A brief Report on the article “Interpolation theory of anisotropic finite elements and applications”

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Abstract: The authors provide us with a new technique to analyse the convergence of the error interpolation of anisotropic finite element methods.
In their technique, the error interpolation is obtained thanks to the use of Newton's formula of the interpolation polynomials on the reference element. The error interpolation on general element could be obtained, as usual thanks to transformation inequalities of function seminorms between a general element and the reference element. These interpolation errors are used by the authors to derive some error estimates of anisotropic finite element methods.

1 Introduction and statement of the main result of the article

It is known that in conforming finite element method, i.e. the finite element space $\mathcal{V}_h$ is included in the solution space $\mathcal{V}$ of the original variational space, the error is reduced to the interpolation error of the finite element space thanks to Cea's Lemma, see [CIA 78].
For non–conforming finite elements, $\mathcal{V}_h \not\subset \mathcal{V}$, the consistent error is, at least, reduced to the interpolation error.
Therefore the interpolation error is a basic fact to derive the error estimate in finite element methods.
To derive interpolation error, we need two key points:

- transformation inequalities of function semi–norms between a general element and a reference element,
- interpolation error on the reference element.

It is needed to compute interpolation error on the reference element the following key points that:

- the finite element is affine or equivalent,
- interpolation operator is bounded,

- interpolation operator is reduced to the identity operator over some polynomial space

After we compute the error on the reference element, we pass using transformation inequalities of function semi–norms to the interpolation error on the elements forming the triangulation. In the interpolation error on an element, it appears, in general, constant depends on the ratio between of the diameter of the element and the biggest ball included in that element. Therefore, we need to bound this ratio in uniform with the mesh parameters. The he ratio is bounded in uniform with the mesh parameter, this condition is called the *regular condition*, see for instance in [CIA 78], and *nondegenerate condition*, see for instance [BRE 94, Page 106–107, (4.4.16)].
The elements which do not satisfy the regular condition are called *isotropic elements*.
The regular condition is sometime equivalent to the minimal angle condition, see [CIA 78, Page 128].
It is found that the *regular condition* is not necessary to obtain the convergence of the interpolation error, see [BAB 76].
There are many works devoted to analyze the anisotropic finite element methods. The path followed in almost of these works is based on two points mentioned above, i.e., getting the error in a reference element and then, we use some inequalities between semi–norms, to obtain the error on a general element.
I think that the path followed by the authors is the same one followed in the previous stated literature, but the their idea is to use Newton's formula concerning polynomial interpolation. With the use of Newton's formula, they could derive an error on reference element.
To derive error estimate on the reference element, the authors use the following Lemma (according to the authors, the application of this Lemma could be replaced by the application of the classical interpolation theorem)

Lemma 1.1 (Assumption to be assumed for interpolation error on reference element) Let α be a multi–index, $|\alpha| = m$ and $\widehat{T}$ be a reference element. Let $\widehat{\mathcal{P}}$ be the shape function space and $\widehat{\mathcal{I}}$: $\mathcal{C}^k(\widehat{T}) \to \widehat{\mathcal{P}}$. Suppose that the following property holds:

$$\widehat{\mathcal{I}}(\widehat{p}) = \widehat{p},\ \forall \widehat{p} \in \widehat{\mathcal{P}}_l(\widehat{T}). \tag{1}$$

Assume that $r = \dim \widehat{D}^\alpha \widehat{\mathcal{P}}$. Suppose that $\widehat{\mathcal{I}} \in \mathcal{L}(\mathcal{W}^{l+1,p}(\widehat{T}), \mathcal{W}^{m,q}(\widehat{T}))$, $\mathcal{W}^{l+1,p}(\widehat{T}) \hookrightarrow \mathcal{W}^{m,q}(\widehat{T})$.
If there is an operator $\widehat{\mathcal{S}} : \mathcal{W}^{l+1-m,p}(\widehat{T}) \to \widehat{D}^\alpha \widehat{\mathcal{P}}$, $\widehat{\mathcal{S}} \in \mathcal{L}(\mathcal{W}^{l+1-m,p}(\widehat{T}), L^q(\widehat{T}))$ such that

$$\widehat{D}^\alpha \widehat{\mathcal{I}} \widehat{v} = \widehat{\mathcal{S}} \widehat{D}^\alpha \widehat{v}. \tag{2}$$

