A brief Report on the article [CHA 11]“A robust grid equidistribution method for a one-dimensional singularly perturbed semilinear reaction-diffusion problem ”

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Abstract: The authors consider a one dimensional second order semilinear singularly perturbed equation. The diffusion parameter is denoted by ε^2. On an arbitrary mesh and thanks to a posteriori estimate, it remarked that it is possible to obtain second order accurate uniformly after a suitable choice for the mesh. To get this convenient mesh, the authors used the monitor function equidistribution. The aim of this paper is then to resolve two questions. The fisrt question is to discuss the existence of a solution to equidistribution problem, and the second question is to suggest an algorithm which yields second order accurate uniformly w.r.t. the singular parameter for the discrete solution.

It is first established the existence of a solution to the equidistribution problem. This is done in a framework which can be applied to a more general equidistribution problem. An algorithm is suggested which yields second order accurate uniformly, when the equation is linear and under further mild assumptions, after $O(|\ln\varepsilon|/\ln N)$ iterations, where $N+1$ is the number of mesh points.

Key words and phrases: one dimension; second order semilinear singularly perturbed equation; equidistribution method; second order accurate

Subject Classification :65L10

1 An overview on some standard numerical methods for singularly perturbed equations

Let us consider the following simple example (which is given in [FEI 04, Pages 342–344]):

$$-\varepsilon^2 u''(x) + \nu\, u'(x) = 1,\ x \in \Omega = (0,1), \qquad [1]$$

with

$$u(0) = u(1) = 0, \tag{2}$$

where $\varepsilon > 0$ and $\nu \neq 0$ are two constants. The solution of [1]–[2] is defined by

$$u(x) = \frac{1}{\nu}\left\{x - \frac{\exp(\nu\, x/\varepsilon) - 1}{\exp(\nu\,/\varepsilon) - 1}\right\},\ x \in [0,1]. \tag{3}$$

If $\varepsilon \to 0$ and $\nu > 0$, then $u(x) \to x/\nu$ for $x \in [0,1)$. The limit function is the solution of

$$\nu\, u'(x) = 1,\ x \in (0,1) \text{ and } u(0) = 0. \tag{4}$$

1.1 Standard linear finite element methods and Gibs phenomenon

Let us apply the linear finite element method to approximate [1]–[2]. We then consider the uniform mesh $\mathcal{T}_h = \{K_i; i = 0, \ldots, N\}$ with $K_i = [x_i, x_{i+1}]$ and $x_{i+1} - x_i = h$. The approximate solution u_h as well as the test functions are in the space:

$$\mathcal{V}_h = \left\{\varphi_h \in \mathcal{C}(\overline{\Omega}); \varphi_h|_{K_i} \in \mathcal{P}_1,\ \forall\, K_i \in \mathcal{T}_h,\ \varphi_h(0) = \varphi_h(1) = 0\right\}. \tag{5}$$

Multiplying both sides of [1] by $\varphi \in \mathcal{C}^1(\overline{\Omega})$, with $\varphi(0) = \varphi(1) = 0$, we get

$$\int_0^1 (\varepsilon\, u'(x)\varphi'(x) + \nu\, u'(x)\varphi(x))dx = \int_0^1 \varphi(x)dx,\ \forall\varphi \in \mathcal{C}^1(\overline{\Omega}),\ \varphi(0) = \varphi(1) = 0. \tag{6}$$

The approximate finite element solution is then defined by: find $u_h \mathcal{V}_h$ such that

$$\int_0^1 (\varepsilon\, u_h'(x)\varphi_h'(x) + \nu\, u_h'(x)\varphi_h(x))dx = \int_0^1 \varphi_h(x)dx,\ \forall\varphi \in \mathcal{V}_h. \tag{7}$$

This is equivalent to the following linear system:

$$-\frac{\varepsilon}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{\nu}{2h}(u_{i+1} - u_{i-1}) = 1, \tag{8}$$

with

$$u_0 = u_{N+1} = 0. \tag{9}$$

Note that [8] is also a finite difference scheme by approximating:

1. $u''(x_i)$ by
$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}. \tag{10}$$
2. $u'(x_i)$ by the central quotient
$$\frac{u(x_{i+1}) - u(x_{i-1})}{2h}. \tag{11}$$

