A brief Report on the article "High order finite volume methods for singular perturbation problems"

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1 To be read!!

In the same context of the present article, that is the use of piecewise high order polynomes for singularly perturbed equations, there is already article appeared in 2006:
Liu S., Xu Y. Galerkin methods based on Hermite splines dor singular perturbation problems. *SIAM J Numer Anal*, **43** 2607–2623, 2006.

2 Equation to be solved

It is considered the following fourth order linear operator:

$$(\mathcal{L}_\varepsilon u)(x) = \varepsilon^2 u^{(4)}(x) - \big(p(x)u'(x)\big)' + q(x)u'(x) + r(x)u(x) \qquad [1]$$

and the following boundary value problem of fourth–order

$$\left(\mathcal{L}_\varepsilon u\right)(x) = f(x),\ x \in (0,1), \qquad [2]$$

$$u^{(j)}(0) = u^{(j)}(1),\ j \in \{0,1\}. \qquad [3]$$

Setting the following notations:

- $$a(u,v) = (u'', v''), \qquad [4]$$

- $$b(u,v) = \Big(-\big(p(x)u'\big)' + qu' + ru, v\Big), \qquad [5]$$

- $$A_\varepsilon(u,v) = \varepsilon^2 a(u,v) + b(u,v). \qquad [6]$$

The variational problem for [2]–[3] is: Find $u \in \mathcal{V} = H_0^2(I)$ such that

$$A_\varepsilon(u,v) = (f,v), \ \forall v \in \mathcal{V}. \tag{7}$$

Of course, we should assume some conditions on the data p, q, f, r in order that [7] admits a unique solution, namely

$$p \in \mathcal{W}^{1,\infty}(I), \ q, r \in L^\infty(I). \tag{8}$$

$$p(x) \geq p_{\text{min}} > 0, \ \text{a. e. } x \in I. \tag{9}$$

Remark 1

- It useful to remark that $u \in \mathcal{V} = H_0^2(I)$ yields that $u(0) = u(1) = 0$ and $u'(0) = u'(1) = 0$ which give the boundary conditions [3].
- Was useful if the authors mentioned how to justify the ellpiticity of the operator $A_\varepsilon(\cdot,\cdot)$ given by [6], and then using Lax–Milgram,we prove the existence and uniqueness of the solution $u \in H_0^2(I)$ of problem [7]. Indeed:

$$\begin{aligned} A_\varepsilon(u,u) &= \varepsilon^2 a(u,u) + b(u,u) \\ &= \varepsilon^2 (u'', u'') + b(u,u) \\ &= \varepsilon^2 \int_I (u'')^2 (x) dx + b(u,u) \end{aligned} \tag{10}$$

Let us now compute the second term in the right hand side of [10]

$$\begin{aligned} b(u,u) &= \left(-(pu')', u\right) + (qu', u) + (ru, u) \\ &= \left(p(x)u', u'\right) + (qu', u) + (ru, u) \\ &= \int_I p(x) (u')^2 (x) dx + \int_I q(x) u'(x) u(x) dx + \int_I r(x) u^2(x) dx \\ &= \int_I p(x) (u')^2 (x) dx + \frac{1}{2} \int_I q(x) \left(u^2\right)' (x) dx + \int_I r(x) u^2(x) dx \\ &= \int_I p(x) (u')^2 (x) dx + \int_I \left(r(x) - \frac{1}{2} q(x)'\right) u^2(x) dx \\ &\geq p_{\min} \int_I (u')^2 (x) dx + \int_I \left(r(x) - \frac{1}{2} q(x)'\right) u^2(x) dx \end{aligned} \tag{11}$$

This last inequality with [11] yields

$$A_\varepsilon(u,u) \geq \varepsilon^2 \int_I (u'')^2 (x) dx + p_{\min} \int_I (u')^2 (x) dx + \int_I \left(r(x) - \frac{1}{2} q(x)'\right) u^2(x) dx. \tag{12}$$

Perhaps there is other condition on q and r, e.g.,

$$q \in W^{1,\infty}(I) \text{ and } r(x) - \frac{1}{2} q'(x) \geq 0, \ \text{for a. e. } x \in I. \tag{13}$$

The bilinear form $b(\cdot,\cdot)$ defined by [5] could be also defined by

$$b(u,v) = \left(p(x)u', v'\right) + \left(qu' + ru, v\right). \tag{14}$$

3 Finite volume approximation for [7]

To define a finite volume approximation for [7], we have to define two spaces. The first space is the so called the trial space in which the finite volume approximate solution (is expected to approximate the solution u of [7]). The second space is the so called the test space which approximates the space $\mathcal{V}$.

