A brief Report on the article [DUP 10] "On the necessity of Nitsch term"

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Abstract: The authors consider the Laplace equation with non homogeneous Dirichlet boundary conditions in one and two dimensions. A finite element scheme is suggested using a special variational formulation involving the boundary condition data as a term in the equation, whereas the space of the solution and the space of test functions are $\{u \in H^1(\Omega); \Delta\, u \in L^2(\Omega)\}$.

In the one dimensional case and using the above stated variational formulation, the authors suggested a piecewise linear space in order to approximate the exact solution and the space of the test functions. A convergence analysis based on the use of inf-sup inequality is provided. It is proved that the convergence order is optimal in the sense that the order is one, in the energy norm, and is two in the average norm.

Unfortunately, the analysis used in the one dimensional case can not be used in higher dimension for some geometrical reason stated by the authors. Consequently, some numerical experiments in dimension two are presented to justify the need of additional stabilization methods to obtain a reliable numerical method. The choice followed by the authors is the Nitsche method: that is to add some additional terms to the discrete bilinear and discrete linear forms. Thanks to some numerical experiments, it is proved that the addition of the Nitsche terms is not sufficient: a compatibility condition between the domain and the discrete grid should be added. Under this compatibility condition, discrete Nitsche problem is well posed and its solution converges to the exact solution by optimal order.

Key words and phrases: Laplace equation with non homogeneous Dirichlet boundary conditions; finite element methods; Nitsche method; geometrical compatibility condition

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1 Some final remark

1. In the statement of the model problem, it is mentioned that Ω is a domain of $\mathbb{R}^d$, $d = 2$ or 3, but the cases treated throughout the paper are $d = 1$ and $d = 2$.

2. Is not clear for me what is the advantage when we use the variational form and its discretization treated in [DUP 10] and the usual variational form and its discretization: it would be

useful if the authors could explain more their approach and more precise the difference of the usual variational [7] with its usual discretization and the discretization based on the formulation [3]. .

2 Problem and its variational form

The following problem is studied in [DUP 10]; let Ω be a domain of $\mathbb{R}^d$, $d = 2$, $d = 3$

$$-\Delta u(x) = f(x),\ x \in \Omega, \qquad [1]$$

with boundary condition

$$u(x) = g(x),\ x \in \partial\Omega, \qquad [2]$$

where $f \in \mathbb{L}^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$ are given functions.

The following variational form is used in [DUP 10]: Find v in appropriate space $\mathcal{H}$ such that

$$\mathcal{B}(v, \varphi) = L_2(\varphi),\ \forall\, \varphi \in \mathcal{H}, \qquad [3]$$

where

$$\mathcal{B}(v, \varphi) = \int_\Omega \nabla v(x) \cdot \nabla \varphi(x) dx - \int_{\partial\Omega} \left(u(x)\partial_n\varphi(x) + \varphi(x)\partial_n u(x)\right) d\gamma(x), \qquad [4]$$

and

$$L_2(\varphi) = \int_\Omega f(x)\varphi(x)dx - \int_{\partial\Omega} g(x)\partial_n\varphi(x)\, d\gamma(x). \qquad [5]$$

Indeed, the bilinear form [4] is well defined for v, φ in $\mathcal{H}$ thanks to the following version of integration by parts, see [BRE 94, Proposition 5.1.6], for all $(u, v) \in \mathcal{H} \times\ H^1(\Omega)$:

$$\int_\Omega (-\Delta\, u(x))v(x)dx = \int_\Omega \nabla u(x) \cdot \nabla v(x)dx - \int_{\partial\Omega} \partial_n u(x)v(x)\, d\gamma(x). \qquad [6]$$

It is useful to note that $\mathcal{H}(1, 1) = 0$ and $\mathcal{H}(u, u) \leq 0$, for all harmonic function u.

The standard formulation is to find $u \in H^1(\Omega)$ such that the trace $\tilde{\gamma}(u) = g$ on $\partial\Omega$:

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)dx = \int_\Omega f(x)v(x)dx,\ \forall\, v \in H_0^1(\Omega) \qquad [7]$$

or equivalently, we seek for $w \in\ H_0^1(\Omega)$ such that

$$\int_\Omega \nabla w(x) \cdot \nabla v(x) = -\int_\Omega \nabla \tilde{g}(x) \cdot \nabla v(x)dx + \int_\Omega f(x)v(x)dx,\ \forall\, v \in H_0^1(\Omega), \qquad [8]$$

where, this is feasible since $g \in H^{\frac{1}{2}}(\partial\Omega)$,

$$\tilde{\gamma}(\tilde{g}) = g. \qquad [9]$$

Typically, the space $\mathcal{H}$ is given by

$$\mathcal{H} = \mathcal{H}_\Delta = \left\{u \in H^1(\Omega); \Delta\, u \in \mathbb{L}^2(\Omega)\right\}. \qquad [10]$$

The following Theorem is given in [DUM 06]

THEOREM 2.1 Problem [3]–[5], where $\mathcal{H}$ is the space defined in [10], has a unique solution.

