A brief Report on the article [KEL 10]"An enriched subspace finite element method for convection–diffusion problems"

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International Journal of Numerical Analysis and Modeling, 7 (3), 477–490, 2010.

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Last update: Saturday 16th October, 2010; not finished yet

Abstract: The authors present a new finite element scheme for a second order one dimensional singularly perturbed equation. The reviewer thinks that the basic idea of this new scheme is also used in the context of the finite element approximation of corner singularities problems. The authors first derive a convenient expansion for the exact solution. This expansion is the sum of two parts. The first part is satisfying some estimate which allows us to approximate it as usual using the standard finite element method. The second part is spanned by some known functions. Therefore, the approximation suggested by the authors is the sum of two terms. The first term is the standard finite element approximation of the first part of the above stated expansion and the second term is spanned by the above known functions. The error is computed in energy norm associated with the problem. The convergence order is proved to be optimal and uniform with respect to the singular perturbation parameter. Numerical results are presented.

Key words and phrases: second order one dimensional singularly perturbed equation; finite element method; uniform mesh; enriched subspace finite element method

Subject Classification : 65N30;65N15;65L60

Remark 1 The letter C denotes a positive number that may be different in different instances, but always independent of ε and the mesh spacing h.

1 Some final remark

1. It seems for me that the idea enriched finite element subspace has been also used in the context of the finite element discretization for problems posed on domains with corners. More precise, let u consider the Poisson's equation on L–shaped region. The exact solution is not smooth enough, namely $w \notin H^2(\Omega)$ (this regularity is required to obtain the usual optimal order when we use linear finite element method. The exact solution is a sum of a smooth term and a known term which contains the irregular part of u. Let write $u = w + c\varphi$, where $w \in H^2(\Omega)$

and φ is a known but nonsmooth function. A convenient scheme is based on the use of the enriched space $V_h + \langle \varphi \rangle$, where V_h is a linear finite element space and $\langle \varphi \rangle$ denotes the space spanned by φ, that is $\langle \varphi \rangle = \{c\varphi \; c \in \mathbb{R}\}$.

2. this note is just for explaing the main idea of the article [KEL 10]. Because of the limited time, the following items have not been detailed in the present note:
   (a) the construction of the basis ϕ of the expansion [22] and more general expansion
   (b) how it proved the order of the new scheme: this point maybe does not requires great effort.

2 Some useful information from the article [KEL 10]

1. Problem solved in the article: let p, q be two smooth functions, and $\varepsilon \in (0, 1]$

$$\mathcal{L}\, u(x) = -\varepsilon\, u_{xx}(x) + p(x)u_x(x) + q(x)u(x) = f(x), \; x \in \; (0, 1), \tag{1}$$

   with

$$u(0) = u(1) = 0. \tag{2}$$

2. Some information on the exact solution [1]–[2]: the exact solution u has boundary layer at $x = 1$.
3. Numerical approximation for [1]–[2]: classical numerical methods do not provide good approximation arround the layer $x = 1$, see Section 3 below.
4. Some robust numerical approximation for [1]–[2]: to get an accurate and robust numerical methods, it is often used the so called "Shishkin mesh" near the boundary point, see [ROO 08, SHI 92].
5. Another issue for robust numerical approximation for [1]–[2]: to get an accurate and robust numerical method, another option is to use h_p finite element method, see [SCH 98].

3 An overview on some standard numerical methods for singularly perturbed equations

Let us consider the following simple example (which is given in [FEI 04, Pages 342–344]):

$$-\,\varepsilon u''(x) + \nu\, u'(x) = 1, \; x \in \Omega = (0, 1), \tag{3}$$

with

$$u(0) = u(1) = 0, \tag{4}$$

where $\varepsilon > 0$ and $\nu \neq 0$ are two constants. The solution of [3]–[4] is defined by

$$u(x) = \frac{1}{\nu}\left\{x - \frac{\exp(\nu\, x/\varepsilon) - 1}{\exp(\nu\,/\varepsilon) - 1}\right\},\ x \in [0,1]. \qquad [5]$$

