A brief Report on the article "[LYA 09]: High–accuracy schemes of the finite element method for systems of degenerate elliptic equations on an interval"

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Examples to introduce to the results of the article

Let us consider the following simple example: find a function real $u(x)$

$$-\left(\sqrt{x}u_x(x)\right)_x = 0,\ x \in \Omega = (0,1), \quad [1]$$

with

$$u(0) = g \text{ and } u(1) = 0. \quad [2]$$

Equation [1] implies that, for some constant C_1

$$\sqrt{x}u_x(x) = C_1, \quad [3]$$

which is equivalent to

$$u_x(x) = C_1 x^{-\frac{1}{2}}. \quad [4]$$

Integrating this last equality leads to, for some constant C_2

$$u(x) = 2C_1 x^{\frac{1}{2}} + C_2,\ x \in \Omega. \quad [5]$$

Replacing $x = 0$ in the previous expansion and using $u(0) = g$, we get

$$C_2 = g. \quad [6]$$

Equation [5] becomes then

$$u(x) = 2C_1 x^{\frac{1}{2}} + g,\ x \in \Omega. \quad [7]$$

Replacing $x = 1$ in the previous expression and using $u(1) = 0$, we get

$$C_1 = -\frac{g}{2}, \quad [8]$$

this with [7] implies

$$u(x) = -g x^{\frac{1}{2}} + g,\ x \in \Omega. \quad [9]$$

As we can remark that $u(x)$ can be written as

$$u(x) = x^{\frac{1}{2}}\hat{u}(x) + g.1,\ x \in \Omega, \quad [10]$$

where

$$\hat{u}(x) = -g,\ \forall\, x \in \Omega, \quad [11]$$

Abstract

This paper deals with a finite element scheme for a system of degenerate second order elliptic equations in one dimension. It is known that the exact solution in such problems have unbounded derivatives near degeneration points. Therefore, the discretization of such problems by the standard finite element method leads to loss of convergence of approximate solution to the exact one near degeneration points of coefficients of the differential operator of the problems. The finite element

scheme suggested by the authors is based on a representation of the exact solution in terms of some known functions and unknown but smooth function. In the case of the Dirichlet problem with homogeneous boundary conditions, the authors prove that the exact solution is a product of two functions. The first function is very simple and describes the behaviour of the exact solution near the the degeneration points of coefficients of the differential equations, and, the second function is treated as a new desired smooth function.
In the case of the Dirichlet problem with non homogeneous boundary conditions, the authors use the so called continuation function (of boundary values into the domain) to reduce the problem to a new problem with homogeneous Dirichlet boundary conditions. A construction for the continuation function is provided. By this way, the authors proved that the exact solution of the original problem can be represented as the sum of two terms. The first term is known and the second one is the product of a known function and a unknown smooth function.
In the both cases of homogeneous and non homogeneous boundary conditions and thanks to the above representation for the exact solution, the authors applied the usual finite element method to approximate the unknown smooth function which is the solution of a new system of smooth second order elliptic equations in one dimension. One remarks that this new unknown function is smooth, one can approximate it by piecewise polynomial function with a higher convergence order. This last finite element approximation allows us, thanks to tha above stated representation, to obtain an approximation, for the exact solution of the original system, with optimal order in weighted Sobolev norms.

1 Comments, and remarks

- I think it is useful to use the framework, which is based on some functional tools, of the present article [LYA 09] to provide a finite volume scheme.
- the boundary conddition [LYA 09, (4), Page 20], it is also the subject of [27], is not so clear; perhaps to asked from the authors.
- definition of the weighted Sobolev spaces is not so clear, see Section of typos below
- why it is chosen to find $\varphi(x)$ as a polynome only on $(0, \delta)$ and not on $(0, 1)$,

