A brief Report on the article "[GAR 09]: Convergence of adaptive finite element methods for eigenvalue problems"

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Abstract

This paper deals with the convergence of adaptive finite element methods for second order elliptic eigenvalue problems. The authors consider the Lagrange finite elements of any degree and they prove the convergence of the simple as well the multiple eigenvalues under a minimal refinement of marked elements for all reasonable marking, strategies, and starting from any initial triangulation.

1 Basic knowledge

In many practical applications, it is of interest to find or to approximate the eigenvalues and eigenfunctions of elliptic equations. Finite element approximations for these problems have been widely used and analysed under a general framework. Optimal *a priori* error estimates for the eigenvalues and eigenfunctions have been obtained, see e.g. [BAB 89, BAB 91, RAV 83, STR 73] and the references therein.

Adaptive finite element methods are an effective tool for making an efficient use of the computational resources; for certain problems, it is even indisponsable to their numerical resolvability. A quite popular, natural adaptive version of classical finite element methods consists of the following loop:

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE},$$

that is: solve for the finite element solution on the current grid, compute the *a posteriori* error estimator, mark with its help elements to be subdivided, and refine the current grid into a new, finer one. The ultimate goal of adaptive methods is to equidistribute the error and the computational effort obtaining a sequence of meshes with optimal complexity. Historically, the first step to prove optimality has always been to understand convergence of adaptive methods. A general result of convergence for linear problems has been obtained by Morin et *al.* [MOR 09], where very geenral conditions on the linear problems and the adaptive methods that guarantee convergence are stated.

The goal of the article under consideration is to analyze the convergence of adaptive finite element methods for the eigenvalue problem consisting in finding $\lambda \in \mathbb{R}$, and $u \not\equiv 0$ such that

$$-\nabla \cdot (\mathcal{A}\nabla u) = \lambda \mathcal{B} u \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \qquad [1]$$

under general assumptions on $\mathcal{A}$, $\mathcal{B}$ and Ω.

Adaptive finite element method is based on *a posteriori* error estimators, that are computable quantities depending on the discrete solution and the data, and indicate a distribution of the error.

2 Problem to be solved

The authors consider the following problem: let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary (in particular Ω could be a polygonal domain if $d = 2$ and a polyhedral domain if $d = 3$). Let $a, b : H_0^1(\Omega) \rightarrow \mathbb{R}$ be the bilinear forms defined by

$$a(u,v) = \int_\Omega \mathcal{A}\nabla u(x) \cdot \nabla v(x) dx, \qquad [2]$$

and

$$b(u,v) = \int_\Omega \mathcal{B}\, u(x)\, v(x) dx, \qquad [3]$$

where $\mathcal{A}$ is a piecewise $W^{1,\infty}$ symmetric matrix valued function which is uniformly positive definite, i.e. there exist two positive constants a_1 and a_2 such that

$$a_1|\xi|^2 \leq \mathcal{A}(x)\xi\cdot \leq a_2|\xi|^2, \ \forall\, \xi \in \mathbb{R}^d, \ \forall x \in \Omega. \qquad [4]$$

and $\mathcal{B}$ is a scalar product such that

$$b_1 \leq \mathcal{B}(x) \leq b_2|\xi|^2, \forall x \in \Omega, \qquad [5]$$

for some two positives constants b_1 and b_2.

Continuous eigenvalue problem: Find $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$ such that

$$\begin{cases} a(u,v) = \lambda\, b(u,v), \ \forall\, v \in H_0^1(\Omega) \\ \|\, u\|_b = 1, \end{cases} \qquad [6]$$

where $\|\, u\|^2 = b(u,u)$.

It well known, cf. [BAB 89], that under assumptions on $\mathcal{A}$ and $\mathcal{B}$, problem [6] has a countable sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \nearrow \infty \qquad [7]$$

and corresponding eigenfunctions

$$u_1, u_2, u_3, \ldots \qquad [8]$$

which can be assumed to satisfy

$$b(u_i,u_j) = \delta_{ij} := \begin{cases} 1, i = j, \\ 0, i \ne j. \end{cases} \qquad [9]$$

where in the sequence $\{\lambda_j\}_{j\in\mathbb{Z}}$, the value λ_j is repeaded according to its geometric multiplicity. Also, the eigenvalue can be characterized as extrema of the Rayleight quotient $\mathcal{R}(u) = \dfrac{a(u,v)}{b(u,v)}$, by the following relationships:

- *Minimum priciple*

$$\lambda_1 = \min_{u\in H_0^1(\Omega)} \mathcal{R}(u) = \mathcal{R}(u_1) \qquad [10]$$

$$\lambda_j = \min_{u\in \mathcal{W}_j} \mathcal{R}(u) = \mathcal{R}(u_j), \ j = 2, 3, \ldots, \qquad [11]$$

  where

$$\mathcal{W}_j = \{u \in H_0^1(\Omega), a(u,u_i) = 0, \ \forall\, i = 1, \ldots, j-1\}. \qquad [12]$$

- *Minimum–Maximum priciple*

$$\lambda_j = \min_{V_j \subset H_0^1(\Omega),\ \dim\ V_j = j} \ \max_{u\in V_j} \mathcal{R}(u) = \max_{u\in\langle\{u_1,\ldots,u_j\}\rangle} \mathcal{R}(u), \ j = 1, 2, \ldots \qquad [13]$$