The linear equations [8]–[9] can be written as

$$\mathcal{A}\, U = 1, \tag{12}$$

where $\mathcal{A}$ is a $N \times N$ matrix and 1 in the r.h.s. of [12] is the vector of $\mathbb{R}^N$ whose components are equal to 1. The matrix $\mathcal{A}$ is nonsymmetric, but for $h < 2\varepsilon/|\nu|$ it is *diagonally dominant* (In the sense $\sum_{j=1, j\neq i}^{N} |\, a_{ij}| \leq |\, a_{ii}|$ for all $i = 1, \ldots, N$ with strict inequality for at least one i.) and

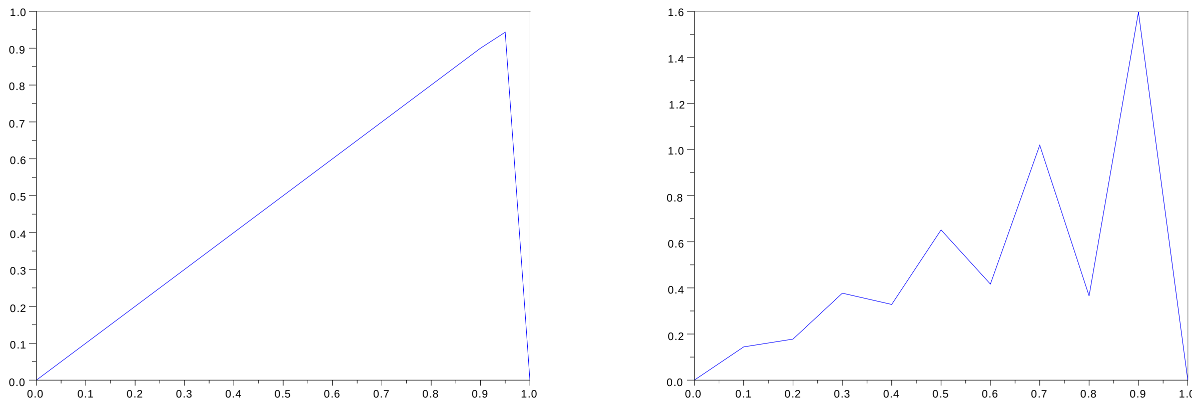

Figure 1: Left graph of the function u, right the finite difference solution [8]–[9] with $h = 1/10$; both simulations are in $\varepsilon = 10^{-2}$ and $\nu = 1$

irreducibly. These previous stated properties of $\mathcal{A}$ guarantees good properties of the approximate solution. But, for $h \geq 2\varepsilon/|\nu|$, the approximate solution do not make sense because of *spurious oscillations* shown in the right Figure 1. This means that the *Gibbs phenomenon* arises here. We see here that *mesh Péclet number* defined by

$$Pe = \frac{h\nu}{2\varepsilon} \qquad [13]$$

must satisfy the condition

$$Pe < 1 \qquad [14]$$

in order to avoid the *Gibbs phenomenon*.

1.2 Gibbs phenomenon and upwind finite difference scheme

An issue to avoid the Gibbs phenomenon is to chose an upwind finite difference scheme instead of the central finite difference scheme [8]–[9], that is to chose

$$-\frac{\varepsilon}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{\nu}{h}(u_i - u_{i-1}) = 1, \qquad [15]$$

with

$$u_0 = u_{N+1} = 0. \qquad [16]$$

So, the upwind scheme [15]–[16] is performed thanks to

1. the approximation of $u''(x_i)$ by

$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}. \qquad [17]$$

2. the approximation of $u'(x_i)$ by the central quotient

$$\frac{u(x_i) - u(x_{i-1})}{h}. \quad [18]$$

Figure 2 represents, using Scilab, the finite difference solution [15]–[16] with $h = 1/10$. As we can see that the finite difference solution [15]–[16] is more reasonable than that of [8]–[9].

