CRA 83] for the fully nonlinear first order Hamilton–Jacobi equations.
- After having introduced the notion of *viscosity solution* by Crandall and Lions [CRA 83] for the fully nonlinear first order Hamilton–Jacobi equations, it is quickly extended to the fully nonlinear second order.
- Existence of *viscosity solution*: What I understood that, using the Jensen's maximum principle [JEN 88] and the Ishii's work [ISH 23] that the classical Perron's method (what is Perron's method) could be used to prove the existence of *viscosity solution*, may be we find more details in [ISH 23].

DEFINITION 2.1 (Definition of the viscosity solutions)

- A convex function $u^0 \in \mathcal{C}(\overline{\Omega})$ satisfying $u^0 = g$ on $\partial\Omega$ is called a viscosity subsolution of [4] if for any function $\varphi \in \mathcal{C}^2(\overline{\Omega})$ such that $u^0 - \varphi$ has a local maximum at $x_0 \in \Omega$, we have $\det(D^2\varphi(x_0)) \geq f(x_0)$.
- A convex function $u^0 \in \mathcal{C}(\overline{\Omega})$ satisfying $u^0 = g$ on $\partial\Omega$ is called a viscosity supersolution of [4] if for any function $\varphi \in \mathcal{C}^2(\overline{\Omega})$ such that $u^0 - \varphi$ has a local minimum at $x_0 \in \Omega$, we have $\det(D^2\varphi(x_0)) \leq f(x_0)$.
- A convex function $u^0 \in \mathcal{C}(\overline{\Omega})$ satisfying $u^0 = g$ on $\partial\Omega$ is called a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution of [4].

2.3 Is it possible to provide finite element approximation for [4]–[5] using the notion of viscosity solution?

It is clear, as said the authors, that the viscosity solution is not variational. (Would be fine from the referee if they could add some sentence explainig the notion of variational.) So, it is straightforward to provide a Galerkin type numerical methods, e.g. finite element method, spectral and discontinuous Galerkin methods, for the viscosity solution of [4]–[5].

2.4 So, we should look for another notion of weak solution for [4]–[5]!!

As said in the previous section, that is not straightforward to provide a Galerkin type numerical methods, e.g. finite element method, spectral and discontinuous Galerkin methods, for the viscosity solution of [4]–[5]. So, it is convenient to look for another notion for a weak solution for [4]–[5]. The new notion is *moment solution*. The *moment solution* is defined using a contructive method, called *vanishing moment method*. The main idea of the *vanishing moment method* is to approximate a fully nonlinear second order PDE by a quasilinear higher order PDE. The notion of moment solution and the vanishing moment method are natural generalizations of the original definition of viscosity solution and the vanishing viscosity method introduced for Hamiton–Jacobi equation in [CRA 83].

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2.5 What is moment solution and the vanishing moment method for [6]?

- *first step*: approximation of [6] by the quasilinear fourth order PDE

$$-\varepsilon\Delta^2 u^\varepsilon(x) + F(D^2 u^\varepsilon, Du^\varepsilon, u^\varepsilon, x) = 0,\ x \in \Omega, \qquad [7]$$

  where $\varepsilon > 0$.

- *second step*: boundary condition of u^ε is the same one of u, that

$$u^\varepsilon(x) = g(x),\ x \in \partial\Omega. \qquad [8]$$

- *third step*: extra boundary conditions of u^ε. Since, boundary condition [8] does not ensure alone the uniqueness of the solution u^ε, so we need to add other extra boundary conditions. The authors [FEN6 09] suggest one of the following boundary conditions:

$$\Delta u^\varepsilon = \varepsilon,\ x \in \partial\Omega, \qquad [9]$$

  or, note that $D^2 u^\varepsilon$ is a $n \times n$ matrix

$$D^2 u^\varepsilon \mathbf{n}(x) \cdot \mathbf{n}(x) = \varepsilon,\ x \in \partial\Omega, \qquad [10]$$

  where $\mathbf{n}(x)$ denotes the unit outward normal to $\partial\Omega$ et the point x. Note that, [10] makes sense since $D^2 u^\varepsilon \mathbf{n}(x)$ and $\mathbf{n}(x)$ are column vectors (are $n \times 1$ matrix), so the inner product makes sense.