If $\varepsilon \to 0$ and $\nu > 0$, then $u(x) \to x/\nu$ for $x \in [0,1)$. The limit function is the solution of

$$\nu\, u'(x) = 1,\ x \in (0,1) \text{ and } u(0) = 0. \qquad [6]$$

3.1 Standard linear finite element methods and Gibs phenomenon

Let us apply the linear finite element method to approximate [3]–[4]. We then consider the uniform mesh $\mathcal{T}_h = \{K_i; i = 0,\ldots,N\}$ with $K_i = [x_i, x_{i+1}]$ and $x_{i+1} - x_i = h$. The approximate solution u_h as well as the test functions are in the space:

$$\mathcal{V}_h = \left\{\varphi_h \in \mathcal{C}(\overline{\Omega}); \varphi_h|_{K_i} \in \mathcal{P}_1,\ \forall\, K_i \in \mathcal{T}_h,\ \varphi_h(0) = \varphi_h(1) = 0\right\}. \qquad [7]$$

Multiplying both sides of [3] by $\varphi \in \mathcal{C}^1(\overline{\Omega})$, with $\varphi(0) = \varphi(1) = 0$, we get

$$\int_0^1 (\varepsilon\, u'(x)\varphi'(x) + \nu\, u'(x)\varphi(x))dx = \int_0^1 \varphi(x)dx,\ \forall \varphi \in \mathcal{C}^1(\overline{\Omega}),\ \varphi(0) = \varphi(1) = 0. \qquad [8]$$

The approximate finite element solution is then defined by: find $u_h \mathcal{V}_h$ such that

$$\int_0^1 (\varepsilon\, u_h'(x)\varphi_h'(x) + \nu\, u_h'(x)\varphi_h(x))dx = \int_0^1 \varphi_h(x)dx,\ \forall \varphi \in \mathcal{V}_h. \qquad [9]$$

This is equivalent to the following linear system:

$$-\frac{\varepsilon}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{\nu}{2h}(u_{i+1} - u_{i-1}) = 1, \qquad [10]$$

with

$$u_0 = u_{N+1} = 0. \qquad [11]$$

Note that [10] is also a finite difference scheme by approximating:

1. $u''(x_i)$ by
$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}. \qquad [12]$$

2. $u'(x_i)$ by the central quotient
$$\frac{u(x_{i+1}) - u(x_{i-1})}{2h}. \qquad [13]$$

The linear equations [10]–[11] can be written as

$$\mathcal{A}\, U = 1, \qquad [14]$$

where $\mathcal{A}$ is a $N \times N$ matrix and 1 in the r.h.s. of [14] is the vector of $\mathbb{R}^N$ whose components are equal to 1. The matrix $\mathcal{A}$ is nonsymmetric, but for $h < 2\varepsilon/|\nu|$ it is *diagonally dominant* (In the sense $\sum_{j=1,j\neq i}^N |\, a_{ij}| \leq |\, a_{ii}|$ for all $i = 1,\ldots,N$ with strict inequality for at least one i.) and *irreducibly.* These previous stated properties of $\mathcal{A}$ guarantees good properties of the approximate solution. But, for $h \geq 2\varepsilon/|\nu|$, the approximate solution do not make sense because of *spurious*

