A brief Report on the article "Convergence analysis for the numerical boundary corrector for elliptic equations with rapidly oscillating coefficients"

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1 Abstract

The authors provide us with a finite element scheme for the elliptic problem $\mathcal{L}_\varepsilon u_\varepsilon = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}u_\varepsilon(x)\right) = f(x)$ in Ω and $u_\varepsilon = 0$ on $\partial\Omega$, where $\Omega = (0,1)^2$, $a(y) = (a_{ij}(y))$ is a periodic symmetric positive definite matrix, and ε is a singular parameter such that $h >> e$. To get a finite element scheme, the authors first derived a convenient asymptotic expansion for the exact solution u_ε. An a priori error estimate between the the exact solution u_ε and this asymptotic expansion is proved under some weak regularity assumption on the exact solution. The previous stated asymptotic expansion contains a corrector term, to capture the homogeneous Dirichlet boundary condition of u_ε and denoted by θ_ε, is satisfying $\mathcal{L}_\varepsilon\theta_\varepsilon = 0$ and its boundary value is highly oscillatory. An analytical approximation for θ_ε is provided. A finite element scheme is suggested using some convenient finite element approximations for each term in the asymptotic expansion and the corrector term θ_ε. Depending on the regularity of the problem and of the functions included in the asymptotic expansion, an a priori error estimate of the suggested finite element scheme is proved. The order of the finite element scheme is $h^2 + \varepsilon^{3/2} + \varepsilon\, h$ in L^2–norm, and $h + \varepsilon^{1+\hat{\delta}}$ in the H_0^1–norm, for some $\hat{\delta} \in (-\frac{1}{4}, 0]$.

2 Key words

elliptic equations; multiscaling; periodic equation; asymptotic expansion; corrector term; analytical approximation of the corrector term; a mixed finite element scheme; composite materials; asymptotic expansion; a priori error estimate; finite element scheme; a priori estimate in the finite element approximation

3 Primary AMS Subject Classification

65N30

4 Secondary AMS Subject Classi35B45fication

35B25; 35B27; 35B40; 35B45; 35B65; 35C20

5 Some basic knowledge

This paper considers the following problem

$$\left(\mathcal{L}_\varepsilon u_\varepsilon\right)(x) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}u_\varepsilon(x)\right) = f(x),\ x\in\Omega, \qquad [1]$$

and

$$u_\varepsilon(x) = 0,\ x\in\partial\Omega. \qquad [2]$$

Here ε is a small scale and $a(y) = (a_{ij}(y))$ is a periodic symmetric positive definite matrix with period $Y = (0,1)^2$, and $\Omega = (0,1)^2$. We assume that $a_{ij} \in L^\infty_{per}(Y)$, i.e., a_{ij} is Y–periodic and $a_{ij} \in L^\infty(\mathbb{R}^2)$, and that there exists a positive constant γ_a such that $\gamma_a\|\xi\|^2 \le a_{ij}(y)\xi_i\xi_j$, for all $\xi = (\xi_i,\xi_j) \in \mathbb{R}^2$ and $y \in (0,1)^2$.
We note that standard finite element methods do not yield good numerical approximation when the mesh size h satisfies $h < \varepsilon$. To overcome this situation, new numerical methods have been recently proposed to solve above problem such as the multiscale finite element methods [EFE 00, HOU 97]. The numerical method used here is based strongly on the use of an asymptotic expansion for ε. It is also used the matrix a to obtain a very efficient method to approximate u_ε.

6 Asymptotic expansion for the exact solution

It is said in the present article that the exact solution ε could be expanded as

$$u_\varepsilon(x) = u_0(x,x/\varepsilon) + \varepsilon u_1(x,x/\varepsilon) + \varepsilon^2 u_2(x,x/\varepsilon) + \dots \qquad [3]$$

It is claimed in the article under consideration that using equation [3] in [1] and matching the terms with the same order in ε, one may define function u_j such that

$$\|u_\varepsilon(x) - u_0(x,x/\varepsilon) - \varepsilon u_1(x,x/\varepsilon)\|_1 \le C\varepsilon^{\frac{1}{2}}\|u_0\|_{2,\infty} \qquad [4]$$

Remark 1 I do not understand well why the left hand side on [4] depends on u_ε, u_0, and u_1, whereas the right hand side depends only on u_0.