2 Typos

- the boundary conddition [LYA 09, (4), Page 20]: there is some typos; perhaps , it the true boundary condition is [27]
- definition of the weighted Sobolev spaces; it is said in [LYA 09, Page 18] that "$H^s_\gamma(\Omega)$ stands for the space of functions for which the seminorm $\| D^s u\|_{L^{2,\gamma}}$ is finite". I think that the right definition is $H^s_\gamma(\Omega)$ stands for the space of functions for which the seminorms $\| D^j u\|_{L^{2,\gamma}}$,

$j \in \{0, \ldots, s\}$ are finite". In fact, if we use the definition of the authors, we can not obtain, for instance, $H^{s+2}_{\gamma-1}(\Omega) \subset H^{s+1}_{\gamma}(\Omega)$. The previous stated result is used in [LYA 09, Line 16, Page 21].

- I think that, cf. [LYA 09, Definition of continuation function, Page 22], the first part in [LYA 09, 6, Page 22] is $\varphi(0) = E$ and not $\varphi(0) = I$,
- I think that, there is a typos [LYA 09, Line after (12), Page 26] to be removed, would say "here $u_h(x) = \sigma(x)\hat{u}_h(x)$".

3 Some literature

It is well known that, see for instance [SMI 81, LIZ 81, KYD 89, TIM 00], that a solution to an elliptic equation, with Dirichlet boundary conditions, with degenerate coefficients has unbounded derivatives near degeneration points. Therefore, the discretization of such problems by the standard Galerkin method leads to loss of convergence of approximate solutions to the exact one near degeneration points of coefficients of the differential operator of the problem. There are several approaches to manage with the numerical approximation of the class of these equations. Among these approaches, the authors, of the article under consideration, quote :

- in [GUS 65], a finite difference scheme is introduced when the variables are chenged to new coordinates.
- in [RUK 87, LYA 93, TIM 94, TIM 94, KAR 99], the authors provide finite element schemes based on the mesh–refinement around the singular points. This technique leads to big systems to solved when the size decreases towards zero.
- *the multiplicative extraction method:* it is used [TIM 00]. It consists the representation of the exact solution as the product of two functions:
  - the first function is very simple and describes the behaviour of the exact solution near the the degeneration points of coefficients of the differential equations,
  - the second function is treated as a new desired function. According to [TIM 03], this function is smooth and then one could approximate it with the help of usual finite element on quasiuniform grid. This method then leads to standard finite element method with *optimal convergence*.

4 Aim of the article under consideration

The previous papers, which related with the numerical approximation of equations having degenerate coefficients, consider homogeneous Dirichlet boundary conditions. The aim of the article under consideration is:

- to consider systems of equations degenerate coefficients with non homogeneous Dirichlet boundary conditions,
- to get high–accuracy schemes: the issue used in the article under consideration, i.e. [LYA 09], is to write the exact solution in terms of known functions and unknown but smooth solution of system of second order elliptic equations.

To use the results of equations with Dirichlet boundary condition, the authors use the so called *function of continuation* in which the domain of definition is boundary and takes its values on the domain. In fact, by this way, the author can apply the results concerning *the multiplicative extraction method*. For a smooth right hand side, the authos suggested a finite element scheme with an *optimal* rate of convergence.

5 Problem to be resolved and some useful results

Let $\Omega = (0,1)$ be the problem domain, and $n \in \mathbb{N}^\star$ be a given integer number. The authors consider the following problem with a degeneration at the point $x = 0$:

$$\mathcal{A}\,u = -\left(x^\alpha\, a(x)u_x(x)\right)_x + x^\alpha\, b(x)u(x) = f(x),\ x \in \Omega, \tag{12}$$

with

$$u(0) = g \ \text{ and } u(1) = 0, \tag{13}$$

where here u is a $n \times 1$ is unknown vector.