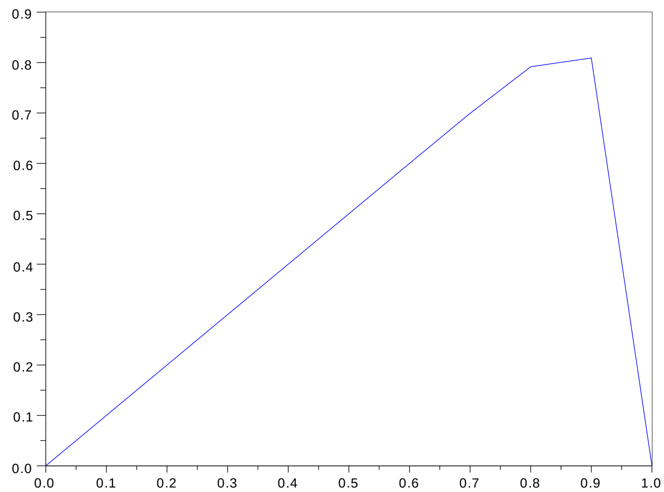

Figure 2: The upwind finite difference solution [15]–[16] with $h = 1/10; \varepsilon = 10^{-2}$ and $\nu = 1$

1.3 Standard linear finite element methods and convergence order

As usual, we use the technique of Cea Lemma to compute the convergence order. The key, for that target is the following equality:

$$a(u - u_h, v_h) = 0, \ \forall\, v_h \in \mathcal{V}_h. \quad [19]$$

This implies that

$$a(u - u_h, u - u_h) = a(u - u_h, u - \pi\, u), \quad [20]$$

where π is the usual interpolation operator defined from $\mathcal{C}(\overline{\Omega})$ into $\mathcal{V}_h$ and

$$a(u, v) = \int_0^1 (\varepsilon\, u'(x) v'(x) + \nu\, u'(x) v(x)) dx. \quad [21]$$

Equality [20] implies that

$$\alpha \parallel u - u_h \|_{1,\Omega}^2 \leq \ M \parallel u - u_h \|_{1,\Omega} \| \ u - \pi\, u \|_{1,\Omega}, \quad [22]$$

which implies in turn, using the known result of the interpolation error

$$\alpha \parallel u - u_h \|_{1,\Omega} \leq \ CMh \max_{x \in \overline{\Omega}} | \, u''(x)|, \quad [23]$$

where C is only depending on Ω, the constants α and M used in [22] are defined by

$$a(v,v) \geq \alpha \| v\|^2_{1,\Omega}, \ \forall\, v \in H^1_0(\Omega), \tag{24}$$

and

$$a(u,v) \leq M\| u\|_{1,\Omega}\| v\|_{1,\Omega}, \ \forall\, u, v \in H^1_0(\Omega). \tag{25}$$

Let us compute M, α, and $\max_{x\in\overline{\Omega}} |\, u''(x)|$

1. *computation of* α: using [21], the fact that $v(0) = v(1) = 0$, and the Poincaré inequality

$$\begin{aligned} a(v,v) &= \varepsilon \int_0^1 (v')^2(x)dx + \nu \int_0^1 v'(x)v(x))dx \\ &= \varepsilon \int_0^1 (v')^2(x)dx + \nu \int_0^1 (v^2)'(x)dx \\ &= \varepsilon \int_0^1 (v')^2(x)dx \\ &\geq \varepsilon\, C(\Omega)\| v\|^2_{1,\Omega}, \end{aligned} \tag{26}$$

where $C(\Omega)$ is only depending on Ω.

2. *computation of* M: using the Cauchy Schwarz inequality, the fact that $\left(\int_0^1 (u')^2(x)dx\right)^{\frac{1}{2}} \leq \| v\|_{1,\Omega}$ to get, assuming that $\varepsilon << 1$

$$\begin{aligned} a(u,v) &= \int_0^1 (\varepsilon\, u'(x)v'(x) + \nu\, u'(x)v(x))dx \\ &\leq \varepsilon\| u\|_{1,\Omega}\| v\|_{1,\Omega} + \nu\| u\|_{1,\Omega}\| v\|_{L^2(\Omega)} \\ &\leq (\varepsilon + \nu)\| u\|_{1,\Omega}\| v\|_{1,\Omega} \\ &\leq (1 + \nu)\| u\|_{1,\Omega}\| v\|_{1,\Omega}. \end{aligned} \tag{27}$$

