A brief Report on the article "Stability and convergence of the spectral Galerkin method for the Cahn–Hilliard equation"

Yinnian He, Yunxian Liu

Numer. Methods Partial Differential Eq. 24, 1485–1500, 2008.

Report done by Dr. Bradji, Abdallah

Provisional home page: http://www.cmi.univ-mrs.fr/~bradji

1 Equation to be solved

It is considered the following Cahn–Hilliard equation:

$$\partial_t u(x,t) + \Delta(u(x,t) - u^3(x,t) + \kappa\Delta u(x,t)) = 0,\ (x,t) \in \Omega \times \mathbb{R}^+, \qquad [1]$$

$$u(\cdot,t) \text{ is } L-\text{periodic for all } t \in \mathbb{R}^+, \qquad [2]$$

$$u(x,0) = u_0(x),\ x \in \Omega. \qquad [3]$$

Here the domain Ω is the open set $(0,L_1)\times(0,L_2)$ of $\mathbb{R}^2$, $\partial_t u = \frac{\partial u}{\partial t}$, κ is a positive constant, (L_1,L_2), $u_0 : \Omega \to \mathbb{R}$ is a given function.
It is to useful to test the conservation of the total mass in the following sense: using equation [1], an integration by part, and [2], we get

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega u(x,t)dx &= -\int_\Omega \Delta(u(x,t) - u^3(x,t) + \kappa\Delta u(x,t))dx\\ &= -\int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}}(u(x,t) - u^3(x,t) + \kappa\Delta u(x,t))dx. \qquad [4]\end{aligned}$$

Would be nice then if it is mentioned in the article if this previous property is satisfied by the spectral Galerkin scheme!

2 Plan of this article

- Definition of a weak solution to [1]–[2]:

Definition 2.1 A function $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is called a weak solution for [1]–[3], if $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^2_{\text{per}}(\Omega))$ and $\partial_t u L^2(0,T;H^{-2}_{\text{per}}(\Omega))$ for all $T > 0$ such that for all $v \in H^2_{\text{per}}(\Omega)$ there holds:

$$(\partial_t u, v) + \big(\nabla(u^3 - u), \nabla v\big) + \kappa(\Delta u, \Delta v) = 0, \ \forall\, 0 < t < T, \tag{5}$$

with the initial condition $u(0) = u_0$, where $(\cdot,\cdot)$ is the usual notation of the inner product in $L^2(\Omega)$.

Remark 1 Under the hypothesis $u \in L^2(0,T;H^2(\Omega))$ and $\partial_t u \in L^2(0,T;H^{-2}_{\text{per}}(\Omega))$, we get, thanks to [Evans, Theorem 3, Page 287], $u \in \mathcal{C}(0,T;L^2(\Omega))$ which gives a sense for the unitial condition $u(0) = u_0$.

- A spectral Galerkin method for [1]–[2]
- Lemma 2.1.: some known results concerning relations between some norms and other results concerning some convergence results in spectral methods
- Lemma 2.2.: a uniform version for Gronwell Lemma
- [YIN 08, Theorem 2.3, Page 1488]: existence and uniqueness of the spectral Galerkin solution (to this end it only assumed $u_0 \in L^2(\Omega)$); it is the subject of [YIN 08, Theorem 2.3, Page 1488] . The techniques used in this item are:
  - some known results in the theory of initial–value problems of the ordinary differential equations
  - the previous stated uniform version for Gronwell Lemma.
- Stability of the spectral Galerkin solution: they are obtained the following stability results:
  - [YIN 08, Theorem 2.4, Page 1489]: first energy inequalities of the spectral Galerkin approximate solution. Techniques used in the Proof consist of some integrations and the use of Lemma 2.1
  - [YIN 08, Theorem 2.5, Page 1490]: second energy inequalities of the spectral Galerkin approximate solution. Techniques used in the Proof consist of some integrations, the use of Lemma 2.1., and Gronwell Lemma.
  - [YIN 08, Theorem 2.6, Page 1492]: second energy inequalities of the spectral Galerkin approximate solution. Techniques used in the Proof consist of some integrations, the use of Lemma 2.1., Young inequality, and Gronwell Lemma.
- Convergence of the spectral Galerkin method:
  - since $u_0 \in H^1_{\text{per}}(\Omega)$, one could apply [YIN 08, Theorem 2.4, Page 1489] to get $u_N \in L^\infty(\mathbb{R}^+,T;L^2(\Omega)) \cap L^2(0,T;H^2_{\text{per}}(\Omega))$ and $\partial_t u_N \in L^2\left(0,T;H^{-2}_{\text{per}}(\Omega)\right)$, and the following a priori estimate