Let us try what I understood from the article.

Example 1 Let us assume that $a_{11}(x/\varepsilon) = x_1/\varepsilon$ and otherwise $a_{ij} = 0$ otherwise. Therefore

$$\begin{aligned}\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}u_\varepsilon(x)\right) &= \frac{\partial}{\partial x_1}(x_1/\varepsilon)\frac{\partial}{\partial x_1}u_\varepsilon(x)\\ &= \frac{1}{\varepsilon}\left(\frac{\partial}{\partial x_1}u_\varepsilon(x) + x_1\frac{\partial^2}{\partial x_1^2}u_\varepsilon(x)\right) \quad [5]\end{aligned}$$

but I do not how to manage after!! It seems that is it is better, as it is advised in the article under consideration, to consult [BEN 78, JIK 94]

The following funtions are used to provide a convenient asymptotic expansion for u_ε; a convenient asymptotic expansion in the sense that we get the arror estimate, in the finite element approximation, given below.

- Let $\chi \in H_p^1er(Y)$, i.e. $\chi \in H_l^1oc(\mathbb{R}^d)$, $d = 2$ or 3, and χ is Y–periodic (Y is a domain in $\mathbb{R}^d$, , $d = 2$ or 3) be the weak solution with zero average over Y of

  $$\nabla_y \cdot a(y)\nabla_y\chi^j = \nabla_y \cdot a(y)\nabla_y y_j = \sum_{i=1}^{d}\frac{\partial}{\partial y_j}a_{ij}(y). \quad [6]$$

  Let us denote by

  $$A_{ij} = \frac{1}{|Y|}\sum_{l,m=1}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi^i)\frac{\partial}{\partial y_m}(y_j - \chi^j)dy. \quad [7]$$

  We can check that the matrix $A = (A_{ij})$ is symmetric positive definite. Define the weak solution u_0 of the following elliptic problem with homogeneous Dirichlet boundary conditions

  $$-\nabla \cdot A\nabla u_0(x) = f(x),\ x \in \Omega. \quad [8]$$

- We define the function u_1 by

  $$u_1(x, \frac{x}{\varepsilon}) = -\sum_{j=1}^{d}\chi^j(\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_j}(x). \quad [9]$$

- Since we need an approximation in terms of linear combination of u_0, u_1 and $u_0 + \varepsilon u_1$ does not satisfy the boundary condition of u, we need to introduce the corrector term $\theta_\varepsilon \in H^1(\Omega)$ as the solution of

  $$-\nabla \cdot a(x/\varepsilon)\nabla\theta_\varepsilon(x) = 0,\ x \in \Omega, \quad [10]$$

  and

  $$\theta_\varepsilon(x) = -u_1(x, \frac{x}{\varepsilon})\ x \in \partial\Omega. \quad [11]$$

  Hence, we have $u_0 + \varepsilon u_1 + \varepsilon\theta_\varepsilon \in H_0^1(\Omega)$.

  Since problem [10] has the highly oscillatory coefficient $a(x/\varepsilon)$, one should find an analytical approximation before going to approximate the solution θ_ε of problem [10] will be discussed later.