  - the boundary condition [9] seems to be more convenient when we use finite element, spectral, or discontinuous Galerkin methods. I guess that the boundary condition [9] is used in [FEN4 10]!!!.

– the boundary condition [10] seems to be more convenient when we use mixed finite element. So, since the authors are interested with mixed finite element method, the authors will be interested to use the boundary condition [10].

2.6 Thanks to the previous subsection, what is the vanishing moment method for the Monge–Ampère problem [4]–[5]?

Thanks to the previous subsection, the vanishing moment method for the Monge–Ampère problem [4]–[5] is the following approximation of [4]–[5]:

$$-\varepsilon\Delta^2 u^\varepsilon(x) + \det(D^2 u^\varepsilon(x)) = f(x),\ x \in \Omega, \qquad [11]$$

where $\varepsilon > 0$,

$$u^\varepsilon(x) = g(x),\ x \in \partial\Omega, \qquad [12]$$

$$D^2\, u^\varepsilon \mathbf{n}(x) \cdot \mathbf{n}(x) = \varepsilon,\ x \in \partial\Omega. \qquad [13]$$

2.7 What about the convergence of the vanishing moment method towards the solution of [4]–[5]?

It is proved in [FEN1 10]:

1. the problem [11]–[13] has a unique strictly convex solution u^ε, provided that $f > 0$,
2. u^ε converges uniformly to the unique viscosity solution of [4]–[5]. This means, as I understood

$$\sup_{x\in\overline{\Omega}} |\, u^\varepsilon(x) - u(x)| \to 0,\ \text{as } \varepsilon \to 0. \qquad [14]$$

3. the following *a priori bounds* also proven:

$$\|\, u^\varepsilon\|_{H^j} = O\left(\varepsilon^{-\frac{j-1}{2}}\right), \quad \|\, u^\varepsilon\|_{W^{2,\infty}} = O\left(\varepsilon^{-1}\right), \qquad [15]$$

$$\|\, D^2 u^\varepsilon\|_{L^2} = O\left(\varepsilon^{-\frac{1}{2}}\right), \quad \|\operatorname{cof}(D^2 u^\varepsilon)\|_{L^\infty} = O\left(\varepsilon^{-1}\right), \qquad [16]$$

for $j = 2, 3$, where $\operatorname{cof}(D^2 u^\varepsilon)$ is the cofactor matrix of the Hessian matrix $D^2 u^\varepsilon$.

3 A weak formulation for the moment solution

3.1 Some philosophy

There are several popular mixed formulations for fourth–order problems, see [BRE 91]. Authors [FEN6 09] say "Howver, since the Hessian matrix, D^2u^ε, appears in [11], in a nonlinear fashion, we can not use $\Delta^2 u^\varepsilon$ alone as our additional variable, but rather we are forced to use $\sigma^\varepsilon = D^2u^\varepsilon$ as a new variable. Because of this, we rule out the family of Ciarlet–Raviart mixed element (see [CIA 78]). On the other hand, this observation suggests to use Hermann–Miyoshi and Hermann–Johnson mixed elements (see [BRE 91]) which both seek σ^ε as an additional variable. In this paper, we shall focus on developing Hermann–Miyoshi elements. "

3.2 A weak formulation

The vanishing moment method [11]–[13] can be written as

$$\sigma^\varepsilon - D^2u^\varepsilon = 0, \qquad [17]$$

$$-\varepsilon\Delta\,\mathrm{tr}(\sigma^\varepsilon) + \det(\sigma^\varepsilon) = f, \qquad [18]$$

Let us consider the following spaces:

$$V := H^1(\Omega), \quad W := \left\{\mu \in \left(H^1(\Omega)\right)^{n\times n} : \mu_{ij} = \mu_{ji}\right\}, \qquad [19]$$

$$V_0 := H_0^1, \quad V_g := \left\{v \in H^1 : v|_{\partial\Omega} = g\right\}, \qquad [20]$$

$$W_\varepsilon := \{\mu \in W, \mu\,\mathbf{n}\cdot\mathbf{n} = \varepsilon\}, \quad W_0 := \{\mu \in W, \mu\,\mathbf{n}\cdot\mathbf{n} = 0\}. \qquad [21]$$