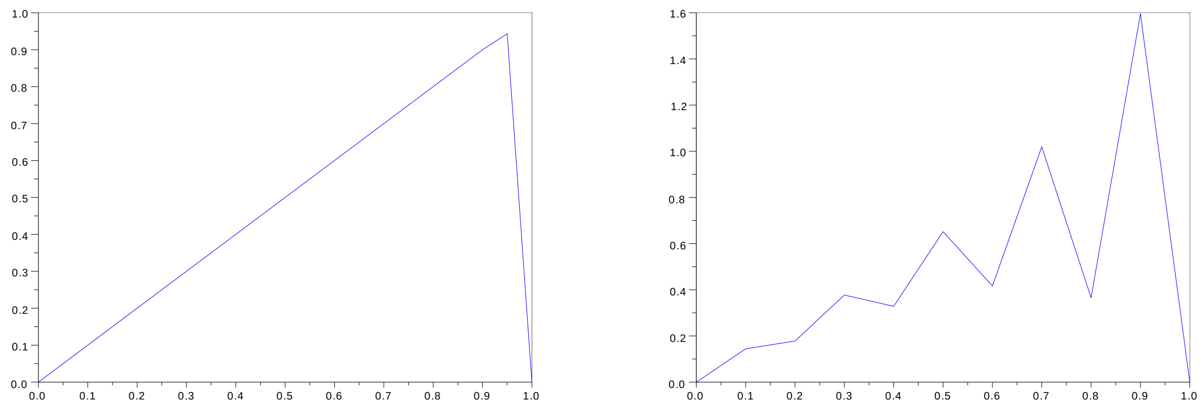

Figure 1: Left graph of the function u, right the finite difference solution [10]–[11] with $h = 1/10$; both simulations are in $\varepsilon = 10^{-2}$ and $\nu = 1$

oscillations shown in the right Figure 1. This means that the *Gibbs phenomenon* arises here. We see here that *mesh Péclet number* defined by

$$Pe = \frac{h\nu}{2\varepsilon} \qquad [15]$$

must satisfy the condition

$$Pe < 1 \qquad [16]$$

in order to avoid the *Gibbs phenomenon.*

4 The main idea of the article

1. The authors use an alernate way to obtain an accurate and robust numerical method.
2. The method is based on finite element method with uniform mesh.
3. The finite element scheme is based on an expansion for the exact solution. This expansion is spanned by some known functions, more details will be given later.
4. The finite element space used in the new scheme is spanned by an usual finite element space and the functions stated in the previous item, see Section 1.

5 A simple particular case of the mean idea of the article

Let us first apply the standard finite element on [1]. A linear finite element method gives us, thanks to Cea lemma

$$\| u - u_h\|_{1,\varepsilon} \leq Ch| u|_{2,(0,1)}, \tag{17}$$

where

$$\| v\|_{1,\varepsilon}^2 = \int_0^1 \left\{ \varepsilon\, v_x^2(x) + v^2(x)\right\} dx. \tag{18}$$

The following formula, on the estimate of the exact solution u, is provided in [KEL 10]

$$| u^{(k)}| \leq C\| f\|_{k,\infty} \left(1 + \varepsilon^{-k} \exp\left(-p_{\min}(1-x)/\varepsilon\right)\right), \tag{19}$$

which implies that, with $k = 2$

$$| u|_{2,(0,1)} \approx \varepsilon^{-3/2}, \text{ as } \varepsilon \approx 0. \tag{20}$$

This with [17] gives

$$\| u - u_h\|_{1,\varepsilon} \approx C\varepsilon^{-3/2}h. \tag{21}$$

Using the idea of [KEL 10], we can find an approximation, u_h for u in which the $\| u - u_h\|_{1,\varepsilon}$ is of order h uniformly. The idea is sketched as follows:

1. Convenient expansion for u

   $$u = r_1 + \sum_{i=0}^{2} \gamma_i \phi_i, \tag{22}$$

   where

   $$| r_1|_{2,(0,1)} \leq C\left(1 + \varepsilon^{3/2}\right), \tag{23}$$

   $$\phi_i = (1-x)^i \exp -p_0(1-x)/\varepsilon, \tag{24}$$

   and $\{\gamma_i,\ i = 0, \dots, 2\}$ with $p_0 = p(1)$.

2. The enriched finite element space: the enriched finite element space is given by

   $$\bar{\mathcal{V}}_h = \mathcal{V}_h + \left\{\sum_{i=0}^{2} \beta_i\phi_i,\ \beta_i \in \mathbb{R}\right\}, \tag{25}$$

   where $\mathcal{V}_h$ is the piecewise linear finite element space with spacing h. So, we look for an approximation $u_h \in \bar{\mathcal{V}}_h$ instead of $\mathcal{V}_h$.

3. The convergence order: we prove that the new approximation, via the use of the Galerkin method, in $\bar{\mathcal{V}}_h$ is of order h uniformly.

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