A Posteriori Error Analysis and Adaptive Finite Element Methods for Electromagnetic and Acoustic Problems

Zhiming Chen[1]

LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, People's Republic of China. zmchen@lsec.cc.ac.cn

1 Introduction

The objective of this paper is to report some of our recent efforts in exploring the possibility of extending the general framework of adaptive finite element methods based on *a posteriori* error estimates initiated in [BR78] to resolve Maxwell singularities. *A posteriori* error estimates are computable quantities in terms of the discrete solution and known data that measure the actual discrete errors without the knowledge of exact solutions. They are essential in designing algorithms for mesh modification which equi-distribute the computational effort and optimize the computation. The ability of error control and the asymptotically optimal approximation property make the adaptive finite element methods attractive for complicated physical and industrial processes.

The first problem we consider is the time-harmonic Maxwell equation in the bounded domain, that is, the time-harmonic Maxwell cavity problem. It is well-known that the solution of the time-harmonic Maxwell equations could have much stronger singularities than the corresponding Dirichlet or Neumann singular functions of the Laplace operator when the computational domain is non-convex or the coefficients of the equations are discontinuous. For example, for the domains that have "screen" or "crack" parts as indicated in Fig 1, the regularity of the solution is only in $\mathbf{H}^s$ with $s < 1/2$. In this case the $\mathbf{H}^1$-conforming discretization cannot be used directly to solve the time-harmonic cavity problem. One way to overcome the difficulty is to use the so-called singular field method which decomposes the solution into a regular part that can be treated by $\mathbf{H}^1$-conforming Lagrangian finite elements and an explicit singular part [ACS98], [DHL99]. For the mathematical analysis of the singularities of the solutions of Maxwell equations, we refer to [BS87], [BS94], [CD00], and the references therein.

A posteriori error estimates for Nédélec $\mathbf{H}(\mathbf{curl})$-conforming edge elements are obtained in [M98] for Maxwell scattering problems and in [BHHW00] for eddy current problems. The key ingredient in the analysis is the orthogonal Helmholtz decomposition $\mathbf{v} = \nabla\varphi + \Psi$, where for any $\mathbf{v} \in \mathbf{H}(\mathbf{curl};\,\Omega)$, $\varphi \in H^1(\Omega)$, and $\Psi \in \mathbf{H}(\mathbf{curl};\,\Omega)$. Since a stable edge element interpolation operator is not available for functions in $\mathbf{H}(\mathbf{curl};\,\Omega)$, some kind of regularity result for $\Psi \in \mathbf{H}(\mathbf{curl};\,\Omega)$ is required. This regularity result is proved in [M98] for domains with smooth boundary and in [BHHW00] for convex polyhedral domains. The key observation in our analysis is that if one removes the orthogonality requirement in the Helmholtz decomposition, the regularity $\Psi \in \mathbf{H}^1(\Omega)$ can be proved in the decomposition $\mathbf{v} = \nabla\varphi + \Psi$ for a large class of non-convex polygonal domains or domains having screens [BS87], [BS94], see also [DHL99]. Our extensive numerical experiments for the lowest order edge element indicate that for the cavity problem with very strong singularities $\mathbf{H}^s$ $(s < 1/2)$, the adaptive methods based on our *a posteriori* error estimates have the very desirable quasi-optimality property

$$\|\mathbf{E} - \mathbf{E}_k\|_{\mathbf{H}(\mathbf{curl};\,\Omega)} \le C\, N_k^{-1/3},$$

where N_k is the number of elements of the k-th adaptive mesh $\mathcal{M}_k$, and $\mathbf{E}_k$ is the finite element solution over $\mathcal{M}_k$.

The second problem concerns an adaptive perfectly matched layer (PML) technique for solving the time harmonic electromagnetic scattering problem with the perfectly conducting boundary condition. Adaptive PML technique was first proposed in Chen and Wu [CW03] for scattering problem by periodic structures (the grating problem) and in Chen and Liu [CL05] for the acoustic scattering problem in which one uses the a posteriori error estimate to determine the PML parameters. Combined with the adaptive finite element method, the adaptive PML technique provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer.

In the third problem we consider the time-dependent eddy current problems which involve discontinuous coefficients, reentrant corners of material interfaces, and skin effect. Thus local singularities and internal layers of the solution arise. We develop an adaptive finite element method based on reliable and efficient *a posteriori* error estimates for the $\mathbf{H} - \psi$ formulation of eddy current problems with multiply connected conductors. The numerical results indicate that our adaptive method has the following very desirable quasi-optimality property:

$$\eta_{\mathrm{total}} \approx C\, N_{\mathrm{total}}^{-1/4}$$

is valid asymptotically, where η_{total} is the total error estimate (see Theorem 5 below), and $N_{\mathrm{total}} := \sum_{n=1}^{M} N_n$ with M being the number of time steps and N_n being the number of elements of the mesh $\mathcal{T}_n$ at the n-th timestep.

In extending our general methodology of using adaptive PML technique for solving time-domain Maxwell scattering problems, we need to consider the convergence and stability of the time-domain PML methods for Maxwell scattering problems. As a first step we consider here the stability and convergence of the time-domain PML method for acoustic scattering problems. We will consider the well-posedness and the stability of the time-dependent acoustic scattering problem with the radiation condition at infinity, the well-posedness of the unsplit-field PML method for the acoustic scattering problems, and the exponential convergence of the non-splitting PML method in terms of the thickness and medium property of the artificial PML layer. The stability of the time-domain PML method can be proved by combining the stability of original scattering problem and the convergence of the PML method.

2 The time-harmonic Maxwell cavity problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded polygonal domain with two disjoint connected boundaries Γ and Σ. Given a current density $\mathbf{f}$, we seek a time-harmonic electric field $\mathbf{E}$ subject to the perfectly conducting boundary condition on Γ and the impedance boundary condition on Σ

$$\begin{aligned} \nabla \times (\mu_r^{-1} \nabla \times \mathbf{E}) - k^2 \varepsilon_r \mathbf{E} &= \mathbf{f} \quad \text{in} \quad \Omega, && (1)\\ \mu_r^{-1} (\nabla \times \mathbf{E}) \times \mathbf{n} - \mathbf{i}\, k\lambda \mathbf{E}_t &= \mathbf{g} \quad \text{on} \quad \Sigma, && (2)\\ \mathbf{E} \times \mathbf{n} &= 0 \quad \text{on} \quad \Gamma, && (3) \end{aligned}$$

where $\mathbf{n}$ is the unit outer normal of the boundary, $\mathbf{E}_t := (\mathbf{n} \times \mathbf{E}|_\Sigma) \times \mathbf{n}$, ε_r is the complex relative dielectric coefficient, $\mu_r > 0$ is the relative magnetic permeability of the material in Ω, $k > 0$ is the wave number, and $\lambda > 0$ is the impedance on Σ.

Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{L}^2(\Sigma)$ satisfying $\mathbf{g} \cdot \mathbf{n} = 0$ on Σ. The weak formulation of (1) – (3) is: Find $\mathbf{E} \in H_\Gamma(\mathbf{curl};\, \Omega)$ such that

$$a(\mathbf{E}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \overline{\mathbf{v}} + \int_\Sigma \mathbf{g} \cdot \overline{\mathbf{v}}_t \quad \forall \mathbf{v} \in H_\Gamma(\mathbf{curl};\, \Omega), \tag{4}$$

where $H_\Gamma(\mathbf{curl}; \Omega) = \{\mathbf{v} \in H(\mathbf{curl};\, \Omega) \mid \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma \text{ and } \mathbf{v}_t \in \mathbf{L}^2(\Sigma)\}$ and

$$a(\mathbf{E}, \mathbf{v}) := (\mu_r^{-1} \nabla \times \mathbf{E}, \nabla \times \mathbf{v}) - (k^2 \varepsilon_r \mathbf{E}, \mathbf{v}) - \mathbf{i} \int_\Sigma k\, \lambda\, \mathbf{E}_t \cdot \overline{\mathbf{v}}_t\, .$$

The existence and uniqueness of the solution of the problem (4) under various conditions on the domain Ω, the coefficients ε_r, μ_r have been studied in [M03]. Here for the sake of simplicity we simply assume that the problem (4) has a unique solution. Thus there exists a constant $\beta > 0$ depending only on Ω, ε_r, μ_r, λ and the wave number k such that [BA73, Chapter 5]

