A brief Report on the article [BRE 10] "An *a posteriori* error estimator for a quadratic $\mathcal{C}^0$–interior penalty method for the biharmonic problem"

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Abstract: The authors consider the biharmonic equation posed on two dimensional bounded polygonal domain . The finite element discretization presented by the authors is based on the use of a quadratic $\mathcal{C}^0$–interior penalty method. They derived an error estimator, denoted by η_h. This error estimator η_h is reliable (resp. efficient) in the sense that it is bounded below (resp. above) by the error between the exact solution and the finite element approximate solution. Numerical examples are presented.

Key words and phrases: biharmonic equation; efficient and reliable error estimator; quadratic $\mathcal{C}^0$–interior penalty method

Subject Classification : 65N30; 65N15

1 Some final remarks

1. in my opinion, the present article [BRE 10] is a useful work: some nice literature (around 44 references provided) is provided as well as nice details for the results provided by the article. My hope I find some time to come back to this article!!

2. the discrete formulation is based on the use of the expression involved in the continuous formulation plus some terms and a penalty term. I think that, these additional terms have been added because of the nonregularirty of the discrete functions. These terms yeild some consistensy, stability, and symmetrization. I understood that more information about these additional terms as well as the properties of consistensy and stability have been detailed in the previous papers [ENG 02, BRE 05].

2 Useful knowledge

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $f \in \mathbb{L}^2(\Omega)$. A weak formulation of the biharmonic problem is to find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v), \ \forall\, v \in \ H_0^2(\Omega), \quad [1]$$

where

$$a(u,v) = \int_\Omega D^2 u(x) : D^2 v(x) dx, \quad [2]$$

$$\begin{aligned} D^2 u : D^2 v &= \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \\ &= u_{xx} v_{xx} + u_{yy} v_{yy} + 2 u_{xy} v_{xy} \\ &= \Delta\, u \Delta\, v + (\, 2 u_{xy} v_{xy} - u_{xx} v_{yy} - u_{yy} u_{xx}) \end{aligned} \quad [3]$$

and $(\,\cdot,\cdot)$ denotes the $\mathbb{L}^2$–inner product.
So, $D^2 u$ is the Hessian matrix and $D^2 u : D^2 v$ denotes the summation of each corresponding component product.

1. what problem represents [1]–[3]: according to [BRE 08, Theorem 5.9.6, Page 144], if $u \in H^4(\Omega)$ and $f \in \mathbb{L}^2(\Omega)$, the weak unique solution of [1]–[3] satisfies (the idea behind of such result is, maybe, the use of the integration by parts which yields verious versions of the cross derivatives u_{xxyy}) the biharmonic equation in the $\mathbb{L}^2(\Omega)$–sense:

$$\Delta^2 u = f. \quad [4]$$

2. standard methods for the biharmonic equation: Conforming finite element methods for [1] requires $\mathcal{C}^1$–finite element spaces, which are complicated to construct and involve a large number of degree of freedom.

3. another issue: another issue is to use mixed finite element methods.

4. what about mixed finite element for [1]: according to the authors of [BRE 10], the design of stable (would be fine from the authors to give more details here as the notion of stability) mixed finite element methods is a delicate task and highly nontrivial for more complicated fourth order equations.

5. path followed by the authors of [BRE 10]: the path followed by the authors is to use an interior penalty which preserves the symmetric positive definitness of [1] and at the same time uses only $\mathcal{C}^0$–finite elements for second–order problems was proposed, for instance, in [ENG 02].

6. some literature:

   (a) multigrid and domain for $\mathcal{C}^0$–interior penalty methods were studied in [BRE 05] and [BRE 05]

(b) the authors said (I enjoyed this nice sentence) "In this paper we develop a simple residual–based *a posteriori* error estimator for a quadratic $\mathcal{C}^0$–interior penalty method for [1]. "

(c) while there is a vast literature on error estimators for conforming finite element methods for second order elliptic problems, see for instance [VER 92] and [VER 95].

(d) there are also quite a few papers on error estimators for nonconforming finite element methods.

(e) there are only a hanful of papers on error estimators for fourth order elliptic equations.

3 Some mathematics: finite element discretization

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω. The interior (resp. boundary) edges are denoted by $\mathcal{E}_h^{\rm i}$ (resp. $\mathcal{E}_h^{\rm b}$) and define $\mathcal{E}_h = \mathcal{E}_h^{\rm i} \cup \mathcal{E}_h^{\rm b}$. Let $h_T = {\rm diam(T)}$ and $h = \max\{h_T : T \in \mathcal{T}_h\}$. The length of an edge $e \in \mathcal{E}$ is denoted by h_e. The following Sobolev space associated to $\mathcal{T}_h$ is introduced

$$H^k(\Omega, \mathcal{T}_h) = \Big\{ v \in H_0^1(\Omega) : v_T = v|_T \in H^k(\Omega),\ \forall T \in \mathcal{T}_h \Big\}. \qquad [5]$$

The $\mathcal{C}^0$–finite element space is

$$\mathcal{V}_h = \left\{ v_h \in H_0^1(\Omega) : v_h|_T \in \mathcal{P}^2(\Omega),\ \forall T \in \mathcal{T}_h \right\}. \qquad [6]$$

