A brief Report on the article [CHO 10]“Numerical solution for space fractional dispersion equations with nonlinear source terms ”

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Abstract: The aim of the article is to provide a finite difference scheme for the fractional differential dispersion equation with a nonlinear forcing term. Thanks to a right shifted Grünwald formula (Note that unshifted Grünwald formula for the fractional derivative of Riemann-Liouville type is unstable regardless a finite difference scheme is either explicit or implicit.) , the authors present an implicit Euler approximation for the problem under consideration. Existence of the finite difference approximate solution is proved thanks to the use of the Leray-Schauder fixed point theorem. To prove the convergence of the approximate solution, the authors first justified the sability of the finite difference scheme and then they combined this stability together with consistency of the finite difference scheme. as usual, the stated stability result can be used to prove the uniqueness of the approximate finite difference solution. It is proved that the convergence order is $h + k$ in a discrete L^2-norm, where h (resp. k) is the mesh size of space (resp. time) discretization. Some numerical experiments are presented.

Key words and phrases: fractional differential dispersion equation; nonlinear forcing term; finite difference scheme; one dimensional spatial domain; Euler approximation; right shifted Grünwald formula: stability; consistency; convergence order

Subject Classification : 65M06; 65M12; 65M15

1 What i have learned from the article

1. some overwiew: even the ideas of the article are simple and treat the case of spatial one dimensional domain but they are they are interesting. Maybe, the results can presented in general framework in which the problem is multidimensional.

2. utility: The fractional order diffusion equations have been discussed by many authors as generalizations of classical diffusions equation in order to treat sub and super–diffusive process. These equations appear for instance in *applications fluid flows*, *finance*, *biological sciences*, see [BAE 98, MEE 06].

3. what about linear case: [MEE 06] provides us with a finite difference scheme in the linear case [1], i.e., $f(u) := f(x,t)$

4. problem to be approximated: the following problem is considered

$$u_t(x,t) = \frac{\partial u^\alpha}{\partial x^\alpha}(x,t) + f(x,t,u), \ (x,t) \in (0,1) \times (0,T), \tag{1}$$

   where $T > 0$ is given and $1 < \alpha < 2$.

   The following Dirichlet boundary conditions are assumed:

$$u(0,t) = u(1,t) = 0, \ \forall t \in (0,T). \tag{2}$$

   Initial condition is

$$u(x,0) = u_0(x), \ \forall x \in (0,1). \tag{3}$$

5. what is fractional derivative? for arbitrary non integer β, the fractional derivative is defined by

$$\frac{\partial \varphi^\beta}{\partial x^\beta}(x,t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_0^x \frac{\varphi(\xi)}{(x-\xi)^{\beta+1-n}}d\xi, \tag{4}$$

   where n is the integer satisfying $n-1 < \beta < n$ and Γ is the function defined by

$$\Gamma(\beta) = \int_0^\infty \exp(-x)x^{\beta-1}dx. \tag{5}$$

6. some literature: existence and uniqueness of the solution of [1]–[3] is proved for instance using semi group theory when f is globaly Lipschitz continuous, see for instance [BAE 98].

7. right shifted Grünwald formula: in order to perform an approximation for equation [1], we have to approximate the fractional derivative given by [4]. The followingright shifted Grünwald formula yields an approximation for fractional derivative given by [4] by order h:

$$\frac{\partial^\alpha u}{\partial x^\alpha}(x,t) = \frac{1}{\Gamma(-\alpha)}\lim_{N\to+\infty}\frac{1}{h^\alpha}\sum_{k=0}^{N}\frac{\Gamma(k-\alpha)}{\Gamma(k+1)}u(x-(k-1)h,t) + O(h). \tag{6}$$

   It is useful to write the previous approximation in the following weight form

$$\frac{\partial^\alpha u}{\partial x^\alpha}(x,t) = \lim_{N\to+\infty} g_{\alpha,k}u(x-(k-1)h,t) + O(h), \tag{7}$$

   where

$$g_{\alpha,k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}. \tag{8}$$

2 Finite difference scheme

the finite difference suggested in [CHO 10] is

$$\frac{u_i^n - u_i^{n-1}}{k} = \frac{1}{h^\alpha}\sum_{k=0}^{i+1} g_{\alpha,k}u_{i-k+1}^{n+1} + f(u_i^{n+1}). \tag{9}$$

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