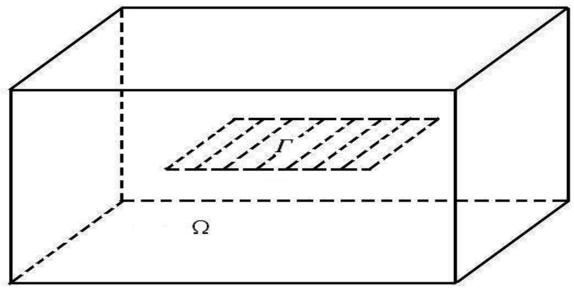

Fig. 1. A domain with screen Γ.

$$\sup_{0 \neq \mathbf{v} \in H_\Gamma(\mathbf{curl};\,\Omega)} \frac{a(\mathbf{E},\,\mathbf{v})}{\|\mathbf{v}\|_{H_\Gamma(\mathbf{curl};\,\Omega)}} \geq \beta \|\mathbf{E}\|_{H_\Gamma(\mathbf{curl};\,\Omega)}. \tag{5}$$

For definiteness we assume in this section that Γ is a Lipschitz screen such that $\Omega \cup \Gamma$ is a Lipschitz domain (see Figure 1) and refer to the discussion of general cases to [CWZ07]. We recall that a surface $\digamma$ is called a Lipschitz screen, if it is a bounded open part of some two-dimensional C^2-smooth manifold such that its boundary $\partial\digamma$ is Lipschitz continuous and $\digamma$ is on one side of $\partial\digamma$. The following decomposition theorem whose proof can be found in [BS87], [BS94], [DHL99], [CWZ07] plays an important role in the forthcoming a posteriori error analysis.

Theorem 1. *For any* $\mathbf{v} \in H(\mathbf{curl};\,\Omega)$ *satisfying* $\mathbf{v} \times \mathbf{n} = 0$ *on* Γ*, there exists a function* $\mathbf{v}_s \in \mathbf{H}^1(\Omega)$ *satisfying* $\mathbf{v}_s \times \mathbf{n} = 0$ *on* Γ *and* $\varphi \in H^1_\Gamma(\Omega)$ *such that*

$$\mathbf{v} = \nabla\varphi + \mathbf{v}_s \quad \text{in } \Omega,$$
$$\|\mathbf{v}_s\|_{1,\Omega} + \|\varphi\|_{1,\Omega} \leq C\,\|\mathbf{v}\|_{H(\mathbf{curl};\,\Omega)}.$$

Here $H^1_\Gamma(\Omega)$ *is the subspace of* $H^1(\Omega)$ *whose functions have zero traces on* Γ*.*

Let $\mathcal{M}_h$ be a regular tetrahedral triangulation of Ω and $\mathcal{F}_h$ be the set of faces not lying on Γ. The finite element space $\mathbf{U}_h$ over $\mathcal{M}_h$ is defined by

$$\mathbf{U}_h := \big\{\mathbf{u} \in H(\mathbf{curl};\,\Omega) \;:\; \mathbf{u} \times \mathbf{n}|_\Gamma = \mathbf{0} \quad \text{and}$$
$$\mathbf{u}|_T = \mathbf{a}_T + \mathbf{b}_T \times \mathbf{x} \quad \text{with} \quad \mathbf{a}_T,\,\mathbf{b}_T \in \mathbb{R}^3, \quad \forall T \in \mathcal{M}_h\big\}.$$

Degrees of freedom on every $T \in \mathcal{M}_h$ are $\int_{E_i} \mathbf{u} \cdot d\,\mathbf{l}, \quad i = 1, \cdots, 6$, where $E_1, \cdots, E_6$ are six edges of T. For any $T \in \mathcal{M}_h$ and $F \in \mathcal{F}_h$, we denote the diameters of T and F by h_T and h_F respectively.

The finite element approximation to (4) is: Find $\mathbf{E}_h \in \mathbf{U}_h$ such that

$$a(\mathbf{E}_h,\,\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \overline{\mathbf{v}} + \int_\Sigma \mathbf{g} \cdot \overline{\mathbf{v}}_t, \quad \forall\, \mathbf{v} \in \mathbf{U}_h. \tag{6}$$

Let $\mathbf{E}$ and $\mathbf{E}_h$ be the solutions of (4) and (6) respectively. Define the total error function by $\mathbf{e}_h := \mathbf{E} - \mathbf{E}_h$. By (5), we know that

$$\|\mathbf{e}_h\|_{H_\Gamma(\mathbf{curl};\,\Omega)} \le \beta^{-1} \sup_{\mathbf{v}\in H_\Gamma(\mathbf{curl};\,\Omega)} \frac{a(\mathbf{e}_h,\,\mathbf{v})}{\|\mathbf{v}\|_{H_\Gamma(\mathbf{curl};\,\Omega)}}.$$

To derive *a posteriori* error estimates, we require the Scott-Zhang interpolant $\mathcal{I}_h :\ H^1_\Gamma(\Omega) \to V_h$ [SC94] and the Beck-Hiptmair-Hoppe-Wohlmuth interpolant $\Pi_h :\ \mathbf{H}^1(\Omega) \cap \mathbf{H}_\Gamma(\mathbf{curl};\,\Omega) \to \mathbf{U}_h$ [BHHW00], where V_h is the standard piecewise linear H^1_Γ-conforming finite element space over $\mathcal{M}_h$. It is known that $\mathcal{I}_h$ and Π_h satisfy the following approximation and stability properties: for any $T \in \mathcal{M}_h$, $F \in \mathcal{F}_h$, $\phi_h \in V_h$, $\phi \in H^1_\Gamma(\Omega)$,

$$\begin{aligned}
&\mathcal{I}_h\phi_h = \phi_h, \quad \|\nabla \mathcal{I}_h\phi\|_{0,T} \le C|\phi|_{1,D_T}\\
&\|\phi - \mathcal{I}_h\phi\|_{0,T} \le Ch_T|\phi|_{1,D_T}, \quad \|\phi - \mathcal{I}_h\phi\|_{0,F} \le C\, h_F^{1/2}\, |\phi|_{1,D_F},
\end{aligned}$$

and for any $T \in \mathcal{M}_h$, $F \in \mathcal{F}_h$, $\mathbf{w}_h \in \mathbf{U}_h$, $\mathbf{w} \in H_\Gamma(\mathbf{curl};\Omega)$,

$$\begin{aligned}
&\Pi_h\mathbf{w}_h = \mathbf{w}_h, \quad \|\Pi_h\mathbf{w}\|_{H(\mathbf{curl};\,T)} \le C\,\|\mathbf{w}\|_{1,D_T},\\
&\|\mathbf{w} - \Pi_h\mathbf{w}\|_{0,T} \le C\,h_T\,|\mathbf{w}|_{1,D_T}, \quad \|\mathbf{w} - \Pi_h\mathbf{w}\|_{0,F} \le C\,h_F^{1/2}\,|\mathbf{w}|_{1,D_F},
\end{aligned}$$

where D_A is the union of elements in $\mathcal{M}_h$ with non-empty intersection with A, $A = T$ or F.

By Theorem 1, for any $\mathbf{v} \in H_\Gamma(\mathbf{curl};\,\Omega)$, there exist a $\varphi \in H^1_\Gamma(\Omega)$ and a $\mathbf{v}_s \in \mathbf{H}^1(\Omega) \cap H_\Gamma(\mathbf{curl};\,\Omega)$ such that

$$\begin{aligned}
\mathbf{v} &= \nabla\varphi + \mathbf{v}_s,\\
\|\varphi\|_{1,\Omega} + \|\mathbf{v}_s\|_{1,\Omega} &\le C\,\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)},
\end{aligned}$$

where the constant C depends only on Ω. Since $\nabla\mathcal{I}_h\varphi$ and $\Pi_h\mathbf{v}_s$ belong to $\mathbf{U}_h$, by the Galerkin orthogonality, we have

$$a(\mathbf{e}_h,\,\mathbf{v}) = a(\mathbf{e}_h,\,\nabla\varphi - \nabla\mathcal{I}_h\varphi) + a(\mathbf{e}_h,\,\mathbf{v}_s - \Pi_h\mathbf{v}_s) \quad \forall\,\mathbf{v} \in H_\Gamma(\mathbf{curl};\,\Omega).$$

For any face $F \in \mathcal{F}_h$, assuming $F = T_1 \cap T_2$, $T_1, T_2 \in \mathcal{M}_h$ and the unit normal $\mathbf{n}$ points from T_2 to T_1, we denote the jump of a function v across F by $[\![v]\!]_F := v|_{T_1} - v|_{T_2}$. The following theorem is proved in [CWZ07].

