Some notes on the article " Sparsity optimized high order finite element functions for H(div) on simplices"

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Abstract: .The authors investigate the space of vector valued–functions with square–integrable divergence and conforming h_p finite element discretization for open bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$. A new set of of basis functions for H(div)-conforming h_p finite element spaces, which yields an optimal sparsity pattern for system matrices derived from the discretization of some convenient bilinear form, is introduced. The construction of basis functions relies on some known construction principles in the literature which concerns the sparsity of H(div)-conforming discretizations. More precise, the construction principles are related to the use of Raviart–Thomas elements, mixed–weighted Jacobi polynomials, and Dubiner basis. This construction implies the L_2-orthogonality of the fluxes of the basis. The proof of sparsity of the mass matrix requires some symbolic computation. The stated finite element basis form a hierarchical set of H(div)-conforming basis functions and hence are applicable also in general settings on unstructured (curved) simplicial meshes.

Key words and phrases: finite element methods; sparsity; high order finite element functions; simplices

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1 some remarks...

1. the article presents, in its introduction a nice overview on the disadvantages of classical high order finite element methods. These disadvantages a related to the sparsity of matrices.
2. thesis [ZAG 06] seems interesting

2 aim and literature...

The authors investigate the space of vector valued–functions with square–integrable divergence:

$$H(\mathrm{div}, \Omega) = \{u \in \mathbb{L}^2(\Omega)^d : \nabla \cdot u \in \mathbb{L}^2(\Omega)\}, \qquad [1]$$

and conforming h_p finite element discretization for open bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$. For than 20 years, spectral methods [CAR 99], as well as the p-version and the h_p-version of finite element method, see e.g. [DEM 06, DEM 08, SCH 98, SOL 03], and the references therein, have become more and more popular.

1. In the classical (h-version) finite element method, convergence of piecewise polynomial discrete solution (typically of low fixed degree) to the exact solution is achieved by decreasing the mesh size h.

2. In the p-version of finite element method, the mesh is fixed and the convergence is achieved by increasing the degree p.

3. In the h_p-version of finite element method, we combine the previous two methods: we allow the mesh refinement in h and increase the degree p, see [BAB 90].
   By the proper combination of local h-refinement and local p-enrichment the h_p-version achieve faster convergence rates with respect to the number of unknowns in case of piecewise smooth solution even in the presence of singularities.

But a naive approach to higher order finite element methods in general suffers from dense element matrices with up to order p^{2d} nonzero entries. Moreover, when using standard numerical quadrature, the calculation of each entry of the element matrix requires number of operations up to order p^d. This results in the total cost of order p^{3d} for assembling one element matrix. In addition to the question of fast and efficient solvers, also fast assembling techniques, stable condition numbers (with respect to p) and sparsity of the element matrices become an issue in spectral and high order finite element method.

The author introduce a new set of of basis functions for H(div)-conforming h_p finite element spaces, which yields an optimal sparsity pattern for system matrices derived from the discretization of the bilinear form:

$$a(u,v) = (\varepsilon\nabla\cdot u, \nabla\cdot v) + (\kappa\, u, v), \qquad [2]$$

for piecewise constant parameters ε and κ. The construction of basis functions relies on some known construction principles in the literature which concerns the sparsity of H(div)-conforming discretizations. More precise, the construction principles are related to the use of Raviart–Thomas elements, mixed–weighted Jacobi polynomials, and Dubiner basis.

The finite element basis $\{\Psi^{(0)}, \Psi^{(1)}, \Psi^{(2)}\}$ is constructed under the following principles:

1. As common, the basis $\{\Psi^{(0)}$ of low-order space is chosen as the classical Raviart–Thomas elements of zeroth order (it useful to explain this point more)

2. The set of divergence–free basis functions $\{\Psi^{(1)}$ are chosen as the curl of H(curl)-conforming completion functions.

3. The set of cell–based functions $\{\Psi^{(2)}$ spanning the non-divergence free subspaces are chosen such that $\{\nabla\cdot\Psi^{(2)}$ coincides with the so called Dubiner basis.

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