A brief Report on the article [TAM 10] "A parameter uniform numerical method for a system of singularly perturbed convection diffusion equations with discontinuous convection coefficients"

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Abstract: The authors develop a parameter uniform numerical method for a system of singularly differential equations of second order with discontinuous convection and source terms. The solution exhibit a strong layer. Numerical tests are presented to justify the theoretical results.

Key words and phrases: system of singularly differential equations of second order; discontinuous convection and source terms; strong layer

Subject Classification (to be checked because these subjects have ben taken from the article and I do not know if they are subject classification 2010): 65L10, CR G1.7?

1 Some final remark

1. It seems for me that there are several techniques or results referred to [FAR 04, FAR 06] in which it is treated single equation with non smooth data. So, may be it is useful to begin with the previous stated references.

2. Since the authors are interested with uniform convergence with respect to ε, so the mesh should have a relation with singular parameter ε. The mesh is not so clear for me yet: what is the relationship between the mesh and the singular parameter ε in order to get the uniform convergence with respect to ε.

3. Numerical results are presensted when $h = 1/256$ whereas $\varepsilon \in \{1/32, 1/64, 1/128, 1/256\}$, so these results are presented when $h \leq \varepsilon$. But, which is interesting is to test the case $\varepsilon << h$ because the convergence is ensured even by the standard numerical methods when $h \leq \varepsilon$ (would say h is sufficiently small), is not it? .

2 What I learned from this nice article!

1. (area of singularly perturbed problems): Singularly perturbed problems appear in many branches of applied mathematics, like fluid dynamics, quantum mechanics, turbulent interaction of waves and currents, electro analytic, chemistry, etc.
2. (what about standard numerical methods for singularly perturbed problems): The solutions of such problems have boundary and interior layers. So, is not straightforward to get an approximation in neighbouring of these layers. The convergence of numerical approximations generated by standard numerical methods applied to such problems depends adversely on the singular perturbation parameter.
3. (literature): Robust parameter uniform numerical methods have been developed over the last 20 years.
   (a) Most of this literature has been devoted to singularly perturbed problems involving single differential equations.
   (b) Only a few authors have developed numerical methods for singularly perturbed system of ordinary differential equations.
   (c) Various methods available in the literature have been interested with systems with smooth source terms.
   (d) Some authors have developed numerical methods for single equation with non smooth data, see [FAR 04, FAR 06].
   (e) In [TAM 07], the authors developed a numerical method for singularly perturbed weakly (in what sens, word “weakly” means here!!) coupled system of two second order ordinary differential equations with discontinuous source term. In the same paper [TAM 07] (this what I understood from the introduction of the paper under consideration [TAM 10]!!), the authors also developed a numerical method for a system of two second order odinary differentil equations with a discontinuous source term. The solution of this type of equation exhibits weak interior layer (what does it mean the word “weak” here).
4. (objective of the paper under consideration, that is [TAM 10]): is to develop a parameter uniform numerical method for a system of singularly differential equations with discontinuous convection source term. The authors [TAM 10] say that the solution exhibit a strong (what does it mean the word “strong” here) layer.

3 Some questions!!

1. I have not understood the words (sentences) may it will be useful to ask question from the corresponding author Ramanujam matram@bdu.ac.in :
   (a) weak layer

(b) strong layer

(c) weakly coupled system

4 Continuous problem

Let $\Omega = (0,1)$ and let d be a given point in Ω. We define the following subintervals:

$$\Omega^- = (0,d) \text{ and } \Omega^+ = (d,1). \qquad [1]$$

The studied problem in [TAM 10] is

$$Py(x) = -\varepsilon\, y''(x) + A(x)y' + B(x)y(x) = f(x),\; x \in \Omega \setminus \{d\} = \Omega^- \cup \Omega^+, \qquad [2]$$

where

$$A = \begin{pmatrix} a_{11}(x) & 0 \\ 0 & a_{21} \end{pmatrix},\; x \in \Omega^-, \qquad [3]$$

and

$$A = \begin{pmatrix} a_{12}(x) & 0 \\ 0 & a_{22} \end{pmatrix},\; x \in \Omega^+, \qquad [4]$$

where $a_{11}(x), a_{21}(x) \geq \alpha_1 > 0,\; \forall\, x \in \Omega^-$ and $a_{12}(x), a_{22}(x) \leq -\alpha_2 < 0,\; \forall\, x \in \Omega^+$,

$$B = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22} \end{pmatrix}, \qquad [5]$$

with $b_{12}(x), b_{21}(x) \leq 0$, $b_{11}(x) > |\, b_{12}(x)|$, $b_{22}(x) > |\, b_{21}(x)|$, $\forall\, x \in \Omega$.