Theorem 2. *Let* $\mathbf{g} \in \mathbf{L}^2(\Sigma)$ *satisfying* $\mathrm{div}_\Sigma\,\mathbf{g} \in L^2(\Sigma)$ *and* $\mathbf{g}\cdot\mathbf{n} = 0$ *on* Σ. *Then there exists a constant* C *depending on* β *and the mesh* $\mathcal{M}_h$ *such that*

$$\begin{aligned}
\|\mathbf{e}_h\|^2_{H_\Gamma(\mathbf{curl};\,\Omega)} \le\ & C \sum_{T\in\mathcal{M}_h} h_T^2\|\mathbf{f} + k^2\varepsilon_r\,\mathbf{E}_h - \nabla\times(\mu_r^{-1}\nabla\times\mathbf{E}_h)\|^2_{0,T}\\
&+ C \sum_{T\in\mathcal{M}_h} h_T^2\,\|\mathrm{div}\,(k^2\varepsilon_r\,\mathbf{E}_h)\|^2_{0,T}\\
&+ C \sum_{F\in\mathcal{F}_h} h_F\|[\![\mu_r^{-1}\,(\nabla\times\mathbf{E})_h\times\mathbf{n}]\!]_F\|^2_{0,F}\\
&+ C \sum_{F\in\mathcal{F}_h} h_F\,\|[\![k^2\varepsilon_r\,\mathbf{E}_h\cdot\mathbf{n}]\!]_F\|^2_{0,F}
\end{aligned}$$

$$+ C \sum_{F\subset\Sigma} h_F \|\mathbf{g} + \mathbf{i}\,\mathbf{k}\,\lambda\,\mathbf{E}_{k,t} + \mathbf{n}\times\mu_r^{-1}\,(\nabla\times\mathbf{E})_h\|_{0,F}^2$$
$$+ C \sum_{F\subset\Sigma} h_F \|\mathrm{div}_\Sigma(\mathbf{g} + \mathbf{i}\,k\,\lambda\,\mathbf{E}_{k,t})\|_{0,F}^2.$$

Based on the a *posteriori* error estimates in above theorem, an adaptive multilevel method for solving (1)-(3) is designed and implemented. The extensive numerical experiments in [CWZ07] for the lowest order edge element indicate that the adaptive methods based on our *a posteriori* error estimates can efficiently capture the Maxwell singualrity and achieve the following very desirable quasi-optimality property

$$\|\mathbf{E}-\mathbf{E}_h\|_{H(\mathbf{curl};\,\Omega)} \le C\,N^{-1/3},$$

where N is the number of elements of the mesh $\mathcal{M}_h$. Fig. 2 shows an adaptive mesh of 2,947,848 elements after 11 adaptive iterations for solving a time-harmonic problem containing an inner screen $\Gamma := \{(x,\,y,\,z) : -0.5 \le x,\,z \le 0.5,\ y=0\}$. In the example $\Omega = (-1,\,1)^3\setminus\Gamma$, $\Sigma = \partial\Omega\setminus\Gamma$, $\mu_r = \varepsilon_r = \lambda = 1$, and

$$\mathbf{f} := \mathbf{0}, \qquad \mathbf{g} := (\nabla\times\mathbf{E}_i)\times\mathbf{n} - \mathbf{i}\,k\,\mathbf{E}_{i,t},$$

where $\mathbf{E}_i = (e^{\mathbf{i}\,y},\,0,\,e^{\mathbf{i}\,y})^T/\sqrt{2}$ perpendicular to the perfect conducting "screen". Thus (1)–(3) models the scattering by Γ under the incident field $\mathbf{E}_i$. In this case, only $\mathbf{H}^s$-regularity $(s<1/2)$ of the solution is guaranteed.. We observe that the mesh is locally refined near the boundary of the "screen". We refer to [CWZ07] for more information on the adaptive multilevel algorithm and more numerical examples.

3 The time-harmonic electromagnetic scattering problem

In this section we consider the time harmonic electromagnetic scattering problem with the perfectly conducting boundary condition

$$\nabla\times\nabla\times\mathbf{E} - k^2\mathbf{E} = 0 \quad \text{in } \mathbb{R}^3\backslash\bar{D}, \tag{7}$$
$$\mathbf{n}\times\mathbf{E} = \mathbf{g} \quad \text{on } \Gamma_D, \tag{8}$$
$$|\mathbf{x}|\Big[(\nabla\times\mathbf{E})\times\hat{\mathbf{x}} - \mathbf{i}k\mathbf{E}\Big] \to 0 \quad \text{as } |\mathbf{x}|\to\infty. \tag{9}$$

Here $D\subset\mathbb{R}^3$ is a bounded domain with Lipschitz polyhedral boundary Γ_D, $\mathbf{E}$ is the electric field, $\mathbf{g}$ is determined by the incoming wave, $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and $\mathbf{n}$ is the unit outer normal to Γ_D. We assume the wave number $k\in\mathbb{R}$ is a constant.

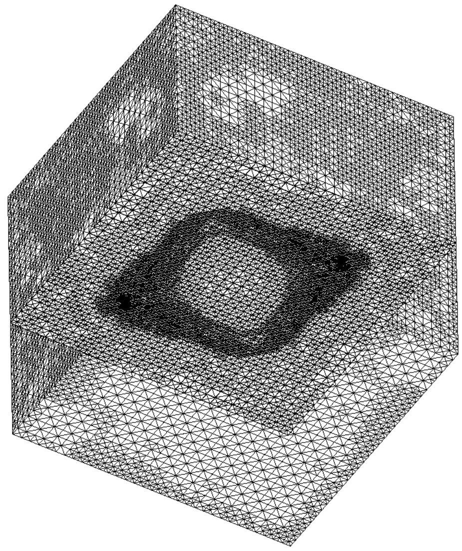

Fig. 2. An adaptively refined mesh of 2,947,848 elements after 11 adaptive iterations.

3.1 The PML equation

Let D be contained in the interior of the ball $B_R = \{\mathbf{x} \in \mathbb{R}^3, |\mathbf{x}| < R\}$ with boundary Γ_R. We first recall the series solution of the scattering problem (7)-(9) outside the ball B_R by following the development in Monk [M03]. Let $Y_n^m(\hat{\mathbf{x}})$, $m = -n, \ldots, n$, $n = 1, 2, \ldots$, be the *spherical harmonics* which satisfies

$$\Delta_{\partial B_1} Y_n^m(\hat{\mathbf{x}}) + n(n+1) Y_n^m(\hat{\mathbf{x}}) = 0 \quad \text{on } \partial B_1, \tag{10}$$

where $\Delta_{\partial B_1} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$ is the Laplace-Beltrami operator for the surface of the unit sphere ∂B_1. The set of all spherical harmonics $\{Y_n^m(\hat{\mathbf{x}}) : m = -n, \ldots, n, n = 1, 2, \ldots\}$ forms a complete orthonormal basis of $L^2(\partial B_1)$.

Denote the *vector spherical harmonics*

$$\mathbf{U}_n^m = \frac{1}{\sqrt{n(n+1)}} \nabla_{\partial B_1} Y_n^m, \qquad \mathbf{V}_n^m = \hat{\mathbf{x}} \times \mathbf{U}_n^m,$$

where $\nabla_{\partial B_1} Y_n^m = \frac{\partial Y_n^m}{\partial\theta}\mathbf{e}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_n^m}{\partial\phi}\mathbf{e}_\phi$, and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ are the unit vectors of the spherical coordinates. The set of all vector spherical harmonics $\{\mathbf{U}_n^m, \mathbf{V}_n^m : m = -n, \ldots, n, n = 1, 2, \ldots\}$ forms a complete orthonormal basis of $\mathbf{L}_t^2(\partial B_1) = \{\mathbf{u} \in L^2(\partial B_1)^3 : \mathbf{u} \cdot \hat{\mathbf{x}} = 0 \text{ on } \partial B_1\}$.

Let $h_n^{(1)}(z)$ be the spherical Hankel function of the first kind of order n. We introduce the *vector wave functions*

$$\mathbf{M}_n^m(r,\hat{\mathbf{x}}) = \nabla\times\{\mathbf{x}h_n^{(1)}(kr)Y_n^m(\hat{\mathbf{x}})\},\qquad \mathbf{N}_n^m(r,\hat{\mathbf{x}}) = \frac{1}{\mathbf{i}k}\nabla\times\mathbf{M}_n^m(r,\hat{\mathbf{x}}),$$

which are the radiation solutions of the Maxwell equation (7) in $\mathbb{R}^3\backslash\{0\}$. In the domain $\mathbb{R}^3\backslash\bar{B}_R$, the solution $\mathbf{E}$ of (7)-(9) can be written as, for $r > R$,

$$\mathbf{E}(r,\hat{\mathbf{x}}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{a_{nm}\mathbf{M}_n^m(r,\hat{\mathbf{x}})}{h_n^{(1)}(kR)\sqrt{n(n+1)}} + \frac{\mathbf{i}kRb_{nm}\mathbf{N}_n^m(r,\hat{\mathbf{x}})}{z_n^{(1)}(kR)\sqrt{n(n+1)}}, \quad (11)$$

where $z_n^{(1)}(kR) = h_n^{(1)}(kR) + kRh_n^{(1)\prime}(kR)$, and a_{nm}, b_{nm} are determined by the trace of $\mathbf{E}$ on Γ_R through $\hat{\mathbf{x}}\times\mathbf{E}|_{\Gamma_R} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_{nm}\mathbf{U}_n^m + b_{nm}\mathbf{V}_n^m$. The series in (11) converges uniformly of $r > R$.

