

A brief Report on the article [MEI 09] “Radially projected finite elements”

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Abstract: The authors developed and analyzed a new method for an exact discretization of the spheroidal domains and for a construction of finite element spaces on such domains. Such method is based on a radial projection mapping defined on the ball into the cube in any space dimensions. The new method is applied on the Laplace–Beltrami equation and an eigenvalue problem posed on the sphere. The convergence order of the method is one in H^1 –norm and then by duality argument is two in L^2 –norm when the elements are linear in the box. So, the order of the method is optimal because it coincides with that when the domain is polyhedral and meshed with triangular elements. In the end of the paper, the authors presented some numerical experiments which illustrate the effectiveness and characteristics of the method.

Key words and phrases: spheroidal domains, exact finite element discretization; Laplace–Beltrami operator; radial projection; eigenvalues; eigenvectors

Subject Classification (to be checked because these subjects have been taken from the article and I do not know if they are subject classification 2010): 65N30, 65N50, 65N25, 35P15, 58J99?

1 Some questions

1. (existence–uniqueness?): It is mentioned in [MEI 09] that the following equation

$$-\Delta_S \mu + \mu = f, \quad [1]$$

where $-\Delta_S$ is the Laplace–Beltrami operator (on the unit sphere \mathcal{S}), see [14]–[16], has a unique solution (it is a classical result). Since I do not know this existence–uniqueness result before, may be it is useful to express equation [1] in the particular case when \mathcal{S} is the unit circle.

2. (polar coordinates?): using polar coordinates, one could map disc (ball in \mathbb{R}^2) into rectangle; could this technique be applied in this article?

May be it is useful to ask a question from the corresponding authors: ajm@auburn.edu or tuncer@efl.edu

2 What I learned from this nice article!

1. (area of pds on sphere): Mathematical weather models and climate models consist of partial differential equations posed on a domain assumed to be a sphere for an obvious reason.
2. (other area of application): pds on spheroidal and cylindrical shells arise frequently in engineering applications, see [LEE 03, KIR 93].
3. (some literature on numerical methods for...): lately, the study of the numerical approximation of solutions of partial differential equations defined on the sphere has garnered much interests: see [DU 03, DU 05] and references therein.
4. (quality of the grid): *Grid generation* (would be nice if there is some short definition for **Grid generation**) is an important part of approximating solutions of pds (partial differential equations), since the accuracy of numerical solutions depends on the *quality of the grid* (**what sense of quality of the grid**), see [DU 05].
5. (approximation of pds on spheres): two main approaches:
 - (a) approximate the sphere, or other surface by a plyhedron (or polyhedral surface), see [DZI 88]. In this approach, the domain is (**or becomes after approximation**) piecewise planar so discretization techniques can be used.
 - (b) construct a mesh on the sphere (path followed in the article under consideration [MEI 09]). In this approach, given a mesh on the sphere, a finite element space (and the associated basis functions) can be constructed, e.g., by constructing the finite element basis directly on spherical triangles by using barycentric coordinate systems, see [BAU 85], or using mapping as done in [MEI 09].

3 Motivation of the article [MEI 09]

Two reasons on which [MEI 09] is motivated:

1. (**exact discretization**): a desire for a exact finite element discretization of the sphere and other spheroidal domains (yielding a conforming finite element approximation),
2. (**codes**): a requirement that this new approach be easy to implement or easy to incorporate into existing finite element codes.

4 Heart idea behind [MEI 09]: radial projection

Let \mathcal{B}_d be the box centered at the origin and of length $2d$ in \mathbb{R}^n , that is

$$\mathcal{B}_d = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \|x\|_\infty = d\}, \quad [2]$$

where $\|x\|_\infty = \max(|x_1|, \dots, |x_n|)$, and \mathcal{S}_r be the sphere of radius r , that is

$$\mathcal{S}_r = \{a = (a_1, \dots, a_n) \in \mathbb{R}^n : \|a\|_2 = r\}, \quad [3]$$

where $\|a\|_\infty = \sqrt{a_1^2 + \dots + a_n^2}$ is the Euclidean norm.

The interior of box \mathcal{B}_d (resp. sphere \mathcal{S}_r) called the cube (resp. ball).

Let us consider the radial projection mapping defined on the box \mathcal{B}_d into the sphere \mathcal{S}_r :

$$\mathcal{P} : \mathcal{B}_d \rightarrow \mathcal{S}_r \quad [4]$$

by

$$\mathcal{P}(x) = \frac{r}{\|x\|_2} x. \quad [5]$$

\mathcal{P} is one to one mapping and its inverse is defined by

$$\mathcal{P}^{-1}(a) = \frac{d}{\|a\|_\infty} a. \quad [6]$$

Let us now, define the mapping from cube centered at origin and of length d into the ball centered at origin and of radius r :

$$\mathcal{M}(x) = \frac{r\|x\|_\infty}{d\|x\|_2} x. \quad [7]$$

\mathcal{M} is one to one mapping and its inverse is defined by

$$\mathcal{M}^{-1}(a) = \frac{d\|a\|_2}{r\|a\|_\infty} a. \quad [8]$$

So, as we can remark that

$$\mathcal{M}|_{\mathcal{B}_d} = \mathcal{P}. \quad [9]$$

This means that \mathcal{M} transform the box \mathcal{B}_d into the sphere \mathcal{S}_r .

4.1 Properties of the radial projection mapping

The following two Lemmata have been used mainly in [MEI 09]

LEMMA 4.1 The radial projection \mathcal{P} defined, on the box \mathcal{B}_d (it is defined by [2]) into the sphere \mathcal{S}_r (it is defined by [3]), by [5] satisfies the following inequalities

$$\|\mathcal{P}(x) - \mathcal{P}(y)\|_2 \leq \frac{2r}{\|x\|_2} \|x - y\|_2, \quad [10]$$

and

$$\|\mathcal{P}^{-1}(a) - \mathcal{P}^{-1}(b)\|_\infty \leq \frac{2d}{\|a\|_\infty} \|a - b\|_\infty. \quad [11]$$

COROLLARY 4.2 The radial projection \mathcal{P} defined, on the box \mathcal{B}_d (it is defined by [2]) into the sphere \mathcal{S}_r (it is defined by [3]), by [5] satisfies the following inequalities

$$\|\mathcal{P}(x) - \mathcal{P}(y)\|_2 \leq \frac{2r}{d} \|x - y\|_2, \quad [12]$$

and

$$\|\mathcal{P}^{-1}(a) - \mathcal{P}^{-1}(b)\|_\infty \leq \frac{2dn}{r} \|a - b\|_\infty. \quad [13]$$

5 Continuous problem: Laplace–Beltrami equation

The model considered in [MEI 09] is the second order elliptic pde posed on the sphere:

$$-\Delta_S \mu + \mu = f, \quad [14]$$

where $-\Delta_S$ is the Laplace–Beltrami operator (on the unit sphere \mathcal{S}). The Laplace–Beltrami equation is defined as

$$\Delta_S = \nabla_S \cdot \nabla_S, \quad [15]$$

and ∇_S is the tangential gradient defined as

$$\nabla_S = \nabla - \mathbf{n}(\nabla \cdot \mathbf{n}), \quad [16]$$

where \mathbf{n} is the unit outward pointing vector normal to the sphere \mathcal{S} .

Existence–uniqueness of a solution for [14] is classical (can also be justified from the fact that \mathcal{S} is smooth), see [AUB 80].

6 Finite element approximation for [14]

Thanks to radial projection mapping [5] and its properties, the authors first constructed a finite elements on the cube (resp. box) and then these elements are transformed to the ball (resp. sphere).

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