3.1 Finite element approximate solution

We introduce the following discrete bilinear form:

$$\begin{aligned} \mathcal{A}_h(u_h, v_h) &= \sum_{T \in \mathcal{T}_h} \int_T D^2 u_h : D^2 v_h dx + \sum_{e \in \mathcal{E}_h} \int_e [\![[\frac{\partial^2 u_h}{\partial^2 \mathbf{n}}]\!]] [\![\frac{\partial u_h}{\partial \mathbf{n}}]\!] \, d\gamma(x) \\ &+ \sum_{e \in \mathcal{E}_h} \int_e [\![\frac{\partial u_h}{\partial \mathbf{n}}]\!] [\![[\frac{\partial u_h}{\partial \mathbf{n}}]\!]] \, d\gamma(x) + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{h_e} \int_e [\![\frac{\partial u_h}{\partial \mathbf{n}}]\!] [\![\frac{\partial v_h}{\partial \mathbf{n}}]\!] \, d\gamma(x), \end{aligned} \qquad [7]$$

where $[\![[\cdot]\!]]$ (resp. $[\![\cdot]\!]$) denotes the mean (resp. the jump) between two neighbouring triangles, $\mathbf{n}$ is the usual normal derivative, and σ is the penalty parameter.

The following properties hold, for more details see the link http://www.math.sc.edu/~fem/fe.html in the home page of Professor Brenner http://www.math.sc.edu/~fem/brenner.html:

1. symmetry:

$$\mathcal{A}_h(u_h, v_h) = \mathcal{A}_h(v_h, u_h),\ \forall\, u_h, v_h \in \mathcal{V}_h. \qquad [8]$$

2. consistensy: let u be the exact solution of [4]

$$\mathcal{A}_h(u, v_h) = (\, f, v_h)\,,\ \forall\, v_h \in \mathcal{V}_h. \qquad [9]$$

3. stability:

$$\mathcal{A}_h(v_h, v_h) \geq \| v_h\|^2_{H^2(\Omega,\mathcal{T}_h)}, \qquad [10]$$

where

$$\| v_h\|^2_{H^2(\Omega,\mathcal{T}_h)} = a_h(v_h, v_h) + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{h_e}\int_e [\![\frac{\partial v_h}{\partial \mathbf{n}}]\!]^2\, d\gamma(x), \forall\, v_h \in \mathcal{V}_h, \qquad [11]$$

with

$$a_h(u_h, v_h) = \sum_{T\in\mathcal{T}_h}\int_T D^2 u_h : D^2 v_h dx, \forall\, u_h, v_h \in \mathcal{V}_h. \qquad [12]$$

So, the finite element appximation is defined by: Find $u_h \in \mathcal{V}_h$ such that

$$\mathcal{A}_h(u_h, v_h) = (\, f, v_h) \quad \forall\, v_h \in \mathcal{V}_h. \qquad [13]$$

3.2 Definition of a reliable estimator

The error estimator η_h is defined by

$$\eta_h = \left(\sum_{T\in\mathcal{T}_h}\eta_T^2 + \sum_{e\in\mathcal{E}_h}\eta_{e,1}^2 + \sum_{e\in\mathcal{E}_h^{\mathrm{i}}}\eta_{e,2}^2\right)^{\frac{1}{2}}, \qquad [14]$$

where

$$\eta_T = h_T^2\| f\|_{\mathbb{L}^2(\Omega)}, \qquad [15]$$

$$\eta_{e,1} = \frac{\sigma}{\sqrt{h_e}}\| [\![\frac{\partial u_h}{\partial \mathbf{n}}]\!]\|^2_{\mathbb{L}^2(e)}, \qquad [16]$$

and

$$\eta_{e,2} = \sqrt{h_e}\| [\![\frac{\partial u_h}{\partial \mathbf{n}}]\!]\|^2_{\mathbb{L}^2(e)}. \qquad [17]$$

The error estimator η_h given by [14]–[17] since it satifies [BRE 10, Theorem 3.1, Page 783] which is:

THEOREM 3.1 Let u (rep. u_h) be the exact solution of [1]–[3] (resp. [13]). Then the following estimate holds

$$\| u - u_h\|_{H^2(\Omega,\mathcal{T}_h)} \leq C\eta_h, \qquad [18]$$

where error estimator η_h is given by [14]–[17].

3.3 Why the error estimator [14]–[17] is efficient

Error estimator [14]–[17] is efficient since η_h satisfies [BRE 10, Theorem 4.1, Page 786]

THEOREM 3.2 Let u (rep. u_h) be the exact solution of [1]–[3] (resp. [13]). Then the following estimate holds

$$\eta_h \leq C\left(\sigma\| u - u_h\|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{T\in\mathcal{T}_h} h_T^4\| f - \bar{f}\|^2_{\mathbb{L}^2(\Omega)}\right)^{\frac{1}{2}}, \qquad [19]$$

where error estimator η_h is given by [14]–[17], and

$$\bar{f} = \frac{1}{\text{meas(T)}}\int_T f(x)dx. \qquad [20]$$

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