Now we turn to the introduction of the absorbing PML layer. We surround the domain $\Omega_R = B_R\backslash\bar{D}$ with a PML layer $\Omega^{\mathrm{PML}} = \{\mathbf{x}\in\mathbb{R}^3 : R < |\mathbf{x}| < \rho\}$. Let $\alpha(r) = 1 + \mathbf{i}\sigma(r)$ be the model medium property which satisfies

$$\sigma\in C(\mathbb{R}),\qquad \sigma\geq 0,\qquad \text{and } \sigma = 0\ \text{ for } r\leq R.$$

Denote by $\tilde{r}$ the complex radius defined by

$$\tilde{r} = \tilde{r}(r) = \begin{cases} r & \text{if } r\leq R,\\ \int_0^r\alpha(t)dt = r\beta(r) & \text{if } r\geq R.\end{cases}$$

It is easy to check that the vector wave functions satisfy

$$\begin{aligned}\mathbf{M}_n^m(r,\hat{\mathbf{x}}) &= h_n^{(1)}(kr)\nabla_{\partial B_1}Y_n^m(\hat{\mathbf{x}})\times\hat{\mathbf{x}},\\ \mathbf{N}_n^m(r,\hat{\mathbf{x}}) &= \frac{1}{\mathbf{i}k}\nabla\times\mathbf{M}_n^m\\ &= \frac{\sqrt{n(n+1)}}{\mathbf{i}kr}z_n^{(1)}(kr)\mathbf{U}_n^m(\hat{\mathbf{x}}) + \frac{n(n+1)}{\mathbf{i}kr}h_n^{(1)}(kr)Y_n^m(\hat{\mathbf{x}})\hat{\mathbf{x}}.\end{aligned}$$

We introduce

$$\begin{aligned}\tilde{\mathbf{M}}_n^m(\tilde{r},\hat{\mathbf{x}}) &= h_n^{(1)}(k\tilde{r})\nabla_{\partial B_1}Y_n^m(\hat{\mathbf{x}})\times\hat{\mathbf{x}},\\ \tilde{\mathbf{N}}_n^m(\tilde{r},\hat{\mathbf{x}}) &= \frac{1}{\mathbf{i}k}\tilde{\nabla}\times\tilde{\mathbf{M}}_n^m\\ &= \frac{\sqrt{n(n+1)}}{\mathbf{i}k\tilde{r}}z_n^{(1)}(k\tilde{r})\mathbf{U}_n^m(\hat{\mathbf{x}}) + \frac{n(n+1)}{\mathbf{i}k\tilde{r}}h_n^{(1)}(k\tilde{r})Y_n^m(\hat{\mathbf{x}})\hat{\mathbf{x}},\end{aligned}$$

where $\tilde{\nabla}\times$ is the curl operator with respect to the complex spherical variables $(\tilde{r},\theta,\phi)$, that is, for $\mathbf{\Phi} = \mathbf{\Phi}_r\mathbf{e}_r + \mathbf{\Phi}_\theta\mathbf{e}_\theta + \mathbf{\Phi}_\phi\mathbf{e}_\phi$,

$$\begin{aligned}\tilde{\nabla}\times\mathbf{\Phi} = &\frac{1}{\tilde{r}\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\mathbf{\Phi}_\phi) - \frac{\partial\mathbf{\Phi}_\theta}{\partial\phi}\right)\mathbf{e}_r\\ &+\frac{1}{\tilde{r}}\left(\frac{1}{\sin\theta}\frac{\partial\mathbf{\Phi}_r}{\partial\phi} - \frac{\partial(\tilde{r}\mathbf{\Phi}_\phi)}{\partial\tilde{r}}\right)\mathbf{e}_\theta\\ &+\frac{1}{\tilde{r}}\left(\frac{\partial(\tilde{r}\mathbf{\Phi}_\theta)}{\partial\tilde{r}} - \frac{\partial\mathbf{\Phi}_\phi}{\partial\theta}\right)\mathbf{e}_\phi.\end{aligned}$$

It is easy to check that $\tilde{\nabla}\times\mathbf{\Phi} = A\nabla\times B\mathbf{\Phi}$, where $A = \mathrm{diag}(\beta^{-2}, \alpha^{-1}\beta^{-1}, \alpha^{-1}\beta^{-1})$ and $B = \mathrm{diag}(\alpha, \beta, \beta)$ are 3×3 diagonal matrices.

We follow [M03] to derive the PML equation. For any

$$\lambda = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_{nm}\mathbf{U}_n^m + b_{nm}\mathbf{V}_n^m \in \mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_{\mathrm{R}}),$$

let $\mathbb{E}(\lambda)(\tilde{r}, \hat{\mathbf{x}})$ be the PML extension given by

$$\mathbb{E}(\lambda)(\tilde{r}, \hat{\mathbf{x}}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{a_{nm}\tilde{\mathbf{M}}_n^m(\tilde{r}, \hat{\mathbf{x}})}{h_n^{(1)}(kR)\sqrt{n(n+1)}} + \frac{\mathbf{i}kRb_{nm}\tilde{\mathbf{N}}_n^m(\tilde{r}, \hat{\mathbf{x}})}{z_n^{(1)}(kR)\sqrt{n(n+1)}}. \quad (12)$$

For the solution $\mathbf{E}$ of the scattering problem (7)-(9), let $\tilde{\mathbf{E}} = \mathbb{E}(\hat{\mathbf{x}} \times \mathbf{E}|_{\Gamma_R})$ be the PML extension of $\hat{\mathbf{x}}\times\mathbf{E}|_{\Gamma_R}$. Since $\tilde{r} = r$ on Γ_R, we know that $\hat{\mathbf{x}}\times\tilde{\mathbf{E}} = \hat{\mathbf{x}}\times\mathbf{E}$ on Γ_R. On the other hand, since $h_n^{(1)}(z) \sim \frac{1}{z}e^{\mathbf{i}(z-\frac{1}{2}n\pi-\frac{1}{2}\pi)}$ asymptotically as $|z| \to \infty$, heuristically $\tilde{\mathbf{E}}(\tilde{r}, \hat{\mathbf{x}})$ will decay exponentially for $r > R$. It is obvious that $\tilde{\mathbf{E}}$ satisfies

$$\tilde{\nabla} \times \tilde{\nabla} \times \tilde{\mathbf{E}} - k^2\tilde{\mathbf{E}} = 0 \quad \text{in} \quad \mathbb{R}^3\backslash\bar{B}_R,$$

which gives the desired PML equation in the spherical coordinates

$$\nabla \times B(A\nabla \times B\tilde{\mathbf{E}}) - k^2A^{-1}\tilde{\mathbf{E}} = 0 \quad \text{in} \quad \mathbb{R}^3\backslash\bar{B}_R.$$

The PML problem is then to find $\hat{\mathbf{E}}$, which approximates $\mathbf{E}$ in Ω_R and $B\tilde{\mathbf{E}}$ in $\Omega^{\mathrm{PML}} = B_\rho\backslash\bar{B}_R$, as the solution of the following system

$$\nabla \times BA(\nabla \times \hat{\mathbf{E}}) - k^2(BA)^{-1}\hat{\mathbf{E}} = 0 \ \text{ in } \Omega_\rho = B_\rho\backslash\bar{D}, \quad (13)$$

$$\mathbf{n} \times \hat{\mathbf{E}} = \mathbf{g} \ \text{ on } \Gamma_D, \quad \hat{\mathbf{x}} \times \hat{\mathbf{E}} = 0 \ \text{ on } \Gamma_\rho. \quad (14)$$

The first hint of why the PML method should work is the following estimate for the PML extension.