Then the following interpolation error holds

$$\|\widehat{D}^\alpha(\widehat{v} - \widehat{\mathcal{I}}\widehat{v})\|_{0,q,\widehat{T}} \le C(\widehat{\mathcal{I}}, \widehat{T}) |\widehat{D}^\alpha \widehat{v}|_{l+1-m,p,\widehat{T}}. \tag{3}$$

Using Newton's formula of the interpolation error, it is proved the following anisotropic interpolation error

Lemma 1.2 (Interpolation error on the reference element $[0,1]^2$ and $[0,1]^3$) There exists a constant $\widehat{C}$ such that

$$\|\widehat{D}^{\alpha}(\widehat{v}-\widehat{\Pi}_j\widehat{v})\|_{L^2(\widehat{T})} \leq \widehat{C}|\widehat{D}^{\alpha}\widehat{v}|_{H^{m+1-|\alpha|}(\widehat{T})},\ j\in\{2,3\}, |\alpha|\leq m, \quad [4]$$

and

$$\|\widehat{D}^{\alpha}(\widehat{v}-\widehat{L}_j\widehat{v})\|_{L^2(\widehat{T})} \leq \widehat{C}|\widehat{D}^{\alpha}\widehat{v}|_{H^{m+1-|\alpha|}(\widehat{T})},\ j\in\{2,3\}, |\alpha|\leq m, \quad [5]$$

where $\widehat{\Pi}_2$ (resp. $\widehat{\Pi}_3$) is the bi–m interpolation operator on $[0,1]^2$ (resp.the bi–m interpolation operator on $[0,1]^3$) and $\widehat{L}_2$ (resp. $\widehat{L}_3$) is the Hermite interpolation operator on $[0,1]^2$ by polynomials $\mathcal{Q}_{2m-1}$ (resp.the bi–m interpolation operator on $[0,1]^3$ by polynomials $\mathcal{Q}_{2m-1}$).

Thanks to the previous Lemma, the authors obtained the following Theorem:

Theorem 1.3 (Interpolation error on a general ractangle element) There exists a constant C independent of the regular condition such that

$$\|v-\Pi_j v\|_{H^k(T)} \leq C \sum_{|\beta|=m+1-k} h_T^{\beta}|D^{\alpha}v|_{H^k(T)},\ j\in\{2,3\}, \quad [6]$$

and

$$\|v-L_j v\|_{H^k(T)} \leq C \sum_{|\beta|=2m-k} h_T^{\beta}|D^{\alpha}v|_{H^k(T)},\ j\in\{2,3\}, \quad [7]$$

where Π_j and L_j are given in the previous Lemma and $h_T^{\beta}=h_1^{\beta_1}h_2^{\beta_2}$, when T is a rectangle with sides of lengths h_1 and h_2, $h_T^{\beta}=h_1^{\beta_1}h_2^{\beta_2}h_3^{\beta_3}$, when T is a cube sides of lengths h_1, h_2 and h_3.

Remark 1 The estimates of the previous Theorem yields

$$\|v-\Pi_j v\|_{H^k(T)} \leq C h^{m+1-k}|v|_{H^{m+1}(T)},\ j\in\{2,3\}, \quad [8]$$

and

$$\|v-L_j v\|_{H^k(T)} \leq C h^{2m-k}|v|_{H^{2m}(T)},\ j\in\{2,3\}, \quad [9]$$

where $h=\max_i h_i$

Theorem 1.4 (Interpolation error on a general triangle or tetrahedron element) There exists a constant C independent of the regular condition such that

$$\|v-I_j v\|_{H^k(T)} \leq C\|B_0^{-T}\left(\sum_{|\beta|=m+1-k} l_T^{2\beta}|D^{\alpha}v|_{H_l^k(T)}\right)^{\frac{1}{2}},\ j\in\{2,3\}, \quad [10]$$

where B_0^{-T} is bounded by $\dfrac{1}{\sin\theta}$ for triangle T and by $\dfrac{6}{\sin\alpha_0\sin\alpha_1\sin\varphi_1}$ for tetrahedron T and θ is some angle in T and

Π_j and L_j are given in the previous Lemma and $h_T^{\beta}=h_1^{\beta_1}h_2^{\beta_2}$, when T is a rectangle with sides of lengths h_1 and h_2, $h_T^{\beta}=h_1^{\beta_1}h_2^{\beta_2}h_3^{\beta_3}$, when T is a cube with sides of lengths h_1, h_2 and h_3; and

$$|w|^2_{H_l^k(T)} = \sum_{|\beta|=} \|D_l^{\beta}w\|^2_{L^2(T)}, \quad [11]$$

where D_l^{β} is some directional derivative.

References

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