Remark 2 It seems that for the first time that equations [1] and [10] have the same difficulty of highly oscillatory data, namely the oscillatory coefficient in [1] and oscillatory coefficient and oscillatory boundary condition in [10], and therefore expansion $u_0 + \varepsilon u_1 + \varepsilon\theta_\varepsilon$, which expected to approximate u_ε and contains θ_ε, does not serve us. It is nice to make the difference between equations [1] and [10].
A first remark that [1] and [10] have the same coefficients. Indeed, assume for the sake of simplicity, that $d = 2$. Therefore

$$\nabla \cdot a\,(x/\varepsilon)\,\nabla\theta_\varepsilon(x) = \frac{\partial\left(a_{11}(x/\varepsilon)\frac{\partial\theta_\varepsilon}{\partial x_1} + a_{12}(x/\varepsilon)\frac{\partial\theta_\varepsilon}{\partial x_2}\right)}{\partial x_1} + \frac{\partial\left(a_{21}(x/\varepsilon)\frac{\partial\theta_\varepsilon}{\partial x_1} + a_{22}(x/\varepsilon)\frac{\partial\theta_\varepsilon}{\partial x_2}\right)}{\partial x_2} \qquad [12]$$

Therefore the difference between [1] and [10] may be occur one remarks that the right hand side [1] equal to f and the boundary conditions vanish, whereas the right hand side in [10] vanishes and the boundary conditions does not vanish.

6.1 Approximation of θ_ε

The function θ_ε is decomposed into functions $\tilde{\theta}_\varepsilon$ and $\bar{\theta}_\varepsilon$ in the following way:

$$-\nabla \cdot a\,(x/\varepsilon)\,\tilde{\nabla}\theta_\varepsilon(x) = 0,\ x \in \Omega \text{ and } \tilde{\theta}_\varepsilon = \left(\sum_{j=1}^{d} \chi^j\,(x/\varepsilon)\,\eta_j - \chi^\star\right)\partial_\eta u_0 \text{ on } \partial\Omega. \qquad [13]$$

and

$$-\nabla \cdot a\,(x/\varepsilon)\,\bar{\nabla}\theta_\varepsilon(x) = 0,\ x \in \Omega \text{ and } \bar{\theta}_\varepsilon = \chi^\star\partial_\eta u_0 \text{ on } \partial\Omega. \qquad [14]$$

where $\chi^\star|_{\Gamma_k}$, $k \in \{e, w, n, s\}$, are properly chosen constants, where $\Gamma_e = \{1\}\times[0,1]$, $\Gamma_w = \{0\}\times[0,1]$, $\Gamma_n = [0,1]\times\{1\}$, and $\Gamma_s = [0,1]\times\{0\}$ are the edges of the domain $\Omega = (0,1)^2$. On remarks that $\sum_{j=1}^{d} \chi^j\,(x/\varepsilon)\,\eta_j$.
The constants $\chi^\star$ are given in Subsection 2.2.1., page 549, of the paper under consideration.
The following result (is the subject of Theorem 3.1., Page 561 of the paper under consideration) gives an error estimate between $u_0 + \varepsilon u_1 + \varepsilon\theta_\varepsilon$ and the exact solution u_ε.

THEOREM 6.1 Let u_0, u_1, ϕ_ε be respectively given in [8], [9] and [SAR 08, (2.16), Page 550]. Under some assumptions on the coefficients a_{ij}, u_0, and χ^j, there exists a negative constant δ and a positive constant C, independent of ε such that

$$\|u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon\, u_1(\cdot, \cdot/\varepsilon) - \varepsilon\,\phi_\varepsilon\|_1 \le C\varepsilon^{1+\delta}\|u_0\|_{2,p}, \qquad [15]$$

and

$$\|u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon\, u_1(\cdot, \cdot/\varepsilon) - \varepsilon\,\phi_\varepsilon\|_0 \le C\varepsilon^{3/2}\|u_0\|_{3,p}. \qquad [16]$$

7 Finite element approximation for u_ε

After the authors [SAR 08] having obtained an analytical approximation, namely $u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon\, u_1(\cdot, \cdot/\varepsilon) - \varepsilon\, \phi_\varepsilon$, for u_ε, the authors moved to obtain a finite element approximation, denoted by u_ε^h, for u_ε. The finite element approximation u_ε^h is based on the approximation of the terms existing in the expansion $u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon\, u_1(\cdot, \cdot/\varepsilon) - \varepsilon\, \phi_\varepsilon$.
It seems that some additional reading I should do before arriving to understand how it is approximated

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