The following assumptions are assumed to be fulfilled

Assumption 5.1

- $a(x)$ and $b(x$ are $n \times n$ smooth matrices. Perhaps, we should assume that $a(x)$ and $b(x$ are bounded in the sense that $a(x)$ and $b(x)$ are in $\left(L^\infty(\Omega)\right)^{n^2}$.
- g is$n \times 1$ vector,
- $b(x) \geq 0$ and there exists a non–negative real number $a_0 > 0$ such that

$$a(x) \geq a_0, \ \text{ a. e. } x \in \Omega. \tag{14}$$

  (where the matrix notation $A \geq B$ means, as usual, $A\zeta \cdot \zeta \geq B\zeta \cdot \zeta$, for all $\zeta \in \mathbb{R}^n$ and $\zeta \cdot \eta$ denotes the scalar product in $\mathbb{R}^n$ of the two vectors ζ and η.)
- the given number α is assumed to be in $(-1,1)$ (Note that when $\alpha = 0$, equation becomes regular.).

The authors first considered the homogeneous case $g \equiv 0$. The non homogenous case, i.e. $g \not\equiv 0$, can be reduced, thanks to *function continuation*, to the homogenous case.

5.1 The homegeneous case

5.1.1 Existence and uniqueness

We need to define the space $\mathcal{V} = \mathring{H}^1_{-\alpha/2}(\Omega)$. To this end, we need to define $L^{2,\gamma}(\Omega)$. For a real γ, the space $L^{2,\gamma}(\Omega)$ is given by

$$L^{2,\gamma}(\Omega) = \left\{ u : \Omega \to \mathbb{R}^n : \int_\Omega |x^{-\gamma} u(x)|^2 dx < \infty \right\}. \qquad [15]$$

We define the space $H^s_\gamma(\Omega)$, for s nonnegative integer, as follows:

$$H^s_\gamma(\Omega) = \left\{ u \in L^{2,\gamma}(\Omega) : \left(u_1^{(j)}, \dots, u_n^{(j)} \right)^t \in \left(L^{2,\gamma}(\Omega) \right)^n,\ j = 0, \dots, s \right\}. \qquad [16]$$

The space $\mathring{H}^s_\gamma(\Omega)$ stands for the closure (in the norm of $H^s_\gamma(\Omega)$) of the set $\mathcal{C}^\infty_0(\Omega)$ of the infinitely differentiable functions with compact support in Ω.
We also need the space $\dot{H}^s_\gamma(\Omega)$ defined by

$$\dot{H}^s_\gamma(\Omega) = \left\{ u \in H^s_\gamma(\Omega) :\ u^{(j)}(1) = 0,\ j = 0, \dots, s-1 \right\}. \qquad [17]$$

It is cited in [LYA 09, Page 18] that the space $\mathcal{V} = \mathring{H}^1_\gamma(\Omega)$, where $\gamma \le -1/2$, is given by

$$\dot{H}^1_\gamma(\Omega) = \left\{ u \in H^s_\gamma(\Omega) :\ u(1) = 0 \right\}. \qquad [18]$$

(where, as usual, the functions considered here are measurable).

To define a weak formulation for problem [12]–[13], we consider the following bilinear form and linear functional:

$$\mathbf{a}(u, v) = \int_\Omega \left(x^\alpha a(x) u_x(x) \cdot v_x(x) + x^\alpha b(x) u(x) \cdot v(x) \right) dx, \qquad [19]$$

and

$$\mathbf{f}(v) = \int_\Omega f(x) \cdot v(x)\, dx. \qquad [20]$$

The authors used Riez–Fischer Theorem (or also Lax Milgram Lemma) to prove the existence and uniqueness of the following problem, under some condition on α: find $u \in \mathcal{V}$ such that

$$\mathbf{a}(u, v) = \mathbf{f}(v),\ \forall v \in \mathcal{V}. \qquad [21]$$

The following Theorem is proven in the considered article:

THEOREM 5.2 ([LYA 09, Theorem 4, Page 20]) Assume $\gamma \ge \alpha/2 - s - 1$, where s is an integer. Then, problem [21] has a unique solution for any $f \in H^s_\gamma(\Omega)$.

