A brief Report on the article [CHE 10] "An anisotropic nonconforming element for fourth order elliptic singular perturbation problem"

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Abstract: The authors consider a fourth order singular perturbation problem posed on two dimensional rectangular domain . An anistroptic nonconforming finite element scheme is presented to approximate the problem. The scheme is based on the use of a technique called the double set parameter method. Thanks to the use of a convenient choice for the first and the second set parameters, a convergence of order one in the energy norm associated to the problem is proved uniformly with respect to the singular parameter. Numerical examples are presented to explain theoretical results.

Key words and phrases: fourth order elliptic problem; singular perturbation problem; anisotropic nonconforming finite element method; double set parameter method; uniform convergence

Subject Classification : 65N12; 65N30

Some final remarks

Because of limited time, I could not finish well reading the article. Next some remarks:

1. convergence order: I order is said is optimal!! I have not understood this. Indeed, the shape function space is spanned by quadratic polynomes and the cubic polynomes $\{\xi^3, \eta^3\}$. So, the interpolation error, so far as I know, is of order $h^{3+1-2} = h^2$ (2 is the order of the norm), see [BRE 94, Theorem 4.4.4, Page 104]. This point is not clear for me!!

2. problem treated: the problem treated here is a special one $\varepsilon^2\Delta^2 u - \Delta\, u = f$; is it possible to consider a more general fourth order equation. Is it possible to consider the domain problem as a polygone instead of a rectangle.

3. weak formulation: the weak formulation followed in [CHE 10] is

$$\varepsilon^2 a(u,v) + b(u,v) = (\,f, v\,)\,,\ \forall\, v \in \mathcal{V}, \tag{1}$$

where

$$\mathcal{V} = \{v \in H^2(\Omega);\ v = \frac{\partial^2 v}{\partial\, \mathbf{n}^2} = 0\}, \tag{2}$$

$$a(u,v)=\int_\Omega D^2u(x):D^2v(x)dx, \qquad [3]$$

and

$$b(u,v)=\int_\Omega \nabla u(x)\cdot\nabla u(x)dx. \qquad [4]$$

Which I find strange that the space $\mathcal{V}$ given by [2] is not well defined; more precise the elements of $\mathcal{V}$ have with their first derivatives well defined traces, but I do not know if the second derivatives of the elements of $\mathcal{V}$ have well defined trace, would say is $\frac{\partial^2 v}{\partial\,\mathbf{n}^2}$ well defined for only $v\in H^2(\Omega)$?

1 Problem considered

The problem considered in [CHE 10] is

$$\varepsilon^2\Delta^2 u(x)-\Delta\,u(x)=f(x),\;\; x\in\Omega=(0,1)^2, \qquad [5]$$

with

$$u(x)=\frac{\partial^2 u}{\partial\,\mathbf{n}^2}(x)=0, x\in\partial\Omega. \qquad [6]$$

2 Some preliminary computations on the biharmonic equation

This article I'm reviewing now for Zmath has the same subject as that I reviewed since some days ago; i.e. [BRE 94]. Since I'm new in the subject of the discretization of the biharmonic equation, I would like to try to understand at least the basic background on the subject of biharmonic equation and its discretization. The first item of the following items have been taken from a downloaded paper from th link of the Brenner's home page: http://www.math.sc.edu/~fem/fe.html

1. continuous case: let $\mathcal{G}$ be a somooth region in $\mathbb{R}^2$, u, v be two smooth functions defined on $\mathcal{G}$, we have

$$\begin{aligned}
\int_{\mathcal{G}} \Delta^2 u(x) v(x) dx &= \int_{\mathcal{G}} \Delta(\Delta\, u) v(x) dx = \int_{\mathcal{G}} \nabla \cdot \left(\nabla \left(\Delta\, u(x)\right)\right)\, v(x) dx \\
&= -\int_{\mathcal{G}} \nabla \left(\Delta\, u(x)\right)\, \cdot\, \nabla\, v(x) dx + \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x) \\
&= -\sum_{i=1}^{2} \int_{\mathcal{G}} \frac{\partial\, \Delta\, u}{\partial\, x_i}(x) \frac{\partial\, v}{\partial\, x_i}(x) dx + \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x) \\
&= \int_{\mathcal{G}} \Delta\, u(x) \sum_{i=1}^{2} \frac{\partial}{\partial\, x_i} \left(\frac{\partial\, v}{\partial\, x_i}\right)(x) dx - \int_{\partial\mathcal{G}} \Delta\, u(x) \sum_{i=1}^{2} \frac{\partial\, v}{\partial\, x_i}(x) \mathbf{n}_i d\gamma(x) \\
&+ \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x) \\
&= \int_{\mathcal{G}} \Delta\, u(x)\, \Delta\, u(x) dx - \int_{\partial\mathcal{G}} \Delta\, u(x) \frac{\partial\, v}{\partial\, \mathbf{n}}(x) \gamma(x) + \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x) \qquad [7]
\end{aligned}$$

