Some notes on the article "Finite volume element method for second order quasilinear elliptic equations"

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Last update: Saturday 17th September, 2011; sure I come back to this article to learn more...

Abstract: The authors consider a general class of second order quasilinear equations posed on two dimensional polygonal convex domain. This class of equations includes in particular some known models like the equation of prescribed mean curvature, the subsonic flow on irrotational ideal compressible gas, and the Bratu's equation. The discretization is performed using a finite volume element method. A finite volume element scheme is presented. The existence and uniqueness is proved thanks to the introduction of a linearized operator which has a relation with the problem to be approximated. Several estimates have obtained and discussed. It is first proved that the convergence order is h in the energy norm. The convergence order is proved to be $h^{1-\frac{2}{r}}$ in $W^{1,\infty}$ when the exact solution is satisfying $u \in W^{2,r}(\Omega)$, where $2 < r \leq \infty$ but this last order can be improved to h when u is satisfying $u \in W^{2,\infty}(\Omega) \cap H^3(\Omega)$. It is also proved that the convergence order in L^∞ of the approximate solution is $h^2|\ln h|$. In the quadratic norm, it is proved that the convergence order is $h^2|\ln h|$ but this order can be improved to h^2 when the derivatives of some data are satisfying some condition which is satisfied by Bratu's equation. Numerical tests are presented to support the theoretical results. The article under review is nice and merits to be read.

Key words and phrases: Finite volume element method; second order quasilinear elliptic equations; error estimates

Subject Classification: 65M08

1 what i have learned

1. finite element for the same problem : the same subject of this article (would say quasilinear equations) is treated in the context of *finite element method* by the authors Frehse and Rannacher [FRE 78]

2. nice book on finite volume and volume element method: monograph [LI 00] seems interesting to read...

3. some literature: seems interesting to see the articles [CHA 99, CHA2 02, CHA3 05].

4. existence, uniqueness, and analysis of the convergence: they are analysed thanks to the introduction of a linearized operator which has a relation with the problem to be approximated. I think this a nice point to be read, see [XU 96].
5. small h to get optimal order....: this I think is not known in the case when the problem is linear...

2 nice information...

The finite volume element method (also called the finite volume or co–volume method in some literature) is a class of important numerical tools for solving differential equations, especially for those arising from physical conservation laws including mass, momentum and energy. Because this method possesses a local physical conservation property, which is crucial in many applications, it is popular in computational fluid mechanics, groundwater hydrology and reservoir simulation. In the past sveral decades many researchers have studied this method extensively and obtained some important results. We refer to the monograph of Li *et al.* [LI 00] for general presentation of this method, and [CAI 91, CAI2 91, CHA 99, CHA2 02, CHA3 05].
The finite volume element method attempts to combine the flexibility of the finite element method with the local conservation property inherent in finite volume method. With this in mind, the approximate solution is saught in some finite element space living in a domain of interest. Then the local conservation property is preserved in each control volume that is constructed around a vertex belonging to the aformentioned discretization.

3 problem to be solved

The following problem is considered in the article under review:

$$-\nabla\cdot\, F(x,u,\nabla u)+g(x,u,\nabla u)=0,\;\; x\in\Omega, \qquad [1]$$

with Dirichlet boundary condition

$$u(x)=0,\;\; x\in\partial\Omega. \qquad [2]$$

The domain Ω is a convex polygonal domain in $\mathbb{R}^2$.
For the sake of simplicity, the following equation is treated instead of [1]:

$$-\nabla\cdot\, F(x,\nabla u)+g(x,u,\nabla u)=0,\;\; x\in\Omega. \qquad [3]$$

3.1 assumption on the data

it is assumed that, for some positive constant β, the following condition holds

$$|\,DF_1|+|\,DF_2|+|\,D^2F_1|+|\,D^2F_2|+|\,Dg|+|\,D^2g|\le\beta, \qquad [4]$$

where $F=(F_1,F_2)$.

3.2 where arises the problem under study?

Problem [3] arises for instance in

1. the equation of prescribed mean curvature:

$$F(x, \nabla u) = \left(1 + |\nabla u|^2\right)^{-\frac{1}{2}} \nabla u, \; g(x, u, \nabla u) = g(x). \qquad [5]$$

2. subsonic flow on irrational, ideal, compressible gas:

$$F(x, \nabla u) = \left(1 - \frac{\gamma - 1}{2} |\nabla u|^2\right)^{\frac{1}{\gamma - 1}} \nabla u, \; g(x, u, \nabla u) = g(x), \qquad [6]$$

where $\gamma > 1$.

3. Bratu's equation:

$$F(x, \nabla u) = \nabla u, \; g(x, u, \nabla u) = -\lambda \exp(u), \qquad [7]$$

whith $\lambda > 0$.