3. *computation of* $u''(x)$: using expression [3] to get

$$u''(x) = -\frac{\nu}{\varepsilon^2}\frac{\exp(\nu\, x/\varepsilon)}{\exp(\nu\,/\varepsilon) - 1}, \ \forall\, x \in (0,1). \tag{28}$$

So, an estimate for u'' can be provided as, since $0 < \exp(\nu\, x/\varepsilon) \leq \exp(\nu\,/\varepsilon)$:

$$\max_{x\in[0,1]} |\, u''(x)| \leq \frac{\nu}{\varepsilon^2\,(\exp\,(\nu\,/\varepsilon) - 1)}. \tag{29}$$

Gathering [26]–[29] with [23] to get

$$\begin{aligned} \| u - u_h\|_{1,\Omega} &\leq C(\Omega)\frac{M}{\alpha} h \max_{x\in\overline{\Omega}} |\, u''(x)| \\ &\leq C(\Omega)\frac{\nu(1+\nu)}{\varepsilon^3\,(\exp\,(\nu\,/\varepsilon) - 1)} h. \end{aligned} \tag{30}$$

So, estimate [30] depends adversly on ε which is not so good when ε is small.

2 Aim of the article and description for the main results

The work [CHA 11] is interested with the following singularly perturbed reaction–diffusion equation:

$$\mathcal{L}\, u(x) = -\varepsilon^2 u(x) + b(x, u(x)) = 0, \; x \in D = (0,1), \qquad [31]$$

with the following Dirichlet boundary condition:

$$u(0) = u(1) = 0, \qquad [32]$$

where ε is a *small* parameter and b is sufficiently smooth such that, for some two constants γ_0 and $\bar{\gamma}$

$$0 < \gamma_0^2 \leq b_s(x,s) \leq \bar{\gamma}^2, \; \forall\, (x,s) \in [0,1] \times \mathbb{R}. \qquad [33]$$

Under the previous stated assumption on b, problem [31]–[32] has a unique solution which exhibits sharp boundary layers of width $O(\varepsilon \ln(1/\varepsilon))$ (maybe this result can be found in [ROO 08])
Consider an arbitrary mesh $\{x_i\}_{i=0}^N$ with $0 = x_0 < x_1 < \ldots < x_N = 1$, and we define the local mesh size $h_i := x_i - x_{i-1}$, for $i = 1, \ldots, N$.
The finite difference scheme is defined by

$$\mathcal{L}^N\, u_i^N = -\varepsilon^2 \delta^2 u_i^N + b(x_i, u_i^N) = 0, \; i \in [\![1, N-1]\!], \qquad [34]$$

with

$$u_0^N = u_N^N = 0, \qquad [35]$$

where

$$\delta^2 u_i^N = \frac{2}{h_i + h_{i+1}} \left(\frac{u_{i+1}^N - u_i^N}{h_{i+1}} - \frac{u_i - u_{i-1}^N}{h_i} \right). \qquad [36]$$

There exists a unique solution for [34]–[35].

2.1 Some interesting estimates and questions to be answered

A first key in the article is the following estimate given [KOP1 07]:

$$\max_{x \in [0,1]} |u^N(x) - u(x)| \leq \bar{C} \max\{\bar{M}_i^N h_i\}^2, \qquad [37]$$

where

$$\bar{M}_i^N = \min\{|\delta^2 u_{i-1}^N|, |\delta^2 u_i^N|\}^{\frac{1}{2}} + \left(|\delta^3 u_i^N|\right)^{\frac{1}{2}} + 1, \qquad [38]$$

which uses the discrete third derivative $\delta^3 u_i^N = \dfrac{\delta^2 u_i - \delta^2 u_{i-1}^N}{h_i}$.
In [KOP2 07], it is proved that

$$\bar{C} \max\{\bar{M}_i^N h_i\} \leq C \max\{\bar{\bar{M}}_i^N h_i\}, \qquad [39]$$

where

$$\bar{\bar{M}}_i^N = \min\{|\delta^2 u_{i-1}^N|, |\delta^2 u_i^N|\}^{\frac{1}{2}} + 1, \qquad [40]$$

So the third discrete derivative which appears in [38] is unecessary.