Another version for the previous identity which is used [BRE 94] is the following

$$\begin{aligned}
\int_{\mathcal{G}} \Delta^2 u(x) v(x) dx &= \int_{\mathcal{G}} \Delta(\Delta\, u) v(x) dx = \int_{\mathcal{G}} \nabla \cdot \left(\nabla \left(\Delta\, u(x)\right)\right)\, v(x) dx \\
&= -\int_{\mathcal{G}} \nabla \left(\Delta\, u(x)\right)\, \cdot\, \nabla\, v(x) dx + \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x) \\
&= -\int_{\mathcal{G}} \Delta \left(\nabla\, u(x)\right)\, \cdot\, \nabla\, v(x) dx + \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x) \\
&= -\sum_{i=1}^{2} \int_{\mathcal{G}} \Delta\, u_{x_i}(x) v_{x_i}(x) dx + \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x) \\
&= -\sum_{i=1}^{2} \int_{\mathcal{G}} \nabla\, \cdot \nabla\, u_{x_i}(x) v_{x_i}(x) dx + \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x) \\
&= \sum_{i=1}^{2} \int_{\mathcal{G}} \nabla\, u_{x_i}(x) \cdot \nabla\, v_{x_i}(x) dx - \sum_{i=1}^{2} \int_{\partial\mathcal{G}} \mathbf{n}_i \cdot \nabla\, u_{x_i}(x) v_{x_i}(x) d\gamma(x) \\
&+ \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x) \\
&= \sum_{i=1}^{2} \sum_{j=1}^{2} \int_{\mathcal{G}} u_{x_j x_i}(x)\, v_{x_j x_i}(x) dx - \sum_{i=1}^{2} \int_{\partial\mathcal{G}} \mathbf{n}_i \cdot \nabla\, u_{x_i}(x) v_{x_i}(x) d\gamma(x) \\
&+ \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x) \\
&= \int_{\mathcal{G}} D^2 u(x) : D^2 v(x) dx - \int_{\partial\mathcal{G}} \frac{\partial \nabla\, u}{\partial \mathbf{n}}(x) \cdot \nabla\, v(x) dx \\
&+ \int_{\partial\mathcal{G}} \frac{\partial \Delta\, u}{\partial \mathbf{n}}(x) v(x) d\gamma(x). \qquad [8]
\end{aligned}$$

Summing the previous identity, for u smooth and v piecewise smooth over $\mathcal{G} = K \in \mathcal{T}_h$, we get

$$\begin{aligned}
\int_{\Omega} \Delta^2 u(x) v_h(x) dx &= \sum_{K \in \mathcal{T}_h} \int_K D^2 u(x) : D^2 v_h(x) dx - \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla \left(\nabla\, u\right) \cdot \mathbf{n}_{K,\sigma}(x) \cdot \nabla\, v_h(x) dx \\
&+ \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla(\Delta\, u)(x) \cdot \mathbf{n}_{K,\sigma} v_h(x) d\gamma(x), \qquad [9]
\end{aligned}$$

where $\mathbf{n}_{K,\sigma}$ is the unit vector normal to σ outward to K.