3 How it the situation in the one dimensional case?

3.1 A functional result useful for inf–sup result

Th the finite element scheme presented by thr authors is based on the discretization of [3]–[5], where $\mathcal{H}$ is the space defined in [10] using piecewise linear finite element space. The convergence analysis is done thanks to inf–sup inequality. The following result is used to check an inf–sup result:

THEOREM 3.1 Let $\mathcal{V}$ be a Hilbert space endowed with the norm $\|\cdot\|$ decomposed as $\mathcal{V} = \mathcal{V} \oplus \mathcal{V}$ where the angle between $\mathcal{V}_1$ and $\mathcal{V}_2$ has a cosine γ different from 1, where

$$\gamma = \sup_{(v_1,v_2)\in\mathcal{V}_1\setminus\{0\}\times\mathcal{V}_2\setminus\{0\}} \frac{|\langle v_1, v_2\rangle|}{\|v_1\|\,\|v_2\|}. \tag{11}$$

Let $\mathcal{B}(\cdot,\cdot)$ be a continuous bilinear form on $\mathcal{V}\times\mathcal{V}$. If $\mathcal{B}$ satisfies an inf–sup condition on $\mathcal{V}_1\times\mathcal{V}_1$ with a constant α_1, then there exists a constant $\alpha > 0$ such that

$$\inf_{u\in\mathcal{V}} \sup_{v\in\mathcal{V}} \frac{|\mathcal{B}(u,v)|}{\|u\|\,\|v\|} \geq \alpha. \tag{12}$$

3.2 Some discrete subspaces to check inf–sup inequality

Let $\mathcal{V}_h$ be the piecewise linear space, with spacing h, on interval $[0,1]$. The following subspaces of $\mathcal{V}_h$ are useful:

1. the two spaces containing function basis which do not vanish on the boundary mesh:

$$\mathcal{V}_h^1 = \langle\{\varphi_0, \varphi_1\}\rangle, \tag{13}$$

$$\mathcal{B}(u-u_h, u-u_h)\mathcal{V}_h^2 = \langle\{\varphi_N, \varphi_{N+1}\}\rangle. \tag{14}$$

2. interior subspace:

$$\mathcal{V}_h^3 = \langle\{\varphi_2, \dots, \varphi_{N-1}\}\rangle. \tag{15}$$

The previous two subsection allow the authors to justify that discrete problem is well posed.

4 What about the convergence order?

The first key, as usual, that the following equality holds

$$\mathcal{B}(u-u_h, v_h) = 0,\ \forall\, v_h \in \mathcal{V}_h. \tag{16}$$

So, thanks an elementary reasoning based on the fact that $\mathcal{B}(\cdot,\cdot)$ is bilinear, we get

$$\mathcal{B}(u-u_h, u-u_h) = \mathcal{B}(u-u_h, u-\Pi\, u), \tag{17}$$

which implies using the fact that $\mathcal{B}(\cdot,\cdot)$ is continuous

$$\mathcal{B}(u-u_h, u-u_h) \leq C\|u-u_h\|_{1,\Omega}\,\|u-\Pi\, u\|_{1,\Omega}. \tag{18}$$

On the other hand, we remark that

$$\mathcal{B}(u-u_h, u-u_h) = \| \nabla (u-u_h)\|^2_{\mathbb{L}^2(\Omega)}, \qquad [19]$$

which implies uing the Poincaré inequality, for some positive constant C_p

$$\mathcal{B}(u-u_h, u-u_h) \geq C_p\| u-u_h\|^2_{1,\Omega}. \qquad [20]$$

This with [18] and the standard interpolation error, we get

$$\| u-u_h\|_{1,\Omega} \leq Ch| u-u_h|_{2,\Omega}. \qquad [21]$$

I do not know if the duality argument can be applied here to obtain the order two in $\mathbb{L}^2(\Omega)$ norm (The authors just said the order is two in $\mathbb{L}^2(\Omega)$, in Theorem 2.2, without any comment or justification but the authors provided detailed proof on the order one in the enery norm as it is mentioned in [16]–[21].). Because of limited time, I stop here and my hope I come back another time to see details on the average error in the one dimension, and then I pass to the dimension two in which, I think, there are useful knowledge to be learned from this article like the Nitsche method.

References

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