Some notes on the article "SDFEM with non-standard higher-order finite elements for a convection-diffusion problem with characteristic boundary layers"

Franz, Sebastian

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Report done by Professor Bradji, Abdallah

Provisional home page: http://www.cmi.univ-mrs.fr/~bradji

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Abstract: The author considers singularly perturbed convection-diffusion problem with an exponential layer at the outflow boundary and two characteristic layers on a rectangular domain. Some non-standard higher-order finite elements using streamline diffusion finite element method (SDFEM) are proposed. A convergence analysis is provided. In addition to this, for the standard higher-order space Q_p supercloseness of the numerical solution w.r.t. an interpolation of the exact solution in the streamline diffusion norm of order $p + 1/2$ is proved.

Key words and phrases: ; streamline diffusion finite element method (SDFEM); higher-order finite elements; convection-diffusion problem; singularly perturbed equations; characteristic boundary layers

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some remarks

1. it is useful to understand why an exponential layer at the outflow boundary and two characteristic layers appear for [1].
2. the present work is some extension of [STY2 08]

1 problem to be resolved

The article treats the following singularly perturbed convection-diffusion problem with an exponential layer at the outflow boundary and two characteristic layers

$$-\varepsilon\Delta u(x,y) - b(x,y)u_x(x,y) + c(x,y)u(x,y) = f(x,y),\ \ (x,y) \in \Omega = (0,1)^2, \tag{1}$$

with the boundary Dirichlet condition

$$u(x,y) = 0,\ \ (x,y) \in \partial\Omega. \tag{2}$$

The data b, c, and ε satisfy

1. $b \in W^{1,\infty}(\Omega)$ and $c \in L^{\infty}(\Omega)$,

2. $b \geq \beta$ for some positive constant β,

3. ε is a small parameter.

Problem [1]–[2] gives rise to an exponential layer of width $O(\varepsilon)$ at $x = 0$ and to two characteristic layers of width $O(\sqrt{\varepsilon})$ at $y = 0$ and $y = 1$.
Under the following assumption, problem [1]–[2] has a unique solution

$$c + b_x/2 \geq \gamma > 0. \tag{[3]}$$

Note we can always assume [3] by introducing the new transformation $\bar{u}(x, y) = u(x, y) \exp(\rho\, x)$ with a convenient ρ.

2 mesh

Standard discretisation methods on quasi-uniform meshes will not give accurate solutions in the presence of layers. The accuracy is only satisfactory if the mesh width is of the same size as the perturbation parameter ε, which is infeasible in view of computational costs. This is the reason for using layer-adapted meshes based on a priori knowledge of the solution behaviour.
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Back in 1969 Bakhvalov [BAK 69] proposed one of the first layer-adapted meshes. Analysis on these kind of meshes is somewhat difficult. The piecewise uniform Shishkin meshes [MIL 96] proposed in 1996 are easier to handle. See also [ROO 08] for a detailed discussion of their properties and uses. The first analysis of finite element methods on Shishkin meshes was published in [STY 97].
Since the standard Galerkin methods lacks stability even on layer-adapted meshes, see [LIN 01], a stabilisation term will be added to the standard discretisation.
A frequently used stabilisation technique is the streamline diffusion finite element method (SDFEM) proposed by Hughes and Brooks [HUG 79].
The present work is some extension of [STY2 08].

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