Lemma 1. *For any $\lambda \in \mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_{\mathrm{R}})$, let $\mathbb{E}(\lambda)$ be the PML extension in* (12)*. Then, for any $r > R$, we have*

$$\| \hat{\mathbf{x}} \times \mathbb{E}(\lambda) \|_{\mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_{\mathrm{r}})} \le C(1 + kR)e^{-\mathrm{Im}(k\tilde{r})(1-\frac{R^2}{|\tilde{r}|^2})^{1/2}} \| \lambda \|_{\mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_{\mathrm{R}})}.$$

We give a brief description of the proof of the lemma. The full proof can be found in [CC06]. We first recall the following exponential decay estimate of the first Hankel function proved in [CL05] based on the Macdonald formula.

Lemma 2. *For any $\nu \in \mathbb{R}, z \in \mathbb{C}_{++} = \{z \in \mathbb{C} : \mathrm{Im}(z) \ge 0, \mathrm{Re}(z) \ge 0\}$ and $\Theta \in \mathbb{R}$ such that $0 < \Theta < |z|$, we have*

$$|H_\nu^{(1)}(z)| \le e^{-\mathrm{Im}(z)(1-\frac{\Theta^2}{|z|^2})^{1/2}}|H_\nu^{(1)}(\Theta)|.$$

Next by simple calculation we have

$$\hat{\mathbf{x}} \times \mathbb{E}(\lambda) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{h_n^{(1)}(k\tilde{r})}{h_n^{(1)}(kR)} a_{nm} \mathbf{U}_n^m + \frac{R}{\tilde{r}} \frac{z_n^{(1)}(k\tilde{r})}{z_n^{(1)}(kR)} b_{nm} \mathbf{V}_n^m,$$

which together with following estimate for the spherical Hankel functions due to Nedelec [N80, p.195] implies Lemma 1.

Lemma 3. *For any* $\Theta > 0$, $\delta_n(\Theta) = \frac{z_n^{(1)}(\Theta)}{h_n^{(1)}(\Theta)}$ *satisfies* $|\delta_n(\Theta)| \geq \frac{n(n+1)}{2\Theta^2+n+1}$.

3.2 Finite element discretization

We start by introducing the weak formulation of the PML problem (13)-(14). Let

$$b(\mathbf{\Psi}, \mathbf{\Phi}) = \int_{\Omega_\rho} (BA\nabla \times \mathbf{\Psi} \cdot \nabla \times \bar{\mathbf{\Phi}} - k^2 (BA)^{-1} \mathbf{\Psi} \cdot \bar{\mathbf{\Phi}}) d\mathbf{x}.$$

Then the weak formulation of (13)-(14) is: Given $\mathbf{g} \in \mathbf{H}^{-1/2}(\mathrm{Div}; \Gamma_\mathrm{D})$, find $\hat{\mathbf{E}} \in \mathbf{H}(\mathbf{curl}, \Omega_\rho)$, such that $\mathbf{n} \times \hat{\mathbf{E}} = \mathbf{g}$ on Γ_D, $\hat{\mathbf{x}} \times \hat{\mathbf{E}} = 0$ on Γ_ρ, and

$$b(\hat{\mathbf{E}}, \mathbf{\Phi}) = 0, \qquad \forall \mathbf{\Phi} \in \mathbf{H}_0(\mathbf{curl}; \Omega_\rho). \tag{15}$$

Let Γ_ρ^h, which consists of piecewise triangles whose vertices lie on Γ_ρ, be an approximation of Γ_ρ. Let Ω_ρ^h be the subdomain of Ω_ρ bounded by Γ_D and Γ_ρ^h. Let $\mathcal{M}_h$ be a regular triangulation of the domain Ω_ρ^h. We will use the lowest order Nédeléc edge element [N80] for which the finite element space $\mathbf{U}_h$ over $\mathcal{M}_h$ is defined by

$$\mathbf{U}_h = \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}; \Omega_\rho^h) : \mathbf{u}|_K = \mathbf{a}_K + \mathbf{b}_K \times \mathbf{x}, \forall \mathbf{a}_K, \mathbf{b}_K \in \mathbb{R}^3, \forall K \in \mathcal{M}_h\}.$$

Degrees of freedom of functions $\mathbf{u} \in \mathbf{U}_h$ on every $K \in \mathcal{M}_h$ are $\int_{e_i} \mathbf{u} \cdot \mathbf{dl}, i = 1, \ldots, 6$, where $e_1, \ldots, e_6$ are six edges of K. Denote by $\mathring{\mathbf{U}}_h = \mathbf{U}_h \cap \mathbf{H}_0(\mathbf{curl}; \Omega_\rho^h)$. In the following, we will always assume that the functions in $\mathring{\mathbf{U}}_h$ are extended to the domain Ω_ρ by zero so that any function $\mathbf{u} \in \mathring{\mathbf{U}}_h$ is also a function in $\mathbf{H}_0(\mathbf{curl}; \Omega_\rho)$. The finite element approximation to (15) reads as follows: Find $\mathbf{E}_h \subset \mathbf{U}_h$ such that $\mathbf{n} \times \mathbf{E}_h = \mathbf{g}_h$ on Γ_D, $\mathbf{n} \times \mathbf{E}_h = 0$ on Γ_ρ^h, and

$$b(\mathbf{E}_h, \mathbf{\Phi}_h) = 0, \qquad \forall \mathbf{\Phi}_h \in \mathring{\mathbf{U}}_h.$$

Here $\mathbf{g}_h$ is some edge element approximation of $\mathbf{g}$ on Γ_D. Notice that the integral in $b(\mathbf{E}_h, \mathbf{\Phi}_h)$ is actually over Ω_ρ^h since $\mathbf{\Phi}_h = 0$ in $\Omega_\rho \backslash \Omega_\rho^h$ by our convention.

For any $K \in \mathcal{M}_h$, we denote by h_K its diameter. Let $\mathcal{F}_h$ be the set of all faces of the mesh $\mathcal{M}_h$ that do not lie on Γ_D and Γ_ρ^h. For any $F \in \mathcal{F}_h$, h_F stands for its diameter. For any interior face F which is a common face of K_1 and K_2 in $\mathcal{M}_h$, we define the following jump residuals across F

$$[\![\mathbf{n} \times (BA\nabla \times \mathbf{E}_h)]\!] = \mathbf{n}_F \times (BA\nabla \times (\mathbf{E}_h|_{K_1} - \mathbf{E}_h|_{K_2})),$$
$$[\![k^2(BA)^{-1}\mathbf{E}_h \cdot \mathbf{n}]\!] = k^2(BA)^{-1}(\mathbf{E}_h|_{K_1} - \mathbf{E}_h|_{K_1}) \cdot \mathbf{n}_F,$$

using the convention that the unit norm vector $\mathbf{n}_F$ to F points from K_2 to K_1. The local error indicator η_K for any $K \in \mathcal{M}_h$ is defined as

$$\begin{aligned}\eta_K^2 = \quad & h_K^2 \| \, k^2(BA)^{-1}\mathbf{E}_h - \nabla \times BA\nabla \times \mathbf{E}_h \, \|^2_{\mathbf{L}^2(K)} \\ & + h_K^2 \| \, \mathrm{div}(k^2(BA)^{-1}\mathbf{E}_h) \, \|^2_{L^2(K)} \\ & + h_K \| \, [\![\mathbf{n} \times (BA\nabla \times \mathbf{E}_h)]\!] \, \|^2_{\mathbf{L}^2(\partial K)} + h_K \| \, [\![k^2(BA)^{-1}\mathbf{E}_h \cdot \mathbf{n}]\!] \, \|^2_{L^2(\partial K)}.\end{aligned}$$

The following theorem is the main result of this section whose proof can be found in [CC06].

Theorem 3. *There exists a constant C depending only on the minimum angle of the mesh $\mathcal{M}_h$ and $\sigma_0 = \max_{\tau \in \mathbb{R}} \sigma(\tau)$ such that the following a posteriori error estimate is valid*

$$\begin{aligned}& \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{H}(\mathbf{curl};\Omega_R)} \\ \le \; & C\| \, \mathbf{g} - \mathbf{g}_h \, \|_{\mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_\mathrm{D})} + C(1+kR)^3 R^{1/2} \Big(\sum_{K \in \mathcal{M}_h} \eta_K^2 \Big)^{1/2} \\ & + C(1+kR)^3 e^{-\mathrm{Im}(k\tilde{\rho})(1-\frac{R^2}{|\tilde{\rho}|^2})^{1/2}} \| \, \hat{\mathbf{x}} \times \mathbf{E}_h \, \|_{\mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})}.\end{aligned}$$

3.3 A numerical example

The implementation of the adaptive algorithm in this section is based on the adaptive finite element package ALBERT [SS00] and its adaptation to the edge element by Dr. Long Wang. We use the a posteriori error estimate in Theorem 3 to determine the PML parameters. We choose the PML medium property as the power function and thus we need only to specify the thickness $\rho - R$ of the layer and the medium parameter σ_0. Recall from Theorem 3 that the a posteriori error estimate consists of two parts: the PML error and the finite element discretization error. In our implementation we first choose ρ and σ_0 such that the exponentially decaying factor:

$$e^{-k\mathrm{Im}(\tilde{\rho})(1-\frac{R^2}{|\tilde{\rho}|^2})^{1/2}} \le 10^{-8},$$

which makes the PML error negligible compared with the finite element discretization errors. Once the PML region and the medium property are fixed, we use the standard finite element adaptive strategy to modify the mesh according to the a posteriori error estimate (cf. e.g. [CL05]).