4 mesh and schemes

The discretization of the domain Ω is performed using *conforming* finite element triangulation $\mathcal{T}_h$ consisting of closed triangle elements denoted by K such that $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$, where $h = \max_{K \in \mathcal{T}_h} h_K$ and h_K is the diameter of the triangle K. The triangulation $\mathcal{T}_h$ is assumed to be *shape regular*, that is, there exists a positive constant C such that $h_K \leq C \rho_K$ for all $K \in \mathcal{T}_h$, where ρ_K is the diameter of the largest ball contained in K. We denote by $\mathcal{E}_h$, N_h, and N_h^0 the sets of all edges, all vertices and all interior vertices of $\mathcal{T}_h$, respectively.
In order to formulate the finite volume element method we first introduce a dual mesh denoted by $\mathcal{T}_h^\star$, based on $\mathcal{T}_h$, whose elements are called the control volume. The control volumes are contructed as in [CHA3 05]: let Q_K be the barycentre of the element $K \in \mathcal{T}_h$. We connect Q_K with the midpoints of the edges of K; thus partitioning K into three quadrilaterals K_p, with $p \in N_p(K)$ ($N_p(K)$ denotes the vertices of K). Then, for each vertex $p \in N_h$, we associate a control volume $K_p^\star$ that consists of all the quadrilaterals sharing p as vertex. By this way, we obtain a group of control volumes covering the domain Ω denoted by $\mathcal{T}_h^\star$. The new mesh $\mathcal{T}_h^\star$ called the dual mesh of $\mathcal{T}_h$. The dual mesh $\mathcal{T}_h^\star$ is said to be *regular* or *quasiuniform* if there exists some positive constant C such that

$$C^{-1} h^2 \leq \mathrm{meas}(\mathrm{K_p^\star}) \leq \mathrm{Ch}^2, \; \forall\, \mathrm{K_p} \in \mathcal{T}_\mathrm{h}^\star. \qquad [8]$$

The barycentre–type dual mesh can be performed for any triangulation $\mathcal{T}_h$ and leads to relatively simple calculation, the authors said (would fine if the authors justify this using a reference...)
If the primary triangulation is quasiuniform, then the dual mesh $\mathcal{T}_h^\star$ is also quasiuniform.
As usual two spaces can be introduced. One is a finite element space which is defined on the primary

mesh $\mathcal{T}_h$ and the other one is the finite volume space which is defined on the dual mesh $\mathcal{T}_h^\star$. We define the standard piecewise linear finite element space

$$\mathcal{S}_h = \{v \in \mathcal{C}(\overline{\Omega}) : v|_K \in \mathcal{P}_1(K),\ \forall K \in \mathcal{T}_h\} \cap H_0^1(\Omega). \qquad [9]$$

We define the finite volume space

$$\mathcal{S}_h^\star = \{v \in L^2(\Omega) : v|_{K_p^\star} \in \mathcal{P}_0(K),\ \forall K_p^\star \in \mathcal{T}_h^\star \text{ and } v|_{K_p^\star} = 0\ \forall p \in \partial\Omega\}. \qquad [10]$$

4.1 finite volume element scheme

We first introduce the interpolation operator $I_h^\star : \mathcal{C}(\overline{\Omega}) \cap H_0^1(\Omega) \to \mathcal{S}_h^\star$ defined by

$$I_h^\star v_h = \sum_{p \in N_h^0} v_h(p)\xi_{K_p^\star}, \qquad [11]$$

where $\xi_{K_p^\star}$ is the characteristic function of the control volume $K_p^\star$.
The finite volume scheme is given by: find $u_h \in \mathcal{S}_h$ (so it is piecewise linear) such that

$$a_h(u_h, I_h^\star v_h) = 0,\ \forall\, v_h \in \mathcal{S}_h, \qquad [12]$$

where

$$a_h(u_h, I_h^\star v_h) = -\sum_{p \in N_h^0} v_h(p) \int_{\partial K_p^\star} v_h(p) F(x, \nabla u_h) \cdot \mathbf{n}\, ds + \sum_{p \in N_h^0} v_h(p) \int_{K_p^\star} g(x, u_h, \nabla u_h) dx. \qquad [13]$$

They are anylysed thanks to the introduction of a linearized operator which has a relation with the problem to be approximated. I think this a nice point to be read, see [XU 96]

5 what about the convergence analysis?

The convergence analysis of the finite volume scheme is performed thanks to the introduction of a linearized operator which has a relation with the problem to be approximated. I think this a nice point to be read, see [XU 96].

1. optimal order is obtained in H^1, for h sufficiently small

$$\| u - u_h\|_1 \le Ch |\, u|_2. \qquad [14]$$

2. order estimate in $W^{1,\infty}$, when $u \in W^{2,r}$, where $2 < r \le \infty$

$$\| u - u_h\|_{1,\infty} \le Ch^{1-\frac{2}{r}} |\ln h| |\, u|_{2,r}. \qquad [15]$$

3. order estimate in $W^{1,\infty}$, when $u \in W^{2,\infty} \cap H^3(\Omega)$

$$\| u - u_h\|_{1,\infty} \le Ch \left(\| u\|_{2,\infty} + \| u\|_3\right). \qquad [16]$$

4. order estimate in L^∞, when $u \in W^{2,\infty} \cap H^3(\Omega)$

$$\| u - u_h \|_{0,\infty} \leq Ch^2 |\ln h| \left(\| u \|_{2,\infty} + \| u \|_3 \right) \left(1 + \| u \|_{2,\infty} \right). \qquad [17]$$

5. order estimate in L^2, when $u \in W^{2,\infty} \cap W^{3,p}$, where $p > 1$

$$\| u - u_h \| \leq Ch^2 |\ln h| \left(1 + \| u \|_{2,\infty} \right) \| u \|_{3,p}. \qquad [18]$$

6. order estimate in L^2, when $D_{zz}F = 0$ and $D_{zz}g = 0$ (it is the case of, for instance, Bratu's equation

$$\| u - u_h \| \leq Ch^2 |\ln h| \left(1 + \| u \|_{3,\infty} \right) \| u \|_{3,p}. \qquad [19]$$

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