ASSUMPTION 4.1 It is assumed in [TAM 10] that

1. the functions $(\, b_{ij})_{i,j=1}^2$ are smooth on Ω,
2. the functions $(\, a_{ij})_{i,j=1}^2$ and f_1, f_2 are smooth on $\Omega \setminus \{d\}$
3. the functions $(\, a_{ij})_{i,j=1}^2$ and f_1, f_2 and their derivatives have left and right limits at $x = d$.

What smoothness is meant by the authors: may be this smoothness means the required conditions which leads that the results throughout [TAM 10] hold. The following notation is used throughout [TAM 10]:

$$[w]\,(d) = w(d+) - w(d-) = \lim_{h \to 0^+} w(d+h) - \lim_{h \to 0^+} w(d-h). \qquad [6]$$

Remark 1 (Sign condition on A) The sign condition, stated above, imposed on $(\, a_{ij})_{i,j=1}^2$ is motivated by the reference [FAR 04] for single equation.

Before, going to the discretization used in [TAM 10], we first give an idea on the standard numerical methods.

5 An overview on some standard numerical methods for singularly perturbed equations

Let us consider the following simple example (which is given in [FEI 04, Pages 342–344]):

$$-\varepsilon u''(x) + \nu\, u'(x) = 1,\ x \in \Omega = (0,1), \qquad [7]$$

with

$$u(0) = u(1) = 0, \qquad [8]$$

where $\varepsilon > 0$ and $\nu \neq 0$ are two constants. The solution of [7]–[8] is defined by

$$u(x) = \frac{1}{\nu}\left\{x - \frac{\exp(\nu\, x/\varepsilon) - 1}{\exp(\nu\,/\varepsilon) - 1}\right\},\ x \in [0,1]. \qquad [9]$$

If $\varepsilon \to 0$ and $\nu > 0$, then $u(x) \to x/\nu$ for $x \in [0,1)$. The limit function is the solution of

$$\nu\, u'(x) = 1,\ x \in (0,1) \text{ and } u(0) = 0. \qquad [10]$$

5.1 Standard linear finite element methods and Gibs phenomenon

Let us apply the linear finite element method to approximate [7]–[8]. We then consider the uniform mesh $\mathcal{T}_h = \{K_i; i = 0, \dots, N\}$ with $K_i = [x_i, x_{i+1}]$ and $x_{i+1} - x_i = h$. The approximate solution u_h as well as the test functions are in the space:

$$\mathcal{V}_h = \left\{\varphi_h \in \mathcal{C}(\overline{\Omega}); \varphi_h|_{K_i} \in \mathcal{P}_1,\ \forall\, K_i \in \mathcal{T}_h,\ \varphi_h(0) = \varphi_h(1) = 0\right\}. \qquad [11]$$

Multiplying both sides of [7] by $\varphi \in \mathcal{C}^1(\overline{\Omega})$, with $\varphi(0) = \varphi(1) = 0$, we get

$$\int_0^1 (\varepsilon\, u'(x)\varphi'(x) + \nu\, u'(x)\varphi(x))dx = \int_0^1 \varphi(x)dx,\ \forall \varphi \in \mathcal{C}^1(\overline{\Omega}),\ \varphi(0) = \varphi(1) = 0. \qquad [12]$$

The approximate finite element solution is then defined by: find $u_h \mathcal{V}_h$ such that

$$\int_0^1 (\varepsilon\, u_h'(x)\varphi_h'(x) + \nu\, u_h'(x)\varphi_h(x))dx = \int_0^1 \varphi_h(x)dx,\ \forall \varphi \in \mathcal{V}_h. \qquad [13]$$

This is equivalent to the following linear system:

$$-\frac{\varepsilon}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{\nu}{2h}(u_{i+1} - u_{i-1}) = 1, \qquad [14]$$

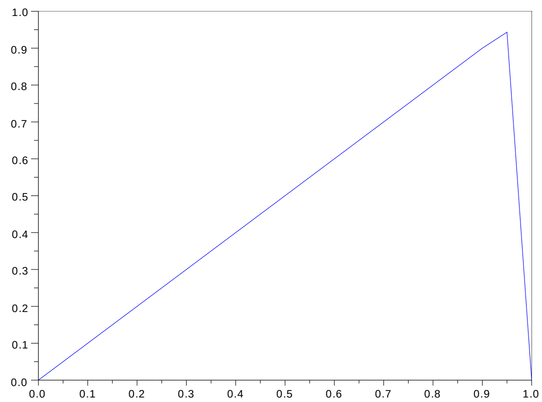
with

$$u_0 = u_{N+1} = 0. \qquad [15]$$

Note that [14] is also a finite difference scheme by approximating:

