Some notes on the article "Construction and convergence study of schemes preserving the elliptic local maximum principle"

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Last update: Monday 16th October, 2011; sure I come back to this article to learn more...

Abstract: The authors consider the finite volume approximation of diffusion equations on very generic meshes. They construct a nonlinear finite volume scheme which satisfies the minimum and maximum principles and is non oscillating. The mentioned scheme is based on a nonlinear combination of linear fluxes, and can be constructed in two or three dimensions and in the presence of strong anisotropy and heterogeneity. An existence result of a solution to the scheme is proved using Brouwer's topological degree.

To prove the convergence of the approximates solutions towards the exact solution, a discrete compactness property is stated. Under a coercivity property on the fluxes, it is proved that the approximates solutions converge towards the exact solution in L^q for all $q < \frac{2d}{d-2}$.

Several numerical tests are presented to support theoretical results and to understand how some parameters in the scheme should be chosen.

Key words and phrases: second order elliptic equations; anisotropic heterogeneous diffusion; finite volume method; local maximum and minimum principles;

Subject Classification: 65N08; 65N12

some remarks...

1. literature: there is a useful literature concerning numerical methods for elliptic PDEs on generic meshes

2. linear or nonlinear scheme?: the scheme presented in the article is nonlinear (even the problem is linear), see [BUE 05, KER 81, KEI 09]

3. convergence of the scheme: a convergence analysis is provided but there is no convergence order analysis

4. extension: the article under review is an extension of the article [LE 09] to very generic grids in any space dimension

5. is not so clear why the scheme presented has no spurious oscillations.
6. is not so clear what is relation between Definition 3.3 and spurious oscillations which can be produced during the implementation of the numerical scheme.

1 introduction, aim of the article...

Let Ω be an open bounded connected polygonal domain of $\mathbb{R}^d$. We consider the following problem:

$$q(x) = -D(x)\nabla u(x), \; x \in \Omega, \tag{1}$$

$$\nabla \cdot q(x) = f(x), \; x \in \Omega, \tag{2}$$

with

$$u(x) = u_\partial(x), \; x \in \partial\Omega. \tag{3}$$

The following assumptions are assumed to be satisfied

1. source term : the source term f is belonging to $L^2(\Omega)$;
2. exact solution u: the exact solution u of [1]–[3] is the concentration of the radioactive element.
3. permeability: permeability D is symmetric tensor–valued function such that:
   (a) D is piecewise Lipschitz–continuous on Ω
   (b) the set of the eigenvalues of D is included in $[\underline{\lambda}, \overline{\lambda}]$ with $\underline{\lambda} > 0$ for all $x \in \Omega$
   (c) boundary condition data u_∂ is given.

The basic equation (or its transient version) is at the core of complex models of flows in porous media, used, for example, in Petroleum egineering or in the framework of nuclear waste disposal. In such situations, it is important to have robust approximations of the solution of [1]–[3]. This robustness is, in particular measured through the respect of the physical bounds; for instance, in models of two–phase flows in porous media [NOR 07], ensuring that the computed concentration stays between 0 and 1 is of utmost importance; this is also the case when coupling transport equations with chemical models.

If the grid used for the discretization of the PDE has specific orhogonality condition (depending on D), the classical finite volume scheme [EYM 00] (in which the fluxes are approximated by a two–point finite difference expression) provides a solution that respects these bounds. However, in practical situations such a very specific grid might not even constructible. Several methods have been recently developed to construct schemes for elliptic PDEs on generic meshes: multipoint flux approximations [AAV1 98, AAV2 98, AGE 10], discontinuous Galerkin [DI 10, RIV 08], discrete duality finite volume methods [BOY 08, DOM 05], mimetic finite difference methods [BRE1 05, BRE2 05, BEI 08], hybrid finite volume methods [EYM 10, EYM 09], and mixed finite volume methods [DRO 06, DRO 09] (it is justified in [DRO 10] that these last methods, i.e. mimetic finite difference methods,hybrid finite volume methods, and mixed finite volume methods are identical). It has proved in [BUE 05, KER 81, KEI 09] that no linear consistent nine–point control volume scheme constructed on square meshes with a very anisotropic tensor (or on very distored quadrangular cells with an isotropic tensor) can respect the maximum or minimum principle. One must look for a nonlinear scheme in order to satisfy the maximum and minimum principles.

In [BUR 04], a nonlinear correction of the classical picewise linear finite element $\mathcal{P}_1$ is proposed. In [BER 05], a nonlinear method is proposed for homogeneous isotropic diffusions. But the positivity is obtained under a restrictive geometric condition.

Recently in [LE 09], a linear scheme satisfying a maximum principle for anisotropic diffusion operators on distored grids in dimension 2. In [LE 09], a new finite volume method for highly anisotropic diffusion operators is introduced on triangular meshes. This scheme satisfies a discrete version of the classical maximum (and minimum) principle for elliptic equations without geometrical conditions on the mesh. The aim of the article is to extend the results of [LE 09] to very generic grids in any space dimension.

The finite volume framework is chosen since it allows us to get conservative schemes which is essential in physics. The maximum and minimum principles are harder to satisfy, and then more rarely to considered in the construction of finite volume schemes. Schemes satisfy maximum and minimum principles are not only satisfying that the approximate concentration stays within the physical bounds but also they do not develop spurious oscillations.

2 mesh...

like in [EYM 10]

3 local maximum structure...

the following defition is useful:

DEFINITION 3.1 (LMP structure) Let $\mathcal{D}$ be a finite volume discretization as in [EYM 10]. We say that a scheme approximating [1]–[3] using the unknowns $(u_K)_{K\in\mathcal{T}}$ has local maximum principle if it can be written in the form:

$$-\sum_{L\in\mathcal{E}_K}\tau_{K,L}(u)(u_L-u_K)-\sum_{a\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}}\tau_{K,a}(u)(u_L-u_a)=\int_K f,\ \forall K\in\mathcal{T}, \tag{4}$$

for some functions $\tau_{K,L}$ and $\tau_{K,a}$ defined on $\mathbb{R}^{\mathrm{Card}(\mathcal{T})}\to\mathbb{R}^+$.

In [4], u_a stands for some value of u_∂ on a; we assume that this value is betwwen the maximum and minimum of u_∂.

LEMMA 3.2 (Principle of maximum and minimum) Assume that a scheme approximating [1]–[3]

and it has local maximum principle in the sense of the previous definition and assume in addition that $u_\partial \equiv 0$.

1. If $f \geq 0$ then $\min_{K\in\mathcal{T}} u_K \geq 0$.
2. If $f \leq 0$ then $\max_{K\in\mathcal{T}} u_K \leq 0$.

Lemma 3.3 (Non oscillating property) Assume that a scheme approximating [1]–[3] and it has local maximum principle in the sense of Definition 3.1 and assume in addition that $f \equiv 0$. We define:

$$V(K) = \{L \in \mathcal{T}; \tau_{K,L}(u) \neq 0\}, \tag{5}$$

$$E(K) = \{a \in \mathcal{E}_{\text{ext}}; \tau_{K,a}(u) \neq 0\}. \tag{6}$$

Then

$$\min(\min_{J\in V(K)} u_J, \min_{a\in E(K)} u_a) \leq u_K \leq \max(\max_{J\in V(K)} u_J, \max_{a\in E(K)} u_a). \tag{7}$$

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