The following numerical example concerns the scattering of the plane wave $\mathbf{E}_i$ perpendicular to the screen described in last section. Figure 3 shows the $\log N_k$-$\log \mathcal{E}_k$ curves, where $\mathcal{E}_k = (\sum_{K\in\mathcal{M}_k} \eta_K^2)^{1/2}$ is the associated a posteriori error estimate. It indicates that the meshes and the associated numerical complexity are quasi-optimal: $\mathcal{E}_k \approx C N_k^{-1/3}$ is valid asymptotically.

Figures 4 shows the far fields in the direction $(1,0,0)$ for different choices of the PML parameters. We observe that the far fields are insensitive to the choices of PML parameters. More numerical examples can be found in [CC06].

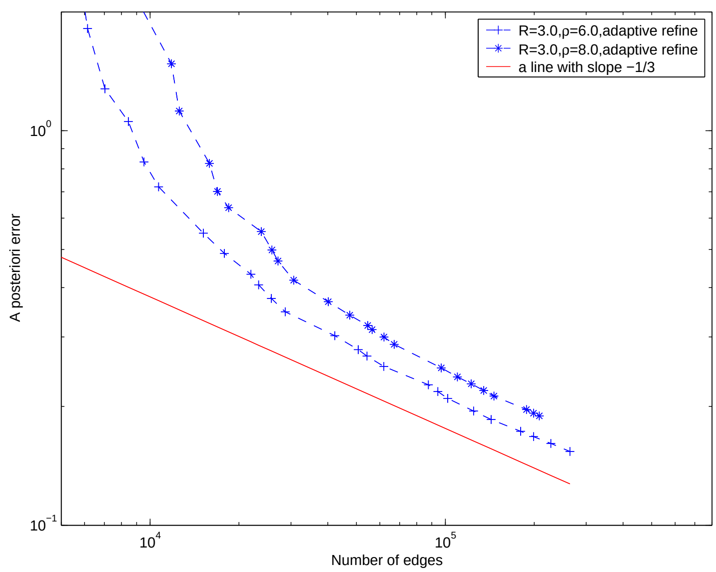

Fig. 3. Quasi-optimality of the adaptive mesh refinements of the a posteriori error estimator

4 The eddy current problem

Three dimensional eddy current problems describe very low-frequency electromagnetic phenomena by quasi-static Maxwell's equations. In this case, displacement currents may be neglected and thus Maxwell's equations become

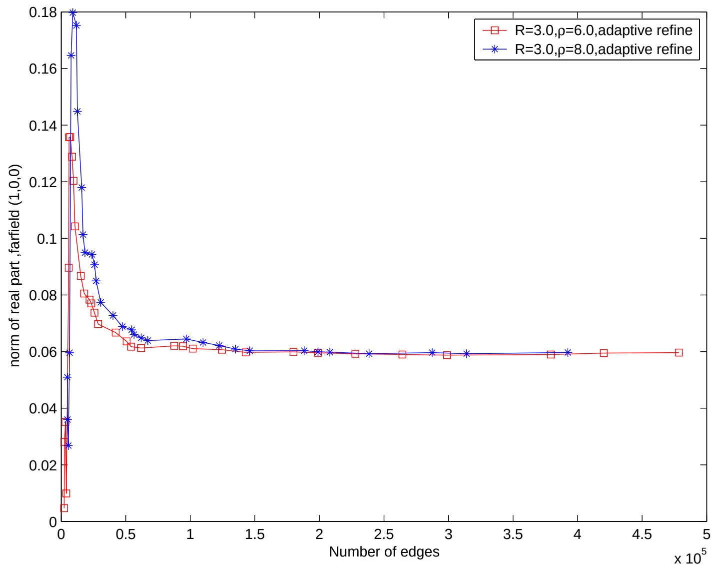

Fig. 4. The module of the real part of the far fields in the direction $(1, 0, 0)$.

$$\begin{cases} \mathbf{curl}\,\mathbf{H} = \mathbf{J} & \text{in } \mathbb{R}^3, \qquad \text{(Ampere's law)} \\ \mu\,\dfrac{\partial \mathbf{H}}{\partial t} + \mathbf{curl}\,\mathbf{E} = 0 & \text{in } \mathbb{R}^3, \qquad \text{(Farady's law)} \\ \operatorname{div}(\mu\mathbf{H}) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{16}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, and $\mathbf{J}$ is the total current defined by:

$$\mathbf{J} = \begin{cases} \sigma\,\mathbf{E} \text{ in } \Omega_c, & \text{(conducting region)} \\ \mathbf{J}_s \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_c}. & \text{(nonconducting region)} \end{cases}$$

Here μ is the magnetic permeability, σ is the electric conductivity, $\mathbf{J}_s$ is the solenoidal source current carried by some coils in the air, and Ω_c is the conducting region which carries eddy currents. To avoid extra complicated constraints on $\mathbf{J}_s$, we assume $\operatorname{supp}(\mathbf{J}_s) \cap \bar{\Omega}_c = \emptyset$.

Let $\Omega \subset \mathbb{R}^3$ be a sufficiently large convex polyhedral domain containing all conductors and coils (see Fig. 5 for a typical model with one conductor and one coil). We assume that μ and σ are real valued $L^\infty(\Omega)$ functions and there exist two positive constants $\mu_{\min}$ and $\sigma_{\min}$ such that $\mu \geq \mu_{\min}$ in Ω and $\sigma \geq \sigma_{\min}$ in Ω_c. Furthermore, we assume $\sigma \equiv 0$ outside of Ω_c.

Since $\operatorname{div}\mathbf{J}_s \equiv 0$, there exists a source magnetic field $\mathbf{H}_s$ such that

$$\mathbf{J}_s = \mathbf{curl}\,\mathbf{H}_s \quad \text{in } \mathbb{R}^3. \tag{17}$$

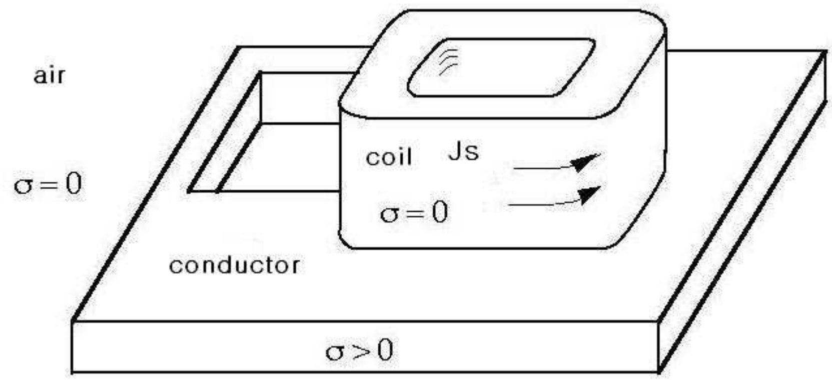

Fig. 5. Setting of the eddy current problems: A conductor with a hole and a coil.

The field $\mathbf{H}_s$ can be written explicitly by the Biot-Savart Law for general coils:

$$\mathbf{H}_s := \mathbf{curl}\,\mathbf{A}_s \quad \text{where} \quad \mathbf{A}_s(\mathbf{x}) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\mathbf{J}_s(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y}.$$

In the following we are going to find the residual $\mathbf{H}_0 := \mathbf{H} - \mathbf{H}_s$. Clearly, by (16) and (4), we have

$$\mathbf{curl}\,\mathbf{H}_0 = 0 \quad \text{in} \ \ \Omega \setminus \overline{\Omega_c}\,.$$

Our goal is to write $\mathbf{H}_0$ as $\nabla\psi$ for some scalar potential ψ. Since $\Omega \setminus \overline{\Omega_c}$ may not be simply connected, ψ may not be unique. To deal with this difficulty, we introduce the following assumption (see [ABDG98, Hypothesis 3.3]).