$$\|u_N\|_{L^\infty(\mathbb{R}^+;L^2(\Omega))} + \|u_N\|_{L^2(0,T;H^{-2}_{\text{per}}(\Omega))} + \|\partial_t u_N\|_{L^\infty\left(0,T;H^{-2}_{\text{per}}(\Omega)\right)} \le C_T, \ \forall T > 0, \tag{6}$$

where C_T is a positive constant depending on (u_0, T, Ω, κ).

This with compactness result, given in [TEM 83] implies the existence of a subsequence of the $\{u_N, N = 1, \ldots, \infty\}$, and a function u such that $u \in L^\infty(\mathbb{R}^+, T; L^2(\Omega)) \cap L^2(0, T; H^2_{\text{per}}(\Omega))$ and $\partial_t u \in L^2\left(0, T; H^{-2}_{\text{per}}(\Omega)\right)$ in which some convergence of u_N towards u, as $N \to \infty$, holds (it is well given in [YIN 08, (3.9)–(3.12)]).

- Passing to the limt in the scheme and using the previous stated convergence, we get that u satisfies the weak formulation given in the Definition 2.1.
- We prove that u satisfies [5], with $u(x, 0) = u_0$, is unique
- The previous two items yields the convergence of the whole sequence $\{u_N\}_1^\infty$ (not only a subsequence) convergence to u, in the sense of [YIN 08, (3.9)–(3.12)], such that $u \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0, T; H^2_{\text{per}}(\Omega))$ and $\partial_t u \in L^2\left(0, T; H^{-2}_{\text{per}}(\Omega)\right)$ and u is the unique weak solution given in the Definition 2.1.
- it is useful to notify here that the convergence of the spectral Galerkin method of CH equation yields an existence of a weak solution to CH equation.
- the previous result is proven when only $u_0 \in H^1_{\text{per}}(\Omega)$. If we assume more regularity on the data (which yields more regularity on the exact solution), $u_0 \in H^4_{\text{per}}(\Omega)$, an error estimate between the exact solution and the spectral Galerkin approximate solution is given in [YIN 08, Theorem 3.1, Page 1497]. It useful to notify that this error estimate is given in the everage norm $L^2(\Omega)$ for all $t \geq 0$.

Remark 2 (Typos)

- In the second line of the Proof of Theorem 3.1, Page 1494, it is written "$u_N \in L^\infty(\mathbb{R}^+, T; L^2(\Omega))$". I think the right statement is "$u_N \in L^\infty(\mathbb{R}^+; L^2(\Omega))$".
- I think that the last term on the left hand side of (3.8), Page 1494, is $\|\partial_t u_N\|_{L^2\left(0,T;H^{-2}_{\text{per}}(\Omega)\right)}$ instead of $\|\partial_t u_N\|_{L^\infty\left(0,T;H^{-2}_{\text{per}}(\Omega)\right)}$
- It is remarked that there is a small typos in the article in the page 1496, line just before (3.21): it is written "Thus, taking the limit $N \to \infty$ in (3.21),...". I think right sentence is "Thus, taking the limit $N \to \infty$ in (3.20),..." (would say that (3.21) should be replaced by (3.20).

Acknowledgement

The author of this review is grateful to the Editor of ZMATH who gave him the opportunity to review the nice article of He and Liu. He really learned some nice and useful things from the review of this article.

References

[TEM 83] Roger Temam: Navier –Stokes equations, theory and numerical analysis. *3rd Ed., North–Holland, Amsterdam.* 1983.

[YIN 08] Yinnian He and Yunxian Liu: Stability and convergence of the spectral Galerkin method for the Cahn–Hilliard equation. *Numer. Methods for Partial Differential Eq.*, **24**, 1485–1500. 2008.