A brief Report on the article "Analysis of linear and quadratic simplicial finite volume methods for elliptic equations"

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Numer. Math (2009), Vol. 11, 469–492.

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Written in Tuesday 21 st July, 2009

Abstract: The present paper provides us with an analysis of some convergence properties of two classes of finite volume methods (FVMs). The first class considered is the linear finite volume methods in any dimension, and the second class is a quadratic simplicial finite volume method in two dimensions.

Concerning the first class, the authors first derived a simple identity between the stifness matrix of the linear FVM and that of the corresponding finite element methods (FEMs) for Poisson equations. Thanks to this identity, the inf-sup condition of the FVM schemes for elliptic equations with variable coefficient is proved, and a superconvergence result is presented. As consequences of the previous stated identity, some a posteriori error estimates are presented and also algebraic solvers for FEM are extended to FVM.

Concerning the second class presented in this paper, the authors constructed and analyzed a general class of two dimensional quadratic simplicial grid FVM schemes. Under some weak condition on the grid, inf-sup condition is established for this class of quadratic simplicial grid FVM schemes.

Key words: linear finite volume methods, linear finite element methods, quadratic finite volume methods, elliptic equations, inf-sup condition, arbitrary dimension, two dimensions

Primary MSC 65N30

Secondary MSC 65N06, 65N12

1 Introduction: known results

The finite volume method (FVM) has been one of the most commonly used numerical methods for solving partial differential equations in practice.

One of the main attractive property of FVM is that, by construction, main physical conservation laws possessed in a given application are prserved in FVM. Besides, similar to the finite element method (FEM), the FVM can be used to deal with domains with complex geometries. For these and other advantages, the FVM has a wide range of applicatnions in scienntific and engineering computations.

For instance, the FVM has been widely used in computational fluid dynamics, heat transfer, hyperbolic equations, and modeling of fuel cells.

Piecewise linear linear finite volume method has been much studied in the literature. Bank and Rose [BAN 87] proved, in two dimensions, that linear FVM is closely to linear FEM. They proved that, in case of Poisson's equation posed on a polygonal domain, the stiffness matrices of linear FVM are identical to those of linear FEM for general grids. For general elliptic equations posed on a polygonal domain in two dimensions, Hackbusch [HAC 89] proved that the difference between the FVM solution and the FEM solution is in general of first order, and is of second order for some special cases of FVM schemes. As a consequence of Hackbusch's result, some known superconvergence results valid in FEM are valid then in FVM. Ewing et al. proved that when the mesh size is sufficicently small, the stiffness matrix of linear FVM is a small perturbation of that of the linear FEM.

2 The aim of the paper and a brief statement of the results

The aim of the article under consideration is to refine and generalize results stated in the previous section to arbitrary space dimensions (recall that some results of the previous section have been shown in the two dimension case).

The main results of the paper under consideration could be stated as follows:

- results concerning linear FVM
  - they authors derived a simple identity between the stifness matrix of FVM and that of FEM.
  - they used previous stated identity to prove the inf–sup of the FVM schemes for elliptic equations with variable coefficients.
  - they obtained a superconvergence result for elliptic equations with variable coefficients.
  - as an application of the previous results, the authors demonstrate:
    - how a discrete FVM stiffness matrix can be preconditioned by a FEM stiffness matrix,
    - some known superconvergence results in FEM could be extended FVM.
- results concerning higher order FVM: it seems that higher order (especially simplicial) FVMs have not been analysed as much or as satisfactorily as linear FVMs in the literature. Some

works have been devoted to quadratic simplicial FVMs, e.g. [CHE 92, TIA 91, LIE 96]. Studies for more general cases can be found in [LI 00]. The contributions of the article under consideration in the present item are the following:

- they treat a general class of quadratic simplicial FVMs in two dimensions. The quadratic simplicial grids considered in [CHE 92, TIA 91, LIE 96] belong to this class. They prove the inf–sup condition of these quadratic simplicial FVMs under conditions that are weaker than all the existing literature.
- they proof stated in the previous item depends strongly on the shape of the partition, namely the minimal angle θ_0 of the partition.
- since the FVMs grids considered in [EMO 92, LIE 96, LI 00] are included in the paper under consideration, the authors provided sufficient conditions in order that the previous shape condition holds in each of the grids considred in [EMO 92, LIE 96, LI 00], more precise, the inf–sup condition holds if
  - $\theta_0 \geq 7.11^\circ$ for FVM considered in [EMO 92],
  - $\theta_0 \geq 9.98^\circ$ for FVM considered in [LIE 96],
  - $\theta_0 \geq 20.95^\circ$ for FVM considered in [LI 00]
- for other schemes considered in the paper, the authors remark that the inf–sup condition may hold for very small θ_0 for some specific quadratic finite volume schemes, for instane, θ_0 can be less than 3° for some FVM schemes.

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