Hypothesis. There exist I connected open surfaces $\Sigma_0, \cdots, \Sigma_I$, called "cuts", contained in $\Omega \setminus \overline{\Omega_c}$, such that

1. each cut Σ_i is an open part of some smooth two-dimensional manifold with Lipschitz-continuous boundary, $i = 1, \cdots, I$;
2. the boundary of Σ_i is contained in $\partial\Omega_c$ and $\overline{\Sigma}_i \cap \overline{\Sigma}_j = \emptyset$ for $i \neq j$;
3. the open set $\Omega^\circ := (\Omega \setminus \overline{\Omega_c}) \setminus (\cup_{i=1}^I \Sigma_i)$ is simply connected and pseudo-Lipschitz (see [ABDG98, Definition 3.1] for the definition of pseudo-Lipschitz domain).

For each Σ_i, we fix its unit normal vector $\mathbf{n}$ pointing to one side. Define

$$\Theta := \{\varphi \in H^1(\Omega^\circ) \ : \ [\varphi]_{\Sigma_j} = \text{const.}, \ \ 1 \leq j \leq I\},$$

where $[\varphi]_{\Sigma_j}$ is the jump of φ across the cut Σ_j. For any $\varphi \in \Theta$, we can extend $\nabla\varphi \in \mathbf{L}^2(\Omega^\circ)$ continuously to a function $\tilde{\nabla}\varphi \in \mathbf{L}^2(\Omega \setminus \overline{\Omega_c})$ such that

$$\tilde{\nabla}\varphi = \nabla\varphi \quad \text{in} \;\; \Omega^\circ \; .$$

It is known [ABDG98, Lemma 3.11] that for any $\varphi \in H^1(\Omega^\circ)$, $\varphi \in \Theta$ if and only if $\mathbf{curl}\,(\tilde{\nabla}\varphi) = 0 \quad \text{in} \;\; \Omega \setminus \overline{\Omega_c}$.

Since Ω° is simply connected, there exists a unique potential $\psi \in \Theta/\mathbb{R}^1$ such that

$$\mathbf{H}_0 = \nabla\psi \quad \text{in} \;\; \Omega^\circ.$$

Thus the second equation in (16) becomes

$$\begin{cases} \mu \dfrac{\partial\,(\mathbf{H}_s + \nabla\psi)}{\partial t} + \mathbf{curl}\,\mathbf{E} = 0 \quad \text{in} \;\; \Omega^\circ, \\ \mu \dfrac{\partial\,(\mathbf{H}_s + \mathbf{H}_0)}{\partial t} + \mathbf{curl}\,\mathbf{E} = 0 \quad \text{in} \;\; \Omega_c \; . \end{cases} \tag{18}$$

For the initial conditions, we set

$$\psi(\cdot, 0) = 0, \quad \mathbf{H}_0(\cdot, 0) = \mathbf{0}. \tag{19}$$

Since the total electro-magnetic energy is finite, we may assume $\mathbf{H} \in \mathbf{L}^2(\mathbb{R}^3)$ which implies $\mathbf{curl}\,\mathbf{E} \in \mathbf{L}^2(\mathbb{R}^3)$. Assuming Ω large enough, we set the following boundary condition on $\partial\Omega$:

$$\nabla\psi \cdot \mathbf{n} = -\mathbf{H}_s \cdot \mathbf{n} \quad \text{on} \;\; \partial\Omega \; . \tag{20}$$

Our next goal is going to derive a weak formula for (16), starting from (18). Since the tangential field $\mathbf{H}_0 \times \mathbf{n}$ is continuous through $\partial\Omega_c$, we add this constraint to the test functions and define

$$\mathbf{X} = \Big\{ \mathbf{v} \;:\; \mathbf{v} = \tilde{\nabla}\varphi \text{ in } \Omega \setminus \overline{\Omega_c} \text{ for some } \varphi \in \Theta/\mathbb{R}^1 \text{ and } \mathbf{v} = \mathbf{w} \text{ in } \Omega_c$$
$$\text{for some } \mathbf{w} \in \mathbf{H}(\mathbf{curl};\, \Omega_c) \text{ such that } \tilde{\nabla}\varphi \times \mathbf{n} = \mathbf{w} \times \mathbf{n} \text{ on } \partial\Omega_c \Big\}.$$

It is clear that $\mathbf{X} \subset \mathbf{H}(\mathbf{curl};\, \Omega)$. For any $\varphi \in \Theta/\mathbb{R}^1$, we multiply the first equation of (18) by $\nabla\varphi$, integrate by part to obtain

$$\frac{\partial}{\partial t} \int_{\Omega^\circ} \mu\,(\nabla\psi + \mathbf{H}_s) \cdot \nabla\varphi = -\int_{\Omega^\circ} \mathbf{curl}\,\mathbf{E} \cdot \nabla\varphi$$
$$= -\int_{\partial\Omega^\circ} \mathbf{curl}\,\mathbf{E} \cdot \mathbf{n}\,\varphi \; . \tag{21}$$

Note that $\partial\Omega^\circ = \partial\Omega \cup \partial\Omega_c \cup (\cup_{j=1}^I \Sigma_j)$. By (18) and (20) we have $\mathbf{curl}\,\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Thus

$$\frac{\partial}{\partial t}\int_{\Omega^\circ}\mu\,(\nabla\psi+\mathbf{H}_s)\cdot\nabla\varphi=\sum_{j=1}^{I}\int_{\Sigma_j}\mathbf{E}\cdot[\mathbf{n}\times\nabla\varphi]_{\Sigma_j}+\int_{\partial\Omega_c}\mathbf{E}\cdot(\mathbf{n}\times\tilde{\nabla}\varphi)\quad(22)$$
$$=\int_{\partial\Omega_c}\mathbf{E}\cdot(\mathbf{n}\times\tilde{\nabla}\varphi),$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega_c$, and we have used the fact that $[\nabla\varphi\times\mathbf{n}]_{\Sigma_j}=\mathbf{0}$ on Σ_j because of $\varphi\in\Theta$. For any $\mathbf{w}\in\mathbf{H}(\mathbf{curl};\,\Omega_c)$, we multiply the second equation of (18) by $\mathbf{w}$ and integrate by part to obtain

$$\frac{\partial}{\partial t}\int_{\Omega_c}\mu\,(\mathbf{H}_s+\mathbf{H}_0)\cdot\mathbf{w}=-\int_{\Omega_c}\mathbf{curl}\,\mathbf{E}\cdot\mathbf{w}$$
$$=\int_{\partial\Omega_c}\mathbf{E}\cdot(\mathbf{n}\times\mathbf{w})-\int_{\Omega_c}\mathbf{E}\cdot\mathbf{curl}\,\mathbf{w}\,.$$

By (2) and the first equation of (16), we have

$$\frac{\partial}{\partial t}\int_{\Omega_c}\mu\,(\mathbf{H}_s+\mathbf{H}_0)\cdot\mathbf{w}+\int_{\Omega_c}\sigma^{-1}\mathbf{curl}\,\mathbf{H}_0\cdot\mathbf{curl}\,\mathbf{w}=\int_{\partial\Omega_c}\mathbf{E}\cdot(\mathbf{n}\times\mathbf{w}),\quad(23)$$

where $\mathbf{n}$ is the unit normal on $\partial\Omega_c$ pointing to the exterior of Ω_c, and we have used (17) and the fact that $\mathbf{J}_s\equiv\mathbf{0}$ in Ω_c. By the tangential continuity of the electric field $\mathbf{E}$, we add (21) to (23) and obtain, for any $\mathbf{v}\in\mathbf{X}$ such that $\mathbf{v}=\tilde{\nabla}\varphi$ in $\Omega\setminus\overline{\Omega_c}$ and $\mathbf{v}=\mathbf{w}$ in Ω_c,

$$\frac{\partial}{\partial t}\int_{\Omega^\circ}\mu\,\nabla\psi\cdot\nabla\varphi+\frac{\partial}{\partial t}\int_{\Omega_c}\mu\,\mathbf{H}_0\cdot\mathbf{w}+\int_{\Omega_c}\sigma^{-1}\mathbf{curl}\,\mathbf{H}_0\cdot\mathbf{curl}\,\mathbf{w}$$
$$=-\frac{\partial}{\partial t}\int_{\Omega}\mu\,\mathbf{H}_s\cdot\mathbf{v}.$$