Multiplying both sides of [18] by $v \in V_0$ and integrating the result over Ω

$$-\varepsilon\int_\Omega \Delta\,\mathrm{tr}(\sigma^\varepsilon)(x)v(x)dx + \int_\Omega \det(u^\varepsilon)(x)v(x)dx = 0. \qquad [22]$$

Using integration by part to get

$$\begin{aligned} -\int_\Omega \Delta\,\mathrm{tr}(\sigma^\varepsilon)(x)v(x)dx &= \sum_{i=1}^n \int_\Omega \Delta\sigma_{ii}^\varepsilon(x)v(x)dx \\ &= \sum_{i=1}^n \int_\Omega \nabla\,\sigma_{ii}^\varepsilon(x)\cdot\nabla\,v(x)dx \\ &= \int_\Omega \nabla\left(\mathrm{tr}(\sigma^\varepsilon)\right)(x)\cdot\nabla\,v(x)dx \end{aligned} \qquad [23]$$

This with [22] gives

$$\varepsilon\int_\Omega \nabla\left(\mathrm{tr}(\sigma^\varepsilon)\right)(x)\cdot\nabla\,v(x)dx - \int_\Omega D^2u^\varepsilon(x)v(x)dx = 0. \qquad [24]$$

But the formulation given in the article is

$$\varepsilon\int_\Omega \mathrm{div}(\sigma^\varepsilon)(x)\cdot\nabla\,v(x)dx + \int_\Omega \det(u^\varepsilon)(x)v(x)dx = 0. \qquad [25]$$

So I ask if

$$\int_\Omega \operatorname{div}(\sigma^\varepsilon)(x)\cdot\nabla v(x)dx = \int_\Omega \nabla\left(\operatorname{tr}(\sigma^\varepsilon)\right)(x)\cdot\nabla v(x)dx \quad ? \qquad [26]$$

Multiplying both sides of [17] by a function $\mu \in W_0$ to get

$$\int_\Omega \sigma^\varepsilon : \mu\, dx + \int_\Omega Du^\varepsilon \cdot \operatorname{div}(\mu)dx = \sum_{k=1}^{n-1}\int_{\partial\Omega} \mu\mathbf{n}\cdot\tau_k \frac{\partial g}{\partial \tau_k} ds, \qquad [27]$$

where $\sigma^\varepsilon : \mu$ denotes the matrix inner product and $\{\tau_k(x),\dots,\tau_{n-1}(x)\}$ denotes the standard basis for the tangent space to $\partial\Omega$ at x.
So, the weak formulation of [17]–[18] is: Find $(u^\varepsilon,\sigma^\varepsilon)\in V_g\times W_\varepsilon$ such that

$$(\sigma^\varepsilon,\mu) + (\operatorname{div}(\mu), Du^\varepsilon) = \langle \bar g,\mu\rangle,\ \ \forall \mu\in W_0, \qquad [28]$$

and

$$(\operatorname{div}(\sigma^\varepsilon), Dv) + \frac{1}{\varepsilon}(\det(\sigma^\varepsilon), v) = \langle f^\varepsilon, v\rangle,\ \ \forall v\in V_0, \qquad [29]$$

where

$$\langle \bar g,\mu\rangle = \sum_{k=1}^{n-1}\langle \frac{\partial g}{\partial\tau_k}, \mu\mathbf{n}\cdot\tau_k\rangle \ \text{ and } \ f^\varepsilon = \frac{1}{\varepsilon} f. \qquad [30]$$

4 The Hermann–Miyoshi mixed element for [28]–[29]

Let $\mathcal{T}_h$ be a quasi–uniform triangular or rectangular partition of Ω if $n=2$ and be a quasi–uniform tetrahedral or 3–D rectangular mesh if $n=3$. Let $V^h \subset H^1(\Omega)$ be the Lagrange finite element space consisting of continuous piecewise polynomials of degree k, $k\geq 2$, associated with the mesh $\mathcal{T}_h$. Let

$$V_0^h := V^h\cap V_0,\ \ V_g^h := V^h \cap V_g, \qquad [31]$$

$$W_\varepsilon^h := \left[V^h\right]^{n\times n}\cap W_\varepsilon,\ \ W_0^h := \left[V^h\right]^{n\times n}\cap W_0. \qquad [32]$$