5.1.2 Expression for the exact solution

We move now to give an expression for the exact solution u for problem [21]. Denote by $\sigma(x) = x^{1-\alpha}$. We consider the space $\hat{\mathcal{V}}$ defined by:

$$\hat{\mathcal{V}} = \dot{H}^1_{\alpha/2-1}(\Omega) = \left\{u \in H^1_{\alpha/2-1}(\Omega) : \; u(1) = 0\right\}. \qquad [22]$$

Let us consider the solution $\hat{u} \in \hat{\mathcal{V}}$ satisfying the following equation:

$$\hat{\mathbf{a}}(\hat{u}, \hat{v}) = \hat{\mathbf{f}}(\hat{v}), \;\; \forall \hat{v} \in \hat{\mathcal{V}}, \qquad [23]$$

where

$$\hat{\mathbf{a}}(\hat{u}, \hat{v}) = \mathbf{a}(\sigma\hat{u}, \sigma\hat{v}) \;\text{ and }\; \hat{\mathbf{f}}(\hat{v}) = \mathbf{f}(\sigma\hat{v}). \qquad [24]$$

We remark that

$$u = \sigma\hat{u}. \qquad [25]$$

We solve then the following problem:

$$\hat{\mathcal{A}}\hat{u}(x) \equiv \mathcal{A}(\sigma\hat{u}) = f(x), \;\; x \in \Omega, \qquad [26]$$

and

$$\hat{u}(1) = 0. \qquad [27]$$

The authors proved the following Theorem:

THEOREM 5.3 (cf [LYA 09, Theorem 5, Page 21]) Assume that $\alpha - s - 3/2 < \gamma < 1/2$, $\mathbf{a} \in W^{s+1}_{\infty}$, and $x\,\mathbf{b} \in W^s_{\infty}$. Then the operator $\hat{\mathcal{A}}$ given by [26] is an isomorphism of the space $H^{s+2}_{\gamma-1}(\Omega) \cap \dot{H}^1_{\gamma}(\Omega)$ on $H^s_{\gamma}(\Omega)$. In addition, a solution to problem [26]–[27] satisfies the boundary conditions

$$\lim_{x\to 0} x^{2-\alpha}\hat{u}_x(x) = 0. \qquad [28]$$

Remark 1 (Boundary condition) In the boundary condition [28], $\hat{u}$ may be bounded and may be not bounded around $x = 0$. As an example, when we take $\hat{u} = \frac{1}{x}$ and $\alpha = 0$. We remark that $\lim_{x\to 0} x^2\hat{u}_x(x)$ whereas $\hat{u} = \frac{1}{x}$ is unbounded.

The following remark is, I think, useful:

COROLLARY 5.4 If the coefficients $a(x)$ and $b(x)$ are belonging to $\mathcal{C}^{\infty}(\overline{\Omega})$, then a solution $\hat{u}(x)$ to [26]–[27] also is a function in $\mathcal{C}^{\infty}(\overline{\Omega})$.

6 Non homogeneous case

6.1 Continuation function

DEFINITION 6.1 (Definition of the continuation function) A vector φ is called a continuation function for the class of right–hand side $H^s_{\gamma}(\Omega)$ of problem [12]–[13] if:

- $$\mathcal{A}\varphi \in H^s_\gamma(\Omega),$$

- $$\varphi(0) = E,$$

- $$\varphi(1) = 0,$$

where E denotes the $n \times 1$ vector whose all the compoenents are equal to 1.

We fix $\delta \in (0,1)$; we compute a continuation function φ on the interval $(0,\delta)$ and then we extend φ to $(0,1)$.

