A brief Report on the article "A posteriori error estimates of a non conforming finite element method for problems with artificial boundary conditions"

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Abstract: The authors consider a second order elliptic equation with boundary (artificial) conditions containing an implicit integral form. This type of the elliptic boundary problems arises, for instance, from the transformation of elliptic problems imposed on unbounded domains. A non conforming finite element method is suggested. A posteriori estimates are presented. Some numerical examples justifying the efficiency of the method are provided.

1 Basic knowledge

Many phusical and engineering problems such as the electrich field and magnetic field, can be modelled by partial differential equations posed on unbounded domains. to efficiency solve such problems by numerical methods, one often introduces proper artificial boundary conditions to translate these problems to bounded domains ones. These artificial boundary conditions often have implicit integral forms, which are quite different from those of explicit boundary conditions Dirichlet, Neumann, or mixed boundary conditions.

Furthermore, when the solutions of the reduced bounded problems have some singularities, e.g., singularities arising from re–entrant corners, singularties of Green's function, and mesh refinement strategy. In this case, a posteriori estimators are often required to identify the regions which need further refinement. There are many methods for the a posteriori estimations, e.g., the residual estimates, the averaging methods, etc., however, they are mostly developed for bounded domains problems imposed with explicit boundary conditions.

The authors develop an efficient a posteriori for non conforming finite element approximation of bounded domain elliptic problems with a boundary condition given in implicit integral form. Such problems come naturally from unbounded domain elliptic problem by imposing proper implicit artificial boundary conditions.

2 Why the problem to be solved is interesting?

Let us consider the following problem

$$-\nabla(\mathcal{A}\,\nabla\,u(\mathbf{x}))+cu(\mathbf{x})=f(\mathbf{x}),\;\;\mathbf{x}\in\Omega, \qquad [1]$$

with the following mixed boundary conditions:

$$\nabla(\mathcal{A}(\mathbf{x})\,\nabla\,u(\mathbf{x}))\cdot\mathbf{n}(\mathbf{x})=g(\mathbf{x}),\;\;\mathbf{x}\in\Gamma_N, \qquad [2]$$

$$u(\mathbf{x})=u_D(\mathbf{x}),\;\;\mathbf{x}\in\Gamma_D, \qquad [3]$$

$$\lim_{\|\mathbf{x}\|\to\infty}u(\mathbf{x})=u_\infty, \qquad [4]$$

where Ω is two dimensional unbounded domain with boundary $\partial\Omega=\Gamma_D\cup\Gamma_N$, and if $c_0=0$ $(c_0\geq c_0$, we assume $u_\infty=0$.

We assume in addition that $\mathrm{supp(f)}\cup\mathrm{supp}(\mathrm{I}-\mathcal{A})\cup\mathrm{supp(c-c_0)}$ is bounded; so there exists sufficiently large R such that $\mathrm{supp(f)}\cup\mathrm{supp}(\mathrm{I}-\mathcal{A})\cup\mathrm{supp(c-c_0)}\subset\mathrm{B(0,R)}$. So $\Gamma_e=\{\mathbf{x}\in\Omega:\|\mathbf{x}\|=R\}$ can be taken as an artificial boundary. Let

$$\Omega_i=B(0,R)\cap\Omega, \qquad [5]$$

$$\Omega_e=B(0,R)^c\cap\Omega. \qquad [6]$$

For $r>R$, we have

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\frac{R}{r}\right)(a_n\cos\,n\theta+b_n\sin\,n\theta)\,, \qquad [7]$$

where

$$a_n=\frac{1}{\pi}\int_0^{2\pi}u(R,\theta)\cos\,n\theta\,d\theta, \qquad [8]$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}u(R,\theta)\sin\,n\theta\,d\theta. \qquad [9]$$

and

$$a_0=\frac{u_\infty}{2}. \qquad [10]$$

Differentiating [7] with respect to r, we get

$$\frac{\partial\,u}{\partial\,r}=-\frac{1}{\pi\,R}\sum_{n=1}^{\infty}\int_0^{2\pi}u(R,\theta)\cos\,n(\theta-\varphi)d\,\varphi\equiv\mathcal{B}\,u(R,\theta), \qquad [11]$$

where $\mathcal{B}$ is a bounded operator from $H^{\frac{1}{2}}$ into $H^{-\frac{1}{2}}$.