Discrete eigenvalue problem We consider a conforming triangulation $\mathcal{T}$ of the domain Ω, that is a partition of Ω into d–simplices such that if two elements intersect, they do so at a full vertex/edge/face of both elements. For any triangulation $\mathcal{S}$ will denote the set of interior sides, where by side we mean an edge if $d = 2$ and a face if $d = 3$. $\kappa_{\mathcal{T}}$ will denote the regularity of $\mathcal{T}$, defined as

$$\kappa_{\mathcal{T}} := \max_{T\in\mathcal{T}} \frac{\mathrm{diam(T)}}{\rho_T}, \qquad [14]$$

where diam(T) is the length of the longest edge of T, and ρ_T is the radius of the largest ball included in T. It is also to define the mesh size $h_{\mathcal{T}} := \max_{T\in\mathcal{T}} h_T$, where $h_T := |\,T|^{\frac{1}{d}}$.
Let $l \in \mathbb{N}$ be fixed, and let $\mathbb{V}_{\mathcal{T}}$ be the element space

$$\mathbb{V}_{\mathcal{T}} := \left\{v \in H_0^1(\Omega) : v|_T \in \mathcal{P}_l(T),\ \forall\, T \in \mathcal{T}\right\}, \qquad [15]$$

Obviously $\mathbb{V}_{\mathcal{T}} \subset H_0^1(\Omega)$ and if $\mathcal{T}_\star$ is a rafinement of $\mathcal{T}$, then $\mathbb{V}_{\mathcal{T}} \subset \mathbb{V}_\star$. We consider then the approximation of the continuous eigenvalue problem [6] as follows: Find $\lambda_{\mathcal{T}} \in \mathbb{R}$ and $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ such that

$$\begin{cases} a(u_{\mathcal{T}}, v_{\mathcal{T}}) = \lambda_{\mathcal{T}}\, b(u_{\mathcal{T}}, v_{\mathcal{T}}),\ \forall\, v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}} \\ \|\, u_{\mathcal{T}}\|_b = 1. \end{cases} \qquad [16]$$

We have similar results to that of [6]. [16] has a countable sequence of eigenvalues

$$1, \mathcal{T}0 < \lambda_{1,\mathcal{T}} \le \lambda_{2,\mathcal{T}} \le \lambda_{3,\mathcal{T}} \le \ldots \le \lambda_{N_{\mathcal{T}},\mathcal{T}}, \qquad [17]$$

where $N_{\mathcal{T}} := \dim\, \mathbb{V}_{\mathcal{T}}$ and corresponding eigenfunctions

$$u_{1,\mathcal{T}}, u_{1,\mathcal{T}}, u_{1,\mathcal{T}}, \ldots, u_{N_{\mathcal{T}},\mathcal{T}} \qquad [18]$$

which can be assumed to satisfy

$$b(u_{i,\mathcal{T}}, u_{j,\mathcal{T}}) = \delta_{ij}. \qquad [19]$$

The discrete eigenvalue can be characterized as extrema of the Rayleight quotient $\mathcal{R}(u) = \dfrac{a(u,v)}{b(u,v)}$, by the following relationships:

- *Minimum priciple*

$$\lambda_{1,\mathcal{T}} = \min_{u\mathbb{V}_{\mathcal{T}}} \mathcal{R}(u) = \mathcal{R}(u_{1,\mathcal{T}}) \qquad [20]$$

$$\lambda_{j,\mathcal{T}} = \min_{u\in\mathcal{W}_j^{\mathcal{T}}} \mathcal{R}(u) = \mathcal{R}(u_{j,\mathcal{T}}),\ j = 2, 3, \ldots, \qquad [21]$$

  where

$$\mathcal{W}_j^{\mathcal{T}} = \{u \in \mathbb{V}_{\mathcal{T}}, a(u, u_i) = 0,\ \forall\, i = 1, \ldots, j-1\}. \qquad [22]$$

- *Minimum–Maximum priciple*

$$\lambda_{j,\mathcal{T}} = \min_{\mathbb{V}_{j,\mathcal{T}}\subset\mathbb{V}_{\mathcal{T}},\ \dim\,\mathbb{V}_{j,\mathcal{T}}=j}\ \max_{u\in\mathbb{V}_{j,\mathcal{T}}} \mathcal{R}(u) = \max_{u\in\langle\{u\in u_{1,\mathcal{T}},\ldots,u_{j,\mathcal{T}}\}\rangle} \mathcal{R}(u),\ j = 1, 2, \ldots \qquad [23]$$

3 The main result

THEOREM 3.1 Let λ_k and u_k be the eigenvalue and eigenfunction of an adaptive finite element algorithm described in [GAR 09]. Then, there exists an eigenvalue λ of the continuous problem such that

$$\lim_{k\to\infty} \lambda_k = \lambda \text{ and } \lim_{k\to\infty} \mathrm{dist}_{H_0^1(\Omega)}(u_k, M(\lambda)) = 0, \qquad [24]$$

where $M(\lambda)$ denotes the set of all eigenfunctions of the continuous problem corresponding to the eigenvalue λ.

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