Let first that if σ is a common edge of two neighbouring element T and L, we have then, since v_h is continuous

$$\begin{aligned}\int_\sigma \nabla(\Delta\, u)(x)\cdot\mathbf{n}_{K,\sigma}v_h(x)d\gamma(x) \quad &+ \quad \int_\sigma \nabla(\Delta\, u)(x)\cdot\mathbf{n}_{L,\sigma}v_h(x)d\gamma(x) = \int_\sigma \nabla(\Delta\, u)(x)\cdot\mathbf{n}_{K,\sigma}v_h(x)d\gamma(x)\\ &- \quad \int_\sigma \nabla(\Delta\, u)(x)\cdot\mathbf{n}_{K,\sigma}v_h(x)d\gamma(x) + \int_\sigma = 0. \qquad [10]\end{aligned}$$

For $\sigma\subset\partial\,\Omega$, we know that $v_h(x)=0$ for all $x\in\partial\Omega$; so $\int_\sigma \nabla(\Delta\, u)(x)\cdot\mathbf{n}_{K,\sigma}v_h(x)d\gamma(x)=0$ for all $\sigma\subset\partial\,\Omega$. Therefore

$$\sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_K}\int_\sigma \nabla(\Delta\, u)(x)\cdot\mathbf{n}_{K,\sigma}v_h(x)d\gamma(x)=0. \qquad [11]$$

Let $\mathbf{t}_\sigma$ be the unite tangente to σ, we have then the following computations to simplify the second term on the right hand side of [11]

$$\begin{aligned}\frac{\partial\nabla\, u}{\partial\mathbf{n}_{K,\sigma}}\cdot\nabla\, v_h \quad &= \quad \frac{\partial}{\partial\mathbf{n}_{K,\sigma}}\left(\frac{\partial\, u}{\partial\mathbf{n}_{K,\sigma}}\cdot\mathbf{n}_{K,\sigma}+\frac{\partial\, u}{\partial\, S}\cdot\mathbf{t}_\sigma\right)\cdot\left(\frac{\partial\, v_h}{\partial\mathbf{n}_{K,\sigma}}\cdot\mathbf{n}_{K,\sigma}+\frac{\partial\, v_h}{\partial\, S}\cdot\mathbf{t}_\sigma\right)\\ &= \quad \left(\frac{\partial^2\, u}{\partial\mathbf{n}^2_{K,\sigma}}\cdot\mathbf{n}_{K,\sigma}+\frac{\partial^2\, u}{\partial\mathbf{n}_{K,\sigma}\partial\, S}\cdot\mathbf{t}_\sigma\right)\cdot\left(\frac{\partial\, v_h}{\partial\mathbf{n}_{K,\sigma}}\cdot\mathbf{n}_{K,\sigma}+\frac{\partial\, v_h}{\partial\, S}\cdot\mathbf{t}_\sigma\right)\\ &= \quad \frac{\partial^2\, u}{\partial\mathbf{n}^2_{K,\sigma}}\frac{\partial\, v_h}{\partial\mathbf{n}_{K,\sigma}}+\frac{\partial^2\, u}{\partial\mathbf{n}_{K,\sigma}\partial\, S}\frac{\partial\, v_h}{\partial\, S}, \qquad [12]\end{aligned}$$

where $\frac{\partial}{\partial\, S}$ denotes the tangential derivative.

For σ is a common edge of two neighbouring element T and L, we have, as in [10]

$$\int_\sigma \frac{\partial^2\, u}{\partial\mathbf{n}_{K,\sigma}\partial\, S}\frac{\partial\, v_h}{\partial\, S}d\gamma(x) \quad + \quad \int_\sigma \frac{\partial^2\, u}{\partial\mathbf{n}_{L,\sigma}\partial\, S}\frac{\partial\, v_h}{\partial\, S}d\gamma(x)=0. \qquad [13]$$

For $\sigma\subset\partial\,\Omega$, we know that $v_h(x)=0$ for all $x\in\partial\Omega$; so $\dfrac{\partial\, v_h}{\partial\, S}$ and therefore $\dfrac{\partial^2\, u}{\partial\mathbf{n}_{K,\sigma}\partial\, S}\dfrac{\partial\, v_h}{\partial\, S}d\gamma(x)$ for all $\sigma\subset\partial\,\Omega$.

Combining now [9] with [10]–[13] and reordering the sum yields that

$$\int_\Omega \Delta^2 u(x)v_h(x)dx=\sum_{K\in\mathcal{T}_h}\int_K D^2u(x):D^2v_h(x)dx-\sum_{\sigma\in\mathcal{E}}\frac{\partial^2\, u(x)}{\partial\mathbf{n}^2_{K,\sigma}}\left[\!\!\left[\frac{\partial\, v_h(x)}{\partial\mathbf{n}_{K,\sigma}}\right]\!\!\right]d\gamma(x), \qquad [14]$$

where $[\![\cdot]\!]$ is the jump of the two neighbouring elements.

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