Using the fact that $\frac{\partial\,u}{\partial\,r}=\frac{\partial\,u}{\partial\,\mathbf{n}}$, [11] implies that

$$\frac{\partial\,u}{\partial\,\mathbf{n}}(R,\theta)=\mathcal{B}\,u(R,\theta). \qquad [12]$$

So problem [1]–[4] is equivalent to

$$-\nabla(\mathcal{A}\,\nabla\,u(\mathbf{x}))+cu(\mathbf{x})=f(\mathbf{x}),\;\;\mathbf{x}\in\Omega_i, \qquad [13]$$

$$\nabla(\mathcal{A}(\mathbf{x})\,\nabla\,u(\mathbf{x}))\cdot\mathbf{n}(\mathbf{x})=g(\mathbf{x}),\;\;\mathbf{x}\in\Gamma_N, \qquad [14]$$

$$u(\mathbf{x}) = u_D(\mathbf{x}),\ \mathbf{x} \in \Gamma_D, \tag{15}$$

$$\frac{\partial u}{\partial \mathbf{n}}(R,\theta) = \mathcal{B}\, u(R,\theta). \tag{16}$$

If $\Gamma_D = \emptyset$, [15] must be replaced with (perhaps because of a compatibility condition, this point is not explained in the article!!)

$$\int_0^{2\pi} u(R,\theta)d\theta = 0. \tag{17}$$

3 Functional spaces and weak formulations

Let us consider the following spaces:

- when $\Gamma_D = \emptyset$:

$$\mathcal{V} = \{v \in H^1(\Omega_i) : \int_0^{2\pi} v(R,\theta)d\theta = 0\}, \tag{18}$$

$$\mathcal{V}_0 = \{v \in\ H^1(\Omega_i) : v|_{\Gamma_D} = 0\}. \tag{19}$$

- when $\Gamma_D \neq \emptyset$:

$$\mathcal{V} = \{v \in\ H^1(\Omega_i) : v|_{\Gamma_D} = u_D\}, \tag{20}$$

$$\mathcal{V}_0 = \{v \in\ H^1(\Omega_i) : v|_{\Gamma_D} = 0\}. \tag{21}$$

We assume that $f \in H^{-1}(\Omega)$, $g \in L^2(\Gamma_N)$, $u_D \in H^{\frac{1}{2}}(\Gamma_D)$. So, there exists $f_0,\ f_1,\ f_2$ such that

$$\langle\, f, v\rangle = \int_{\Omega_i} \left(f_0(\mathbf{x})v(\mathbf{x}) + f_1(\mathbf{x})\frac{\partial v}{\partial x}(\mathbf{x}) + f_2(\mathbf{x})\frac{\partial v}{\partial y}(\mathbf{x}) \right) d\mathbf{x}, \tag{22}$$

where (x,y) are the components of $\mathbf{x} \in \mathbb{R}^2$.

A weak formulation for [13]–[16] may be as: find $u \in \mathcal{V}$ such that

$$a(u,v) + b(u,v) = \langle\, f, v\rangle + \int_{\Gamma_N} g(\mathbf{x})v(\mathbf{x})d\mathbf{x}, \tag{23}$$

where

$$a(u,v) = \int_{\Omega_i} \left(\mathcal{A}(\mathbf{x})\nabla\, u(\mathbf{x}) \cdot\ \nabla\, v(\mathbf{x}) + cu(\mathbf{x})v(\mathbf{x})\right)\, d\mathbf{x}, \tag{24}$$

and

$$b(u,v) = \sum_{n=1}^{\infty} \frac{n}{\pi} \int_0^{2\pi}\int_0^{2\pi} u(R,\theta)\cos\, n(\theta-\varphi)v(R,\theta)d\,\varphi\, d\theta. \tag{25}$$

In practice, only a truncation on the sum [25] to compute an approximation for the solution u of [23], i.e.,

$$b_N(u,v) = \sum_{n=1}^{N} \frac{n}{\pi} \int_0^{2\pi}\int_0^{2\pi} u(R,\theta)\cos\, n(\theta-\varphi)v(R,\theta)d\,\varphi\, d\theta. \tag{26}$$

4 The finite element scheme

The finite element scheme is suggested in the article is a mixed finite element scheme. The element used are the Crouziex–Raviart.

References

[CIA 78] P. G. Ciarlet: The Finite Element Method for Elliptic Problems. *Norhh Holand, Amsterdam*, 1978.