Some notes on the article "A robust *a posteriori* error estimate for hp–adaptive DG methods for convection–diffusion equations"

L. Zhu and D. Schötzau

IMA Journal of Numerical Analysis, 31, 971–1005 (2011)

Report done by Professor Bradji, Abdallah

Provisional home page: http://www.cmi.univ-mrs.fr/~bradji

Last update: Saturday 17th September, 2011.

Abstract: The authors consider a model for convection–diffusion equations in two dimensions. They derived a robust *a posteriori* error estimate for h_p–adaptive DG methods for the problem under consideration. The ratio of the constants in the reliability and efficiency bounds is independent of the Péclet number ε of the equation, and hence the estimate is fully robust. It is applied the estimates as an error indicator in an h_p-adaptive refinement algorithm. Numerical examples show that the indicator is effective in locating and resolving the interior and boundary layers. Once the local mesh size is of the same order as the width of the boundary or interior layer, both the energy error and the error indicator were observed to convergence exponentially.

Key words and phrases: robust a posteriori error estimate; hp–adaptive DG methods; convection–diffusion equations

Subject Classification: 65N30; 65N15

1 what i have learned...

1. boundary or internal layers: it is well known that solutions to convections–diffusion equation may have boundary or internal layers of small width where their gradients change extremely rapidly.

2. how to approximate efficently such cases: is to use adaptive finite element methods that are capable of locally refining the meshes in the vicinity of these layers

3. where we refine: the decision when to refine an element is usually based on *a posteriori* estimates of the error. For excelent surveys, as the authors advise, we refer to [VER 94, AIN 00]

4. robust a posteriori estimates: the design of robust *a posteriori* estimates has attracted a lot of attention. By robustness, we mean that estimates yield upper and lower bounds for the errors measured in suitable norms that differ by a factor independent of the *Péclet number* of the problem.

5. literature: there is already a literature concerning the a posteriori estimates in finite element and discontinuous Galerkin method. However, these methods are based on the use of a fixed, usually low polynomial degree. As a consequence, adaptive h–version methods yield at most algebraic rates of convergence. This is in contrast to h_p–version finite element methods, where the combination of h–refinement and h_p–refinement typically results in exponential rates of convergence; more details can be found in [SCH 98] and the references therein.

6. DG (discontinuous Galerkin) methods and h_p–adaptivity: DG methods are naturally suited for realizing h_p–adaptivity. Indeed, being based on discontinuous finite element spaces, these methods can easily deal with irregularly refined meshes and locally varying polynomial degrees.

7. aim...: the present article is an extension of the work [SCH 09] done by the same author. Indeed, [SCH 09] treats a posteriori estimates of the h version of DG methods.

2 problem to be solved

$$-\varepsilon\Delta \in u + a \cdot \nabla u = f(x),\ x \in \Omega, \qquad [1]$$

with

$$u(x) = 0,\ x \in \partial\Omega, \qquad [2]$$

where Ω is a bounded Lipschitz polygon in $\mathbb{R}^2$.
It is assumed that $\| a\|_{L^\infty(\Omega)}$ and the length scale of Ω are one, so that ε^{-1} is the Péclet number of the problem.

3 meshes

It is assumed that the computational domain can be partioned into shape–regular meshes $\mathcal{T} = \{K\}$ of parallelograms K.

4 headlines of the article

- definition of the scheme
- definition of a robust a posteriori estimate
- Theorem of reliability
- Theorem of efficiency

References

[AIN 00] M. Ainsworth and J. Oden: A posteriori Error Estimation in Finite Element Analysis. *Wiley–Interscience Series in Pure and Applied Mathematics, New York: Wiley*, 2000.

[HOU 06] P. Houston, D. Schötzau, and T. Wihler: An h_p–adaptive discontinuous Galerkin FEM for nearly incompressible linear elasticity. *Comput. Methods Appl. Mech. Eng.*, **195**, 3224–3246 (2006).

[HOU 07] P. Houston, D. Schötzau, and T. Wihler: Energy norm a posteriori error estimation of h_p–adaptive discontinuous Galerkin methods for elliptic problems. *Math. Models Methods Appl. Sci.*, **17**, 33–62 (2007).

[HOU 02] P. Houston, C. Schwab, and E. Süli: Discontinuous h_p–finite element methods for advection–diffusion–reaction problems, *SIAM J. Numer. Anal.*,**39**, 2133–2163 (2002).

[SCH 98] C. Schwab: p and h_p Finite Element Methods–Theory and Application to Solid and Fluid Mechanics. *Oxford: Oxford university Press*, 1998.

[SAN 08] G. Sangalli: Robust a posteriori estimator for advection–diffusion–reaction problems. *Math. Comput.*, **77**, 41–70 (2008).

[SCH 09] D. Schötzau and L. Zhu: A robust a posteriori error estimator for discontinuous Galerkin methods for convection–diffusion equations. *Appl. Numer. Math.*, **59**, 2236–2255 (2009).

[VER 94] R. Verfürth: A posteriori error estimates for nonlinear problems. Finite element discretizations of elliptic equations. *Math. Comput.*, **62**, 445–475 (1994).

[VOH 07] M. Vohralík: A posteriori error estimates for lowest–order mixed finite element discretization for convection–diffusion–reaction equations, *SIAM J. Numer. Anal.*,**45**, 1570–1599 (2007).