Thanks to [38], [39], and [40], we look then for mesh for mesh $\{x_i\}$ and a computational solution such that $\bar{\bar{M}}_i^N h_i \leq CN^{-1}$ and therefore the order of the computational solution is uniformy two.

To look for a mesh $\{x_i\}$ such that $\bar{\bar{M}}_i^N h_i \leq CN^{-1}$, the authors used the so called monitor function equidistribution:

Definition 2.1 (Monitor function equidistribution) Let N be a given integer. A mesh $\{x_i\}_{i=0}^N$ is said to equidistribute a monitor function $M(x) > 0$ if

$$\int_{x_i}^{x_{i+1}} M(x)dx = \frac{1}{N}\int_0^1 M(x)dx, \ \forall i \in [\![1, N]\!]. \qquad [41]$$

The *a posteriori* error estimate [37] with [40] suggest to consider $M(x) = M_i^N$, for all $x \in [x_i, x_{i+1}]$ with $M_i^N = \bar{\bar{M}}_i^N$, for all $x \in [x_i, x_{i+1}]$. So, in this case, [41] is equivalent to

$$M_i^N h_i = \frac{1}{N}\sum_{j=1}^{N} M_j^N h_j, \ \forall i \in [\![1, N]\!]. \qquad [42]$$

Note that a mesh $\{x_i\}_{i=0}^N$ implies $\{u_i^N\}_{i=0}^N$ given by [34]–[35]. This suggests the following problem:

Equidistribution problem: Find $\{x_i\}_{i=0}^N$ and $\{u_i^N\}_{i=0}^N$, with $\{u_i^N\}_{i=0}^N$ is computed from $\{x_i\}_{i=0}^N$ thanks to [34]–[35], such that [42] holds.

It is useful to note also that $\{x_i\}_{i=0}^N$ and $\{u_i^N\}_{i=0}^N$ are *a priori* uknown. Consequently, even [31] is linear, the equidistribution problem [42], which requires the simultaneous solution for [34] and [42], is nonlinear. The following questions to be answered in the article [CHA 11]:

1. Does the equidistribution problem have a solution?
2. Is there an algorithm to solve the stated equidistribution problem which yields to second order accurate?

References

[CHA 11] Chadha, Naresh M.; Kopteva, Natalia: A robust grid equidistribution method for a one-dimensional singularly perturbed semilinear reaction-diffusion problem. *IMA J. Numer. Anal.* , **31**, No. 1, 188–211 (2011).

[FEI 04] M. Feistauer, J. Felcman and I. Straskraba: Mathematical and computational methods for compressible flow. *Oxford Science Publications* , 2004.

[KOP1 07] N. Kopteva: Convergence theory of moving grid methods. *Adaptive Computations: Theory and Algorithms (T. Tang and J. Xu edition. Beijing: Science Press*, 159–210, 2007.

[KOP2 07] N. Kopteva: Maximum norm a posteriori error estimates for a one dimensional singularly perturbed semilinear reaction–diffusion problem. *IMA J. Numer. Anal.*, **27**, 576–592, 2007.

[KOP 08] N. Kopteva: Maximum norm a posteriori error estimates for a 2d singularly perturbed semilinear reaction–diffusion problem. *IMA J. Numer. Anal.*, **46**, 1602–1618, 2008.

[KOP 05] N. Kopteva, Madden, N., Stynes, M.: Grid equidistribution for reaction–diffusion problems in one dimension. *Numer. Algorithms*, **40**, 305–322, 2005.

[KOP 01] N. Kopteva, M. Stynes: A robust adaptive method for a quaslinear one–dimensional convection–diffusion problem. *IMA J. Numer. Anal.*, **39**, 1446–1467, 2001.

[MIL 96] Miller, J. J. H., O'Riodan, E., Shishkin: Fitted Numerical Methods for Singular Perturbation Problems. *World Scientific, Singapore*, 1996.

[ROO 08] Roos, M.G., Stynes, M., and Tobiska, L.: Robust Numerical Methods for Singularly Perturbed Differential Equations. *Second Eddition, Springer, Berlin*, 2008.

[SMA 74] Smart, D.R.: Fixed Point Theorems. *Cambridge: Cambridge University Press*, 1974.