We consider the Taylor expansions of order s for the matrices $a(x)$ and $b(x)$:

$$\mathbb{T}_a(x) = \sum_{i=0}^{s} a_i x^i = \sum_{i=0}^{s} \frac{u^{(i)}}{i!} x^i, \qquad [29]$$

and

$$\mathbb{T}_b(x) = \sum_{i=0}^{s} b_i x^i = \sum_{i=0}^{s} \frac{b^{(i)}}{i!} x^i. \qquad [30]$$

We seek the continuation function $\varphi(x)$ in the form:

$$\varphi(x) = \sum_{k=0}^{s+2} \varphi_k x^k, \qquad [31]$$

and to verify

$$\tilde{\mathcal{A}}\varphi \equiv -\left(x^\alpha \mathbb{T}_a(x)\varphi_x(x)\right)_x + x^\alpha \mathbb{T}_b(x)\varphi(x) = \sum_{i=s+1}^{2s+2} c_i x^{i+\alpha},\ x \in (0,\delta), \qquad [32]$$

$$\varphi(0) = E, \qquad [33]$$

$$x^\alpha \varphi_x(0) = 0. \qquad [34]$$

Substituting [31] in [32] and grouping the terms with identical degrees of x yields correlations for φ_k and c_j.

After having computed $\varphi(x)$ for $x \in (0,\delta)$ as a vector function such that each its components is a polynomial of degree $s+2$ and the differential equation [32] is fulfilled. One continue $\varphi(x)$ onto the segment $[\delta,1]$ for a given in advance natural k such that $k \geq s+2$ by arbitrary function from $\mathcal{C}^k[\delta,1]$ under the following conditions

- $$\varphi(1) = 0, \qquad [35]$$

- $$\lim_{h\to 0, h<0} \varphi^{(j)}(\delta) = \lim_{h \to 0, h>0} \varphi^{(j)}(\delta),\ j = 0,\ldots,k. \qquad [36]$$

For example, one can choose $\varphi(x)$ on $[\delta, 1]$ as a vector function that satisfies the mentioned conditions, provided that its each components is a Hermite polynomial of degree $k+1$. Then the piecewise polynomial vector function $\varphi(x)$ belongs to $\mathcal{C}^k[0,1]$.

Remark 2 (Comments on the choice of the continuation function) I find the previous construction of φ is nice. Since I'm still learning useful knowledge from this article (I would say the article [LYA 09]), I have some remarks

- why it is chosen to find $\varphi(x)$ as a polynome only on $(0,\delta)$ and not on $(0,1)$,
- it is nice to indicate the space to which $\varphi(x)$ belongs. Since ϕ satisfies $\varphi(x) \in \mathcal{C}^\infty[0,\delta]$, and for a given in advance natural k such that $k \geq s+2$, $\varphi(x) \in \mathcal{C}^k[\delta, 1]$, then $\varphi(x) \in \mathcal{C}^k[0,1] \diagdown \{0\}$. This with the conditions [36] implies that $\varphi(x) \in \mathcal{C}^k[0,1]$.
- Let Ψ be the $n \times n$ matrix whose the diagonal equals to $\varphi(x)$ and the other compenents equal to zero. One remarks that g is a $n \times 1$, one could deduce that $\Psi\, g$ is a $n \times 1$ vector. The following properties of $\Psi\, g$ will be used below:
  - regularity of $\Psi\, g$: since $\varphi(x)\mathcal{C}^k[0,1]$, then $\Psi\, g(x)\mathcal{C}^k[0,1]$ (recall that g is $n \times 1$ constant vector) in the sense that each compenent of $\Psi\, g(x)$ belongs to $\mathcal{C}^k[0,1]$
  - boundary value of $\Psi\, g$: since φ built above satisfies [33] and [35], then $\Psi\, g(0) = 0$ and $\Psi\, g(1) = g$,
  - under the conditions that $a(x)$ and $b(x)$ have bounded derivatives of order $s+1$, $\mathcal{A}(\Psi\, g) \in H^s_\gamma(\Omega)$, see [LYA 09, Theorem 6, Page 23] and Theorem 6.3 given below.

