A brief Report on the article [VOL 10]"On a grid-method solution of the Laplace equation in an infinite rectangular cylinder under periodic boundary conditions"

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Abstract: The article deals with a new finite difference scheme to approximate Laplace equation posed on three dimensional infinite rectangular cylinder with periodic boundary conditions. Under the assumption that the boundary value data is a periodic bounded function in the x_3 direction, it is first proved that there exists a unique bounded periodic solution u. A special finite difference scheme is suggested. To derive the error estimate, the authors proved two results on the exact solution. The first result states that the exact solution are bounded. The second result stated some estimate for the fourth derivatives of the exact solution. Thanks to these two stated results on the exact solution and other two lemmas, the author proved that the convergence order of the finite difference scheme is $h^2|\log h|$.

Key words and phrases: three dimensional infinite rectangular cylinder; periodic boundary conditions; Laplace equation; finite difference scheme; error bound

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Brief notes

1. equation sloved let R_∞ be the infinite rectangular cylinder $\{x : 0 \le x_i \le a_i,\ i = 1, 2;\ x_3 \in \mathbb{R}\}$ and consider:

$$-\Delta u(x) = 0,\ x \in R_\infty, \tag{1}$$

with the the boundary condition

$$u(x) = \varphi(x),\ x \in \partial R_\infty. \tag{2}$$

2. numerical method used: finite difference method

3. firt main result: if the boundary value data φ is bounded, then there exist a unique bounded solution u.

4. second main result: if φ is periodic of period two in x_3 direction, then u satisfies the same property, that is u is periodic of period two. In addition, the second derivatives of u are bounded. The fourth derivatives of u are also bounded by some suitable function (some thing like $1/(|x|^2)$ (i do not know in which place this inequality is used).

5. idea on the scheme: the finite difference scheme uses a special operator, taking periodicity in the direction x_3, defined by

$$\begin{aligned} &\tilde{A}u^h(x_1, x_2, x_3) \\ = \ & \frac{1}{6}(u^h(x_1 - h, x_2, x_3) + u^h(x_1 + h, x_2, x_3) + u^h(x_1, x_2 + h, x_3) + u^h(x_1, x_2 - h, x_3) \\ + \ & u^h(x_1, x_2, x_3^{+h}) + u^h(x_1 + h, x_2, x_3^{-h})). \end{aligned} \quad [3]$$

where $x_3^{+h} = x_3 + h$ if $-1 \leq x_3 < 1$ and $x_3^{+h} = -1 + h$ if $x_3 = 1$; $x_3^{-h} = x_3 - h$ if $-1 < x_3 \leq 1$ and $x_3^{-h} = 1 - h$ if $x_3 = -1$

The finite difference scheme suggested in the article is

$$\tilde{A}u^h = u^h, \quad [4]$$

with usual approximation for the boundary conditions.

6. comments.... it seems after Talylor expansion we have

$$\tilde{A}u = u(x_1, x_2, x_3) - \frac{h^2}{2}\Delta\, u(x_1, x_2, x_3) + O(h^4). \quad [5]$$

This implies that

$$\frac{2}{h^2}\left(\tilde{A}u - u(x_1, x_2, x_3)\right) = -\Delta\, u(x_1, x_2, x_3) + O(h^2). \quad [6]$$

So, thanks to the classical idea of finite difference (consistency and approximation), the order of scheme [4]!!!

7. convergence order of the scheme... its proved that the convergence order is $h^2 |\log h|$

References

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