A brief Report on the article [BOU 10] "Theoretical analysis of the upwind finite volume scheme on the counter-example of Peterson "

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Last update: Thursday 14th February, 2010; my hope I come back to this article to learn again

Abstract:The authors consider upwind finite volume method to approximate an advection problem with a constant velocity. By numerous methods, geometric paths counting, recursion, Fourier analysis, generating function, matrix computation, the authors established closed form expressions of an upper bound of the geometric corrector, which has been introduced by the authors some years ago, for the initial (square) Peterson and related (semi infinite, triangular) domains for oblique advection velocity. The most convenient expression appears to be a weighted sum of binomial coefficients. As a result, an explicit upper bound for the geometric corrector propotional to h and θ is proved, where θ is the angle of the advection velocity with the vertical. Therefore, the L^∞ norm of the corrector is of order h for a non vertical advection direction.
As a consequence, the upwind scheme on initial (square) Peterson mesh and on related (triangular, semi–infinite) meshes is therefore of order h for a non vertical advection direction.

Key words and phrases: advection problem with a constant velocity; theoretical analysis; counter-example of Peterson; upwind finite volume; geometric corrector
Subject Classification : 65M08;

1 Motivation

Let us consider the following problem:

$$u_t(x,t) + \nabla.(\mathbf{a}\,u)(x,t) = 0,\ (x,t) \in \Omega \times (0,T) \qquad [1]$$

where $\Omega \subset \mathbb{R}^2$ and $\mathbf{a}$ is a velocity in $\mathbb{R}^2$. This problem is completed with initial and boundary data that satisfy the so called "compatibility conditions.

The space discretization is performed using unstructured mesh composed of control volumes K_j. The surface of K_j is denoted by V_j.

The upwind finite volume scheme is expected to approximate the mean values of u defined by

$$v_j^n = \frac{1}{V_j}\int_{K_j} u(x,t_n)dx, \qquad [2]$$

where $t_n = nk$ with k is the time step discretization.

If the time discretization is for instance the Euler forward scheme, then the unknowns u_j^n which approximates v_j^n satisfy

$$\mathcal{L}(u_j) = u_j^{n+1} - u_j + \frac{k}{V_j} \sum_{k \in \mathcal{N}^-(j)} \mathbf{a} \cdot \mathbf{n}_{jk}(u_k^n - u_j^n) = 0, \qquad [3]$$

where jk is the normal to the interface $K_i \cap K_j$ that points out from K_k to K_k, with a norm equal to the length of the interface $|K_i \cap K_j|$, and $\mathcal{N}^-(j)$ is the set of indices of adjacent volumes to K_j such that $\mathbf{a}$ is inward to K_j, on the common interface, i.e.

$$\mathcal{N}^-(j) = \{k, K_k \text{ adjacentto} K_j \text{ and } \mathbf{a} \cdot\ n_{jk} < 0\}. \qquad [4]$$

It turns out that the above scheme is not consistent in the sense of finite differences: the truncation error $\mathcal{L} - v_j)$, which is obtained by substituting u_j^n by the exact value v_j^n, does not converge to zero as h goes to zero. So, the Lax theorem is helpless. However, numerous theoretical results, see [EYM 00] and references therein, obtained with solutions more or less regular. Specifically, the order of convergence is at least $\frac{1}{2}$ in $\mathbb{L}^2$-norm for H^1 data, see [DES 04] or in $\mathbb{L}^p$ norm for ^{1,p} and BV data, [MER 07] .

In [BOU 05], the concept of geometric corrector is introduced in order to analyse the convergence of the scheme from the the mathematical point of view. This corrector, for which the existence and uniqueness have been proved, depends only the mesh and on the advection velocity $\mathbf{a}$.

2 Definition of the geometric corrector

Let K_j be a non boundary control volume. The corrector $\mathcal{C}_j$ is a point in $\mathbb{R}^2$ which satisfies the following system (for general case see [BOU 05])

$$\sum_{k \in \mathcal{N}^+(j)} \mathbf{a} \cdot \mathbf{n}_{jk}\mathcal{C}_j + \sum_{k \in \mathcal{N}^-(j)} \mathbf{a} \cdot \mathbf{n}_{jk}\mathcal{C}_j = \sum_{k \in \mathcal{N}^+(j)} \mathbf{a} \cdot \mathbf{n}_{jk}(g_{jk} - g_j) + \sum_{k \in \mathcal{N}^-(j)} \mathbf{a} \cdot \mathbf{n}_{jk}(g_{jk} - g_j), \qquad [5]$$

where

- g_j denotes a point inside the volume like for instance the center of gravity of the volume K_j;
- g_{jk} denotes the center of gravity of $K_i \cap K_j$
- $\mathcal{N}^+(j)$ is given by

$$\mathcal{N}^+(j) = \{k, K_k \text{ adjacentto} K_j \text{ and } \mathbf{a} \cdot\ n_{jk} > 0\}. \qquad [6]$$

3 Why the geometric corrector is useful?

For $0 < p \leq 1$, it is proved in [BOU 05] that, when the solution is regular enough, a h^p behavior of the norm of the corrector is of order p then the scheme is of order h^p.

As a result, estimating the corrector is an efficient tool to study the convergence of the scheme.

4 Main result

The following theorem is proved in [BOU 10]:

THEOREM 4.1 For a a non vertical advection direction, the following estimate holds:

$$\max_j |\mathcal{C}_j| \leq Ch. \qquad [7]$$

5 A summary of the main ideas

By numerous methods, geometric paths counting, recursion, Fourier analysis, generating function, matrix computation, the authors established closed form expressions of an upper bound of the geometric corrector for the initial (square) Peterson and related (semi infinite, triangular) domains for oblique advection velocity. The most convenient expression appears to be a weighted sum of binomial coefficients. As a result, an explicit upper bound for the geometric corrector propotional to h and θ is proved, where θ is the angle of the advection velocity with the vertical. Therefore, the L^∞ norm of the corrector is of order h for a non vertical advection direction .

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