So, the Hermann–Miyoshi mixed element for [28]–[29] can be defined as: Find $(u_h^\varepsilon,\sigma_h^\varepsilon)\in V_g^h\times W_\varepsilon^h$ such that

$$(\sigma_h^\varepsilon,\mu_h) + (\operatorname{div}(\mu_h), Du_h^\varepsilon) = \langle \bar g,\mu_h\rangle,\ \ \forall \mu_h\in W_0^h, \qquad [33]$$

and

$$(\operatorname{div}(\sigma_h^\varepsilon), Dv_h) + \frac{1}{\varepsilon}(\det(\sigma_h^\varepsilon), v_h) = \langle f^\varepsilon, v_h\rangle,\ \ \forall v_h\in V_0^h. \qquad [34]$$

It is useful here to mention that, because of the nonlinearity (it occurs because of the presence of $\det(\sigma_h^\varepsilon)$) in the second term on the left hand side of [34], the problem is full nonlinear. So, the authors they first began by a finite element approximation for linearized problem of [34].

5 A main result in this paper

Article [FEN6 09] is full by results. But I think, most important one is

THEOREM 5.1 (Perhaps, some details should be provided here) The aim of the article [FEN6 09] is to provide a mixed finite element scheme, using Hermann–Miyoshi mixed element, for the Monge–Ampère problem [4]–[5]. Let $\Omega \subset R^n$ be a convex domain with smooth boundary $\partial\Omega$, $n \in \{2,3\}$. Assume that $f > 0$ (but which regularity on f?). Then, for each $\varepsilon > 0$, there exists a unique solution u^ε for [11]–[13]. The family of the solutions $\{u^\varepsilon, \varepsilon \in \mathbb{R}_+^\star\}$ satisfy the *a priori bounds* [15]–[16].
In addition to this, u^ε converges uniformly to the unique viscosity solution of [4]–[5], i.e,

$$\sup_{x\in\overline{\Omega}} |\, u^\varepsilon(x) - u(x)| \to 0, \text{ as } \varepsilon \to 0. \qquad [35]$$

A mixed formulation for [11]–[13] is given by [17]–[18]. A weak mixed formulation for [11]–[13] is given by [28]–[30], Let $\mathcal{T}_h$ be a quasi–uniform triangular or rectangular partition of Ω if $n = 2$ and be a quasi–uniform tetrahedral or 3–D rectangular mesh if $n = 3$. Let $V^h \subset H^1(\Omega)$ be the Lagrange finite element space consisting of continuous piecewise polynomials of degree k, $k \geq 2$, associated with the mesh $\mathcal{T}_h$. Consider the spaces $V_0^h, V_g^h, W_\varepsilon^h$, and W_0^h given by [31]–[32]. For sufficiently small h, there exists a unique solution for $(u_h^\varepsilon, \sigma_h^\varepsilon) \in V_g^h \times W_\varepsilon^h$ in some ball (depending on h and ε) included $V_g^h \times W_\varepsilon^h$ (Here it was used the technique of fixed point and a convenient contracting mapping).
In addition to this, the following error estimates hold:

$$\|\, \sigma^\varepsilon - \sigma_h^\varepsilon\|_{\mathbb{L}^2(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\|\, u^\varepsilon - u_h^\varepsilon\|_{H^1(\Omega)} \leq C_1(\varepsilon)h^{l-2}\left(\|\, \sigma^\varepsilon\|_{H^l(\Omega)} + \|\, u^\varepsilon\|_{H^l(\Omega)}\right), \qquad [36]$$

$$\|\, \sigma^\varepsilon - \sigma_h^\varepsilon\|_{H^1(\Omega)} \leq C_2(\varepsilon)h^{l-3}\left(\|\, \sigma^\varepsilon\|_{H^l(\Omega)} + \|\, u^\varepsilon\|_{H^l(\Omega)}\right), \qquad [37]$$

where $C_1(\varepsilon) = C_2(\varepsilon) = O(\varepsilon^{-\frac{9}{2}}$ when $n = 2$, and $C_1(\varepsilon) = C_2(\varepsilon) = O(\varepsilon^{-6}$ when $n = 3$, $l = \min(k+1, r)$.

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