1. $u''(x_i)$ by
$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}. \qquad [16]$$

2. $u'(x_i)$ by the central quotient
$$\frac{u(x_{i+1}) - u(x_{i-1})}{2h}. \qquad [17]$$



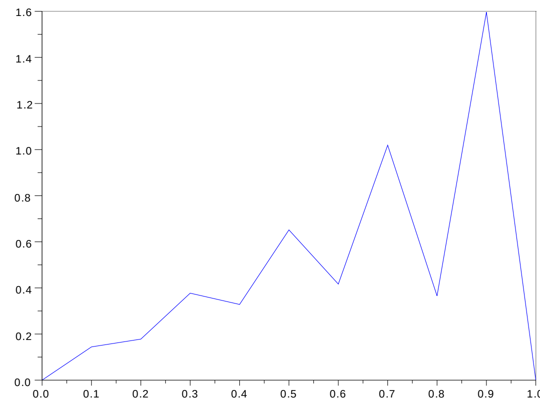


Figure 1: Left graph of the function u, right the finite difference solution [14]–[15] with $h = 1/10$; both simulations are in $\varepsilon = 10^{-2}$ and $\nu = 1$

The linear equations [14]–[15] can be written as

$$\mathcal{A}U = 1, \qquad [18]$$

where $\mathcal{A}$ is a $N \times N$ matrix and 1 in the r.h.s. of [18] is the vector of $\mathbb{R}^N$ whose components are equal to 1. The matrix $\mathcal{A}$ is nonsymmetric, but for $h < 2\varepsilon/|\nu|$ it is *diagonally dominant* (In the sense $\sum_{j=1,j\neq i}^{N} |\, a_{ij}| \leq |\, a_{ii}|$ for all $i = 1, \ldots, N$ with strict inequality for at least one i.) and *irreducibly.* These previous stated properties of $\mathcal{A}$ guarantees good properties of the approximate solution. But, for $h \geq 2\varepsilon/|\nu|$, the approximate solution do not make sense because of *spurious oscillations* shown in the right Figure 1. This means that the *Gibbs phenomenon* arises here. We see here that *mesh Péclet number* defined by

$$Pe = \frac{h\nu}{2\varepsilon} \qquad [19]$$

must satisfy the condition

$$Pe < 1 \qquad [20]$$

in order to avoid the *Gibbs phenomenon.*

5.2 Gibbs phenomenon and upwind finite difference scheme

An issue to avoid the Gibbs phenomenon is to chose an upwind finite difference scheme instead of the central finite difference scheme [14]–[15], that is to chose

$$-\frac{\varepsilon}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{\nu}{h}(u_i - u_{i-1}) = 1, \qquad [21]$$

with

$$u_0 = u_{N+1} = 0. \qquad [22]$$

So, the upwind scheme [21]–[22] is performed thanks to

1. the approximation of $u''(x_i)$ by
$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}. \qquad [23]$$

2. the approximation of $u'(x_i)$ by the central quotient
$$\frac{u(x_i) - u(x_{i-1})}{h}. \qquad [24]$$

Figure 2 represents, using Scilab, the finite difference solution [21]–[22] with $h = 1/10$. As we can see that the finite difference solution [21]–[22] is more reasonable than that of [14]–[15].

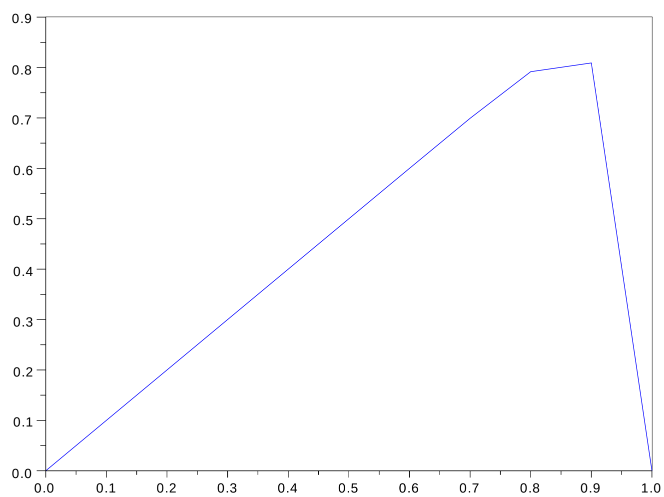


Figure 2: The upwind finite difference solution [21]–[22] with $h = 1/10; \varepsilon = 10^{-2}$ and $\nu = 1$