The following Lemma is used to prove that $\varphi(x)$ is a continuation function for the class H^s_γ:

LEMMA 6.2 ([LYA 09, Lemma 4, Page 23])

Let $p(x)$ be a vector function having a bounded derivative of order $s+1$, and $p^{(j)}(0) = 0$, for all $j \in \{0, \ldots, s\}$. Then the following estimate holds:

$$|\, p^{(j)}(x)| \leq \frac{C}{(s+1-j)!} x^{s+1-j},\ j \in \{0, \ldots, s\}. \quad [37]$$

Proof Since $p^{(s)}(0) = 0$, we have then

$$u^{(s)}(x) = \int_O^x u^{(s+1)}(t)dt, \quad [38]$$

which implies that

$$|\, u^{(s)}(x)| \leq x\, \|\, u^{(s+1)}\|_\infty, \quad [39]$$

where

$$\|\, u^{(s+1)}\|_\infty = \sup_{[0,1]} |\, u^{(s+1)}(t)|. \quad [40]$$

One remarks that, since $p^{(s-1)}(0) = 0$

$$u^{(s-1)}(x) = \int_O^x u^{(s)}(t)dt, \quad [41]$$

and use [39]

$$| u^{(s-1)}(x)| \leq \frac{\| u^{(s+1)}\|_\infty}{2} x^2. \qquad [42]$$

One remarks that, since $p^{(s-2)}(0) = 0$

$$u^{(s-2)}(x) = \int_O^x u^{(s-1)}(t)dt. \qquad [43]$$

This with [42] implies that

$$| u^{(s-2)}(x)| \leq \frac{\| u^{(s+1)}\|_\infty}{6} x^3. \qquad [44]$$

In general case, we obtain for all $j \in \{0, \dots, s\}$

$$| u^{(s-j)}(x)| \leq \frac{\| u^{(s+1)}\|_\infty}{(j+1)!} x^{j+1}, \qquad [45]$$

which completes the proof. □

The following Theorem confirms that the function $\varphi(x)$ described in Subsection 6.1 is a continuation function in the sense of Definition 6.1.

THEOREM 6.3 ([LYA 09, Theorem 6, Page 23]) Assume that $a(x)$ and $b(x)$ have bounded derivatives of order $s+1$. Let γ be a real such that $\gamma < 1/2$ and $\varphi(x)$ be the continuation function descibed in Subsection 6.1 for the class of right hand side $H^s_\gamma(\Omega)$ for problem [12]–[13], in the sense of Definition 6.1.

The following theorem provides with a representation for the exact solution u of [12]–[13]:

THEOREM 6.4 ([LYA 09, Theorem 7, Page 24]) Assume that $a(x)$ and $b(x)$ have bounded derivatives of order $s+1$ and let γ be a real number such that $\gamma < 1/2$. Then the function $\varphi(x)$ descibed in Subsection 6.1 is a continuation function for the class of right hand side $H^s_\gamma(\Omega)$ for problem [12]–[13], in the sense of Definition 6.1. Assume in addition that $\alpha - s - 3/2 < \gamma < 1/2$. Then, for any $f \in H^s_\gamma(\Omega)$, there exists a unique solution u for the problem [12]–[13] and the solution u can be written as:

$$u(x) = \Psi(x)g + x^{1-\alpha}\hat{u}(x), \qquad [46]$$

where

$$\hat{u} \in H^{s+2}_{\gamma-1}(\Omega) \cap \dot{H}^1_\gamma(\Omega), \qquad [47]$$

$$\|\hat{u}\|_{H^{s+2}_{\gamma-1}} \leq C \left(\| f \|_{H^s_\gamma} + | g | \right), \qquad [48]$$

and $\Psi(x)$ is a $n \times n$ matrix whose diagonal equals to $\varphi(x)$, while the rest components equal to zero.

