A brief Report on the article "Superconvergent biquadratic finite volume element method for two dimensional Poisson's equations"

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Abstract: The authors consider the biquadratic finite volume element approximation for the Poisson's equation on the rectangular domain $\Omega = (0,1)^2$. The primal mesh is performed using a ractangular partition. The control volumes are chosen in such a way that the vertices are stress points of the primal mesh. In order to solve the scheme more efficiently, the authors wrote the biquadratic finite volume element scheme as a tensor product form and used the alternating direction technique to solve it.

Thanks to the fact that the primal mesh satisfies a superconvergence property in the interpolatory approximation, the authors prove that the numerical gradients of the method have h^3–superconvergence order at optimal stress points. Using the dual argument technique, the authors also prove that the convergence order in L^2–norm is h^4 at nodal points. A numerical example is presented to support the theoretical results.

Key words and phrases: Poisson's equation, biquadratic finite volume element method, stress points, superconvergence, alternating direction technique

Subject classification: 65N12; 65N15 To be checken if really these subject classification are those of 2010 or not

1 Basic knowledge and motivation

1. (definition): Finite volume element methods, biefly FVEM (called also *box methods* in its early time and *generalized difference methods* in China) discretize integral form of conservation law of differential equations by chosing linear or bilinear finite element spaces as trial spaces.

2. (uses...): FVEM have been widely used in the numerical approximation of partial differential equations because they keep the conservation law of mass or energy.

3. (interpolation...): both finite element and finite volume element methods are both based on the interpolations:

(a) (order of approximation): numerical derivatives have only order k for interpolating polynomials of order k.

(b) (stress points): the previous item does not exclude the possiblity that the approximation of derivatives may have higher order at some points called *stress points*.

(c) (superconvergence): based on the stress points, superconvergence property has been intensively studied.

2 Oulines of the article

2.1 Idea behind the article: stress points

Let consider the reference element $\hat{K} = [-1,1]^2$. Let $\hat{u}$ be a given continuous function defined on $\hat{K} = [-1,1]^2$. $\hat{\pi_2}\,\hat{u}$ denotes the biquadratic interpolation of $\hat{u}$, i.e.

1. $\hat{\pi_2}\,\hat{u} \in \mathbb{Q}_2(\hat{K})$

$$\hat{\pi_2}\,\hat{u} = \sum_{0\le\alpha,\beta\le 2} \xi^\alpha \eta^\beta$$

2.

$$\hat{\pi_2}\,\hat{u}(\hat{a}) = \hat{u}(\hat{a}),\ \forall\,\hat{a} \in \{(-1,\eta),(0,\eta),(1,\eta);\eta \in \{-1,0,1\}\}$$

Let us consider

$$\xi_1 = -\frac{1}{\sqrt{3}},\ \ \xi_2 = \frac{1}{\sqrt{3}} \qquad [1]$$

$$\eta_1 = -\frac{1}{\sqrt{3}},\ \ \eta_2 = \frac{1}{\sqrt{3}} \qquad [2]$$

Some computation leads to

$$\frac{\partial\,\hat{\pi_2}\,\hat{u}}{\partial\,\xi}(\xi_1,\eta_1) = \frac{\partial\ \hat{u}}{\partial\,\xi}(\xi_1,\eta_1) + \frac{\partial^4\ \hat{u}}{\partial\,\xi^4}(\bar{\xi},\bar{\eta}), \qquad [3]$$

where $(\bar{\xi},\bar{\eta})$ is some point in the neigbouring of (ξ_1,η_1).

The previous stated results imply that on an element K, we have

$$\frac{\partial\,\pi_2\,u}{\partial\,x}(x_1,y_1) = \frac{\partial\ u}{\partial\,x}(x_1,y_1) + 0(h^3), \qquad [4]$$

where h is the mesh size.

2.2 Control volumes

Let $\Omega = (0,1)^2$ be the domain problem.

For a given rectangular partition Q_h for Ω, let us denote by (x_i,y_j), $i(j)$, $i = 0,\ldots,2N_x$ ($j = 0,\ldots,2N_y$) denotes the mesh points in x–axis (resp. y–axis). Q_h has N_xN_y elements $E_{ij} = [x_{2i-2},x_{2i}] \times [y_{2j-2},y_{2j}]$. The center of E_{ij} is (x_{2j-1},y_{2j-1}). The control volume associated to

(x_{2j-1}, y_{2j-1}) is the points rectangle formed by the associated points for (ξ_1, η_1), (ξ_1, η_2), (ξ_2, η_1), and (ξ_2, η_2) by the usual bilinear transformation between K and $\hat{K}$.

2.3 Finite volume scheme

Problem is

$$-\Delta u(x) = f(x), \; x \in \Omega, \tag{[5]}$$

with

$$u(x) = 0, \; x \in \partial\Omega. \tag{[6]}$$

The finite volume scheme is based on the integration of [5] on each control volume described in the previous subsecion, and then we consider the finite volume element solution u_h as a bilinear function.

2.4 Superconvergence property

The result [3] is the key of the following result:

1. a bilinear finite volume element approximation for the solution Poisson's problem
2. the vetices of the control volumes are stress points for the primitive biquadratic interpolation
3. the numerical gradients of the method have h^3–superconvergence order at optimal stress points.
4. the convergence order in L^2–norm is h^4 at nodal points.

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