A brief Report on the article [FEN 10] "Finite element methods for the bi–wave equation modeling d–wave superconductors"

X. Feng and M. Neilan

J. Comput. Math. 28 (2010), 331–353.

Report done by Professor Bradji, Abdallah

Provisional home page: http://www.cmi.univ-mrs.fr/~bradji

Written in Tuesday 8th December, 2010

Abstract: The authors consider the Poisson's equation with a singular perturbation by the bi–wave operator. The problem, then, is a singular perturbed fourth order partial differential equation. The problem appears, for instance, in d–wave superconductors. The aim of the article is to provide a finite element approximation for such problem. A weak formulation for the problem as well as the existence and uniqueness of the weak solution are presented and justified. A low order conforming non–C^1 finite element method, inspired from the ideas of nonconforming and discontinuous methods, is introduced. An optimal convergence order is proved. Numerical example to explain the theoretical are presented.

Key words and phrases: fourth order equation; hyperbolic operator; wave operator; bi–wave operator

Subject classification:65N30; 65N12; 65N15

1 Some remarks

1. there is a useful information concerning the physical origine of the so called wave equation. My hope, I get some time in the future to come back again to this article in order to learn more...

2 Basis knowledge

1. equation to be approximate:

$$\varepsilon\Box^2 u(x) - \Delta\, u(x) = f(x),\ x \in \Omega \qquad [1]$$

   with the following boundary conditions

$$u(x) = \partial_{\mathbf{n}}(x) = 0,\ x \in \partial\Omega, \qquad [2]$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary $\partial\Omega$, $\varepsilon << 1$, and $\bar{\mathbf{n}} = (\mathbf{n}_1, -\mathbf{n}_2)$ with $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2)$ denotes the unit outward normal to Ω and

$$\Box\, u = u_{xx} - u_{yy}, \quad \Box^2 u = \Box(\Box\, u). \qquad [3]$$

So

$$\Box^2 u = u_{xxxx} - 2u_{xxyy} + u_{yyyy}. \qquad [4]$$

2. some physics: operator $\Box$ called the (2D) wave operator. So, $\Box^2$ called the b–wave operator

3. point of view: equation [1] can be viewed as a singular (because of the presence of ε perturbation of Poisson's equation by the bi–wave operator $\Box^2$

4. some comparison: the bi–harmonic operator given by

$$\Delta^2 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy}. \qquad [5]$$

Although there is only sign difference in the mixed derivative term, the difference between the bi–harmonic operator Δ^2, given by [5], and the bi–wave operator $\Box^2$, given by [4], is fundamental because Δ^2 is an elliptic operator while $\Box^2$ is a hyperbolic operator.

5. physical origine of the problem [1]–[2]: superconductors are materials that have no resistence to the flow of electricity when the surrounding temperature is below some *critical temperature*. At the superconducting state, the electrons are beleived to "team up pairwise" despite the fact that the electrons have negative charges and therefore they normally repel each other. The Ginzburg–Landau theory [GIN 65] has been well accepted as a good field theory for low (critical temperature) T_c superconductors. However, a theory to explain high T_c superconductivity still needs modern physics.
Equation [1] can be obtained from the Ginzburg–Landau type d–wave model considered in [DU 99] in absence of applied magnetic field by negelecting the zeroth order nonlinear terms

6. aim of the article: the aim of the article is to develop a finite element method for [1]–[2]

7. functional space: the abvious choice for functional space is

$$\mathcal{V} = \left\{ v \in H^1(\Omega) : \ \Box\, v \in L^2(\Omega) \right\}. \qquad [6]$$

The main task then is

   (a) to provide an approximation $\mathcal{V}_h$ for $\mathcal{V}$, which should be as simple as possible

   (b) $\mathcal{V}_h$ should be rich enough to have good properties.

8. conditions on the approximation of the functional space:

   (a) since $\mathcal{V} \subset H^1(\Omega)$, one we would like to get $\mathcal{V}_h \subset \mathcal{V} \subset H^1(\Omega)$. This can be obtained by the classical assumption that $\mathcal{V}_h$ is piecewis polynomial and $\mathcal{V}_h \subset \mathcal{C}(\Omega)$

(b) since $\mathcal{V}$ is a propre subspace of $H^1(\Omega)$, so the classical choice (Lagrange) of $\mathcal{V}_h$ does not guarantee that $\mathcal{V}_h \subset \mathcal{V}$. To answer this question, one remarks that $H^2(\Omega) \subset \mathcal{V}$, one chooses $\mathcal{V}_h \subset H^2(\Omega)$. This can be achieved when we choose $\mathcal{V}_h$ is a $\mathcal{C}^1$ finite element space, e.g. Argyris finite element space. According to the authors, the use of $\mathcal{C}^1$ finite element needs the use of fifth or higher order polynomials with up to second order derivatives as degree of freedom, see [CIA 78, Chapter 6].So, it turns out that the use of $\mathcal{C}^1$ finite elements is expensive for the bi–wave equation. This motivates the authors to construct a new type of finite elements inspired from nonconforming and discontinuous Galerkin methods.

References

[BRE 08] S. C. Brenner and L. R. Scott: The Mathematical Theory of Finite Element Methods. *3rd Edition, Springer*, 2008.

[CIA 78] P. G. Cirlet: The Mathematical Theory of Finite Element Methods for Elliptic Problems. *North Holand, Amsterdam*, 1978.

[DU 99] Q. Du: Studies of Ginzburg–Landau model for d–wave superconductors. *SIAM. J. Appl. Math.*, **59**:4, 1225–1250, 1999.

[FEN 10] X. Feng and M. Neilan: Finite element methods for the bi–wave equation modeling d–wave superconductors. *J. Comput. Math.*, **28**, 331–353, 2010.

[GIN 65] V. L. Ginzburg and L. D. Landau: On the Theory of Superconductivity. *Zh. Ksper. Teoret. Fiz. 20:1064–1082, 1950 (in Russian), in L. D. Landau, and I. D. ter Haar (Eds), Man of Physics, Pergamon, Oxford*, 138–167, 1965 .