7 A finite element scheme

Assume that $\alpha - m < \gamma < 1/2$, $f \in H^{m-1}_\gamma(\Omega)$, $a, b \in W^m_\infty(\Omega)$. Set $s = m-1$ and applying Theorem 6.4, since the assumptions of the theorem hold, to get for $\sigma(x) = x^{1-\alpha}$

$$u(x) = \Psi(x)g + x^{1-\alpha}\hat{u}(x), \quad \text{a e } x \in (0,1), \qquad [49]$$

where $\hat{u}(x)$ is the solution of

$$\mathbf{a}\left(\sigma\hat{u},\sigma\hat{v}\right)=\int_{\Omega}\sigma(x)f(x)\hat{v}(x)dx-\mathbf{a}\left(\sigma\Psi(x)g,\sigma\hat{v}\right),\ \forall\,\hat{v}\in\hat{\mathcal{V}}, \qquad [50]$$

the space $\hat{\mathcal{V}}$ is defined in [22], that is

$$\hat{\mathcal{V}}=\dot{H}^{1}_{\alpha/2-1}(\Omega)=\left\{u\in H^{1}_{\alpha/2-1}(\Omega)\,:\,u(1)=0\right\}. \qquad [51]$$

Let $\cup_{i=0}^{N}I_i=\cup_{i=0}^{N}[x_i,x_{i+1}]$ be a subdivision of the interval $[0,1]$; and consider the space $\mathcal{S}_m$ of piecewise polynomes of degree m. We consider the following approximation of the space $\hat{\mathcal{V}}$:

$$\hat{\mathcal{V}}_h=\left\{u\in\mathcal{S}_m\,:\,u(1)=0\right\}. \qquad [52]$$

We define now an approximation $\hat{u}_h\in\hat{\mathcal{V}}_h$ for $\hat{u}$:

$$\mathbf{a}\left(\sigma\hat{u}_h,\sigma\hat{v}_h\right)=\int_{\Omega}\sigma(x)f(x)\hat{v}_h(x)dx-\mathbf{a}\left(\sigma\Psi(x)g,\sigma\hat{v}_h\right),\ \forall\,\hat{v}_h\in\hat{\mathcal{V}}_h. \qquad [53]$$

We define now an approximation u_h for the exact solution u of the original system [12]–[13] as follows:

$$u_h(x)=\Psi(x)g+x^{1-\alpha}\hat{u}_h(x),\ \forall\,x\in[0,1]. \qquad [54]$$

There following result is the principal one concerning the convergence of the finite element scheme suggested by the authors:

Theorem 7.1 ([LYA 09, Theorem 9 and Corollary 2, Page 26]) Assume that $a(x)$ and $b(x)$ have bounded derivatives of order $s+1$ and let γ be a real number such that $\gamma<1/2$. Then the function $\varphi(x)$ descibed in Subsection 6.1 is a continuation function for the class of right hand side $H^{s}_{\gamma}(\Omega)$ for problem [12]–[13], in the sense of Definition 6.1. Assume in addition that $\alpha-s-3/2<\gamma<1/2$. Then, for any $f\in H^{s}_{\gamma}(\Omega)$, there exists a unique solution u for the problem [12]–[13] and the solution u can be written as [46].
Let $s=m-1$ assume that $a,b\in W^{m}_{\infty}(\Omega)$ and $f\in H^{m-1}_{\gamma}(\Omega)$. Let $\hat{u}_h\in\hat{\mathcal{V}}_h$ be the finite element approximation given by [53]. Let u_h be given [54]. Then the following error estimate hold:

$$\|\,u-u_h\|_{H^{1}_{-\alpha/2}}\le Ch^{\theta}\left(\|\,f\|_{H^{m-1}_{\gamma}}+|\,g|\right), \qquad [55]$$

where

$$\theta=\min(m,m+\gamma-\alpha/2). \qquad [56]$$

If in addition $f\in W^{m-1}_{\infty}(\Omega)$, estimate [55] becomes as

$$\|\,u-u_h\|_{H^{1}_{-\alpha/2}}\le Ch^{m}\left(\|\,f\|_{W^{m-1}_{\infty}(\Omega)}+|\,g|\right). \qquad [57]$$

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