3.1 Trial space

Let us consider a one dimensional finite volume mesh $\mathcal{T}_N$ given by a set of points $0 = x_0 < x_1 < \ldots < x_N$. For any $i \in \mathbb{Z}_N = \{1, 2, \ldots, N\}$, we set $I_i = (x_{i-1}, x_i)$ and $h_i = x_i - x_{i-1}$.
The trial space $\mathcal{U}_N$ is defined as the Hermite cubic element space with respect to Tt_N, namely a function u_N belongs to the space $\mathcal{U}_N$ means that

1. $u_N \in \mathcal{C}^1(I)$ and $u_N^{(j)}(0) = u_N^{(j)}(1)$ for all $j \in \{0, 1\}$
2. $u_N|_{I_i} \in \mathcal{P}_3(I_i)$, for all $i \in \mathbb{Z}_N$.

A basis for $\mathcal{U}_N$ could be defined as:

$$\varphi_{i,0} = \begin{cases} (1 - h_i^{-1}(x_i - x))^2(2h_i^{-1}(x_i - x) + 1), \ x \in [x_{i-1}, x_i], \\ (1 - h_{i+1}^{-1}(x - x_i))^2(2h_{i+1}^{-1}(x - x_i) + 1), \ x \notin (x_i, x_{i+1}], \\ 0, \ x \notin [x_{i-1}, x_{i+1}], \end{cases} \quad [15]$$

$$\varphi_{i,1} = \begin{cases} (x - x_i)(h_i^{-1}(x_i - x) - 1)^2, \ x \in [x_{i-1}, x_i], \\ (x - x_i)(h_{i+1}^{-1}(x - x_i) - 1)^2, \ x \notin (x_i, x_{i+1}], \\ 0, \ x \notin [x_{i-1}, x_{i+1}]. \end{cases} \quad [16]$$

The above basis allows us to write the functions belonging to $\mathcal{U}_N$ as follows, for all $u_N \in \mathcal{U}_N$ (Recall that $\mathbb{Z}_{N-1} = \{1, 2, \ldots, N-1\}$ and that $u(0) = u'(0) = u(1) = u'(1) = 0$.):

$$u_N(x) = \sum_{\mathbb{Z}_{N-1}} \left(u_N(x_i)\varphi_{i,0}(x) + u_N'(x_i)\varphi_{i,1}(x) \right). \quad [17]$$

As we can see that $\dim \mathcal{U}_N = 2(N-1)$.

3.2 Test space

The dual mesh $\mathcal{T}_N^\star$ of $\mathcal{T}_N$ is given by $0 = x_0 < x_{\frac{1}{2}} < \ldots < x_{N-1/2} < x_N = 1$, where $x_{i-1/2} = (x_{i-1} + x_i)/2$, for all $i \in \mathbb{Z}_N$. The trial space $\mathcal{V}_N$ is the piecewise linear polynomial space w.r.t. the mesh $\mathcal{T}_N^\star$.
The basis of $\mathcal{T}_N^\star$ is defined by, for all $j \in \mathbb{Z}_{N-1}$:

$$\psi_{i,0} = \begin{cases} 1, \ x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \\ 0, \ x \notin [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \end{cases} \quad [18]$$

$$\psi_{i,1} = \begin{cases} x - x_i, \ x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \\ 0, \ x \notin [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]. \end{cases} \quad [19]$$

Since the bilinear form $a(\cdot,\cdot)$ is not defined on $\mathcal{U}_N \times \mathcal{V}_N$, it is convenient to find an approximate bilinear form $\tilde{a}(\cdot,\cdot)$ for $a(\cdot,\cdot)$ and defined on $\mathcal{U}_N \times \mathcal{V}_N$. Since $\{\psi_{i,0}, \psi_{i,1},\ i \in \mathbb{Z}_{N-1}\}$ spanned $\mathcal{V}_N$, it suffices then to define $\tilde{a}(u, \psi_{i,0})$ and $\tilde{a}(u, \psi_{i,1})$ for all $i \in \mathbb{Z}_{N-1}$. To define these last quantities, we assume that u is in $H^4(I)$, and then we set $\tilde{a}(u,\psi_{i,0}) = a(u,\psi_{i,0})$ and $\tilde{a}(u,\psi_{i,1}) = a(u,\psi_{i,1})$