5.3 Standard linear finite element methods and convergence order

As usual, we use the technique of Cea Lemma to compute the convergence order. The key, for that target is the following equality:
$$a(u - u_h, v_h) = 0, \ \forall v_h \in \mathcal{V}_h. \qquad [25]$$

This implies that
$$a(u - u_h, u - u_h) = a(u - u_h, u - \pi\, u), \qquad [26]$$

where π is the usual interpolation operator defined from $\mathcal{C}(\overline{\Omega})$ into $\mathcal{V}_h$ and
$$a(u, v) = \int_0^1 (\varepsilon\, u'(x)v'(x) + \nu\, u'(x)v(x))dx. \qquad [27]$$

Equality [26] implies that

$$\alpha \| u - u_h\|_{1,\Omega}^2 \leq M \| u - u_h\|_{1,\Omega}\| u - \pi u\|_{1,\Omega}, \tag{28}$$

which implies in turn, using the known result of the interpolation error

$$\alpha \| u - u_h\|_{1,\Omega} \leq CMh \max_{x\in\overline{\Omega}} | u''(x)|, \tag{29}$$

where C is only depending on Ω, the constants α and M used in [28] are defined by

$$a(v,v) \geq \alpha \| v\|_{1,\Omega}^2, \ \forall v \in H_0^1(\Omega), \tag{30}$$

and

$$a(u,v) \leq M\| u\|_{1,\Omega}\| v\|_{1,\Omega}, \ \forall u, v \in H_0^1(\Omega). \tag{31}$$

Let us compute M, α, and $\max_{x\in\overline{\Omega}} | u''(x)|$

1. *computation of* α: using [27], the fact that $v(0) = v(1) = 0$, and the Poincaré inequality

$$\begin{aligned} a(v,v) &= \varepsilon \int_0^1 (v')^2(x)dx + \nu \int_0^1 v'(x)v(x))dx \\ &= \varepsilon \int_0^1 (v')^2(x)dx + \nu \int_0^1 (v^2)'(x)dx \\ &= \varepsilon \int_0^1 (v')^2(x)dx \\ &\geq \varepsilon C(\Omega)\| v\|_{1,\Omega}^2, \end{aligned} \tag{32}$$

where $C(\Omega)$ is only depending on Ω.

2. *computation of* M: using the Cauchy Schwarz inequality, the fact that $\left(\int_0^1 (u')^2(x)dx\right)^{\frac{1}{2}} \leq \| v\|_{1,\Omega}$ to get, assuming that $\varepsilon << 1$

$$\begin{aligned} a(u,v) &= \int_0^1 (\varepsilon u'(x)v'(x) + \nu u'(x)v(x))dx \\ &\leq \varepsilon\| u\|_{1,\Omega}\| v\|_{1,\Omega} + \nu\| u\|_{1,\Omega}\| v\|_{L^2(\Omega)} \\ &\leq (\varepsilon + \nu)\| u\|_{1,\Omega}\| v\|_{1,\Omega} \\ &\leq (1 + \nu)\| u\|_{1,\Omega}\| v\|_{1,\Omega}. \end{aligned} \tag{33}$$

3. *computation of* $u''(x)$: using expression [9] to get

$$u''(x) = -\frac{\nu}{\varepsilon^2}\frac{\exp(\nu x/\varepsilon)}{\exp(\nu/\varepsilon) - 1}, \ \forall x \in (0,1). \tag{34}$$

So, an estimate for u'' can be provided as, since $0 < \exp(\nu x/\varepsilon) \leq \exp(\nu/\varepsilon)$:

$$\max_{x\in[0,1]} | u''(x)| \leq \frac{\nu}{\varepsilon^2 (\exp(\nu/\varepsilon) - 1)}. \tag{35}$$

Gathering [32]–[35] with [29] to get

$$\begin{aligned} \| u - u_h\|_{1,\Omega} &\leq C(\Omega)\frac{M}{\alpha}h \max_{x\in\overline{\Omega}} | u''(x)| \\ &\leq C(\Omega)\frac{\nu(1+\nu)}{\varepsilon^3 (\exp(\nu/\varepsilon) - 1)}h. \end{aligned} \tag{36}$$

So, estimate [36] depends adversly on ε which is not so good when ε is small.

The authors considered a mesh and then they derived a uniform convergence w.r.t the singlar paramter ε. For the reasons stated in the begin of this document (would say, I have not understood the construction of the mesh and its relation with the singlar paramter ε) I could not continue to understand the article. May it will be soon!!

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