$$\begin{aligned}
\tilde{a}(u,\psi_{i,0}) &= a(u,\psi_{i,0}) \\
&= \int_0^1 u^{(4)}(x)\psi_{i,0}(x)dx \\
&= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^{(4)}(x)dx \\
&= u^{(3)}(x_{i+\frac{1}{2}}) - u^{(3)}(x_{i-\frac{1}{2}}).
\end{aligned} \qquad [20]$$

and, using an integration by parts

$$\begin{aligned}
\tilde{a}(u,\psi_{i,1}) &= a(u,\psi_{i,1}) \\
&= \int_0^1 u^{(4)}(x)\psi_{i,1}(x)dx \\
&= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} xu^{(4)}(x)dx - x_i \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^{(4)}(x)dx \\
&= x_{i+\frac{1}{2}}u^{(3)}(x_{i+\frac{1}{2}}) - x_{i-\frac{1}{2}}u^{(3)}(x_{i-\frac{1}{2}}) - u^{(2)}(x_{i+\frac{1}{2}}) + u^{(2)}(x_{i-\frac{1}{2}}) \\
&- x_i u^{(3)}(x_{i+\frac{1}{2}}) + x_i u^{(3)}(x_{i-\frac{1}{2}}) \\
&= \frac{h_{i+1}}{2}u^{(3)}(x_{i+\frac{1}{2}}) - \frac{h_i}{2}u^{(3)}(x_{i-\frac{1}{2}}) + u^{(2)}(x_{i-\frac{1}{2}}) - u^{(2)}(x_{i+\frac{1}{2}}),
\end{aligned} \qquad [21]$$

Since $x_{i+\frac{1}{2}} \in [x_i, x_{i+1}]$ and $u|_{[x_i,x_{i+1}]} \in \mathcal{P}_3$, for all $u \in \mathcal{U}_N$, then the previous expressions [20]-[21] are well defined for all $u \in \mathcal{U}_N$.

From the computations [20]-[21], we deduce that

$$\tilde{a}(u,v) = (u^{(4)}, v),\ \forall (u,v) \in H^4(I) \times \mathcal{V}_N. \qquad [22]$$

Let us define now the bilinear form $\tilde{A}$

$$\tilde{A}_\varepsilon(u,v) = \varepsilon^2 \tilde{a}(u,v) + b(u,v). \qquad [23]$$

The finite element finite volume solution is defined by : find $u_N \in \mathcal{U}_N$ such that

$$\tilde{A}_\varepsilon(u_N,v) = (f,v),\ \forall v \in \mathcal{V}_N. \qquad [24]$$

4 Optimal mesh

The optimal mesh is contructed thanks to a known expression for the exact solution u of [2]–[3]. Thanks to a known result, the solution of [2]–[3] can be expressed as

$$u = \mathcal{E} + \mathcal{F} + \mathcal{G}, \qquad [25]$$

where $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ are smooth functions satisfy: for all $j = 0, 1, \ldots$, there exists some two constants independent of the singular parameter ε such that

$$|\mathcal{E}^{(j)}(x)| \le c,\ |\mathcal{F}(x)| \le c\varepsilon^{1-j} e^{-\alpha x/\varepsilon},\ |\mathcal{G}(x)| \le c\varepsilon^{1-j} e^{-\alpha(1-x)/\varepsilon}. \qquad [26]$$

It is introduced the following generating function $h^0(x) = \frac{\varepsilon}{N} e^{\frac{\alpha x}{4\varepsilon}}$, and the primal mesh is controled using this generating function as follows:

$$\begin{aligned} h_i &\le \min\left\{ h^0(x_{i-1}), h^0(1-x_i), \frac{1}{N} \right\} \\ &= \min\left\{ \frac{\varepsilon}{N} e^{\frac{\alpha x_{i-1}}{4\varepsilon}}, \frac{\varepsilon}{N} e^{\frac{\alpha(1-x_i)}{4\varepsilon}}, \frac{1}{N} \right\} \end{aligned} \qquad [27]$$

5 The main result

The main result of this paper is the following Theorem

THEOREM 5.1 Let u and u_N be the solutions of [7] and [24] respectively. There exists a positive constant C independent of the mesh parameters and the singular parameter ε such

$$\left\{ \varepsilon^2 \|u - u_N\|_2^2 + \|u - u_N\|_1^2 \right\}^{\frac{1}{2}} \le CN^{-2}, \qquad [28]$$

for a sufficiently large N.

Remark 2 Estimate [28] yields a uniform estimate w.r.t. to ε in the energy norm $\| \cdot \|_1$