For the convenience in notation, we drop the subscript of $\mathbf{H}_0$ and denote the reaction field by $\mathbf{H}$ in the rest of this section. Thus we are led to the following variational problem based on the magnetic reaction field and magnetic scalar potential: Find $\mathbf{H}\in\mathbf{L}^2((0,T);\mathbf{X})$ such that $\mathbf{H}(\cdot,0)\equiv 0$ and

$$\frac{\partial}{\partial t}\int_{\Omega}\mu\,\mathbf{H}\cdot\mathbf{v}+\int_{\Omega_c}\sigma^{-1}\mathbf{curl}\,\mathbf{H}\cdot\mathbf{curl}\,\mathbf{v}=-\frac{\partial}{\partial t}\int_{\Omega}\mu\mathbf{H}_s\cdot\mathbf{v}\quad\forall\,\mathbf{v}\in\mathbf{X}.\quad(24)$$

We use a fully discrete scheme to approximate (24). Let $\{t_0,\cdots,t_M\}$ form a partition of the time interval [0, T] and $\tau_n=t_n-t_{n-1}$ be the n-th timestep. Let $\mathcal{T}_n$ be a regular tetrahedral triangulation of Ω such that $\mathcal{T}_n^c:=\mathcal{T}_n|_{\Omega_c}$ and $\mathcal{T}_n^\circ:=\mathcal{T}_n|_{\Omega^\circ}$ are triangulations of Ω_c and Ω° respectively. Let $\mathcal{T}_{\rm init}$ be the initial regular triangulation of Ω such that each $\mathcal{T}_n$, $n=0,\cdots,M$, is a refinement of $\mathcal{T}_{\rm init}$.

Let $V_n\subset H^1(\Omega)$ and $V_n^\circ\subset H^1(\Omega^\circ)$ be the conforming linear Lagrangian finite element spaces over $\mathcal{T}_n$ and $\mathcal{T}_n^\circ$ respectively, and $\mathbf{V}_n^c\subset\mathbf{H}(\mathbf{curl};\,\Omega_c)$

be the Nédélec edge element space of the lowest order over $\mathcal{T}_n^c$ [N80]. We introduce the finite element space $\mathbf{X}_n \subset \mathbf{X}$ by

$$\mathbf{X}_n = \Big\{ \mathbf{v} \ : \ \mathbf{v} = \tilde{\nabla}\varphi_n \text{ in } \Omega \setminus \overline{\Omega_c} \text{ for some } \varphi_n \in \Theta \cap V_n^\circ/\mathbb{R}^1 \text{ and } \mathbf{v} = \mathbf{w}_n \text{ in } \Omega_c \\ \text{for some } \mathbf{w}_n \in \mathbf{V}_n^c \text{ such that } \tilde{\nabla}\varphi_n \times \mathbf{n} = \mathbf{w}_n \times \mathbf{n} \text{ on } \partial\Omega_c \Big\}.$$

Thus a fully discrete scheme of (24) is: Find $\mathbf{H}_n \in \mathbf{X}_n$ such that $\mathbf{H}_0 \equiv \mathbf{0}$ and

$$\int_\Omega \mu \frac{\mathbf{H}_n - \mathbf{H}_{n-1}}{\tau_n} \cdot \mathbf{v} + \int_{\Omega_c} \sigma^{-1} \mathbf{curl}\, \mathbf{H}_n \cdot \mathbf{curl}\, \mathbf{v} = \int_\Omega \bar{\mathbf{f}}_n \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in \mathbf{X}_n, \tag{25}$$

where $\mathbf{f} := -\mu\, \partial \mathbf{H}_s/\partial t$ and $\bar{\mathbf{f}}_n := \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \mathbf{f}$ is the mean value of $\mathbf{f}$ over $[t_{n-1}, t_n]$. The uniqueness and existence of solutions to (25) follows directly from the Lax-Milgram Lemma.

As in the second section, the key ingredient in the analysis of *a posteriori* error estimates for Maxwell's equations is Helmholtz-type decompositions for functions in $\mathbf{H}(\mathbf{curl}; \Omega)$. In the next, we will introduce an $\mathbf{H}(\mathbf{curl})$-stable decomposition for $\mathbf{X}$. Since both Ω_c and $\Omega \setminus \overline{\Omega_c}$ are multiply connected, it is difficult to find a scalar function ψ with constant jumps across all "cuts" to define the irrotational part. Instead, we represent these discontinuities by the help of some finite element function [ZCW06].

Theorem 4. *Let $\mathbf{X}_{\text{init}}$ be the finite element space over $\mathcal{T}_{\text{init}}$. For any $\mathbf{v} \in \mathbf{X}$, there exists a $\varphi \in H^1(\Omega)/\mathbb{R}^1$, a $\mathbf{v}_{\text{init}} \in \mathbf{X}_{\text{init}}$, and a $\mathbf{v}_s \in \mathbf{H}(\mathbf{curl}; \Omega) \cap \mathbf{H}^1(\Omega_c)$ such that $\mathbf{v}_s = \mathbf{0}$ in $\Omega \setminus \overline{\Omega_c}$ and*

$$\mathbf{v} = \nabla\varphi + \mathbf{v}_{\text{init}} + \mathbf{v}_s.$$

Furthermore, there exists a positive C depending only on Ω and $\mathcal{T}_{\text{init}}$ such that

$$\|\varphi\|_{1,\Omega} + \|\mathbf{v}_s\|_{1,\Omega_c} + \|\mathbf{v}_{\text{init}}\|_{\mathbf{H}(\mathbf{curl};\, \Omega)} \le C \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\, \Omega)}.$$

The following residual based a posteriori error estimate is proved in [ZCW06].

Theorem 5. *There exists a positive constant C depending only on Ω, μ, and σ such that for any $0 \le m \le M$,*

$$\|\sqrt{\mu}\, \mathbf{e}(t_m)\|_{0,\Omega}^2 + \|\mathbf{curl}\, \mathbf{e}\|_{\mathbf{L}^2((0,T);\, \mathbf{L}^2(\Omega))}^2 \le C \sum_{n=1}^{m} \tau_n \left\{ \left(\eta_{\text{time}}^n\right)^2 + \left(\eta_{\text{space}}^n\right)^2 \right\},$$

where the a posteriori error estimates are given by

$$\left(\eta_{\text{time}}^n\right)^2 = \|\mathbf{curl}(\mathbf{H}_n - \mathbf{H}_{n-1})\|_{0,\Omega_c}^2 + \tau_n^{-1}\|\mathbf{f} - \bar{\mathbf{f}}_n\|^2_{\mathbf{L}^2((t_{n-1},t_n);\,\mathbf{L}^2(\Omega))},$$

$$\begin{aligned}\left(\eta_{\text{space}}^n\right)^2 = &\sum_{T\in\mathcal{T}_n} h_T^2 \left\|\text{div}\left(\bar{\mathbf{f}}_n - \mu\frac{\partial \mathbf{H}_h}{\partial t}\right)\right\|_{0,T}^2 \\ &+ \sum_{T\in\mathcal{T}_n^c} h_T^2 \left\|\bar{\mathbf{f}}_n - \mu\frac{\partial \mathbf{H}_h}{\partial t} - \mathbf{curl}(\sigma^{-1}\,\mathbf{curl}\,\mathbf{H}_n)\right\|_{0,T}^2 \\ &+ \sum_{F\in\mathcal{F}_n^{\Omega}} h_F \left\|\left[\left(\bar{\mathbf{f}}_n - \mu\frac{\partial \mathbf{H}_h}{\partial t}\right)\cdot\mathbf{n}\right]_F\right\|_{0,F}^2 \\ &+ \sum_{F\in\mathcal{F}_n^{\Omega_c}} h_F \left\|\left[\sigma^{-1}\mathbf{curl}\,\mathbf{H}_n \times \mathbf{n}\right]_{J,F}\right\|_{0,F}^2 \\ &+ \sum_{F\in\mathcal{F}_n^{\partial\Omega}} h_F \left\|\left(\bar{\mathbf{f}}_n - \mu\frac{\partial \mathbf{H}_h}{\partial t}\right)\cdot\mathbf{n}\right\|_{0,F}^2 .\end{aligned}$$

Here $\mathcal{F}_n^{\Omega}$, $\mathcal{F}_n^{\Omega_c}$, and $\mathcal{F}_n^{\partial\Omega}$ denote the edges in Ω, in Ω_c, and on $\partial\Omega$ respectively.

Based the a posteriori error estimates in above theorem, an adaptive finite element method with variable time-steps and designed and implemented in [ZCW06]. The results indicate that our adaptive method has the following very desirable quasi-optimality property:

$$\eta_{\text{total}} \approx C\,N_{\text{total}}^{-1/4}$$

is valid asymptotically, where η_{total} is the total error estimate (see Theorem 5), and $N_{\text{total}} := \sum_{n=1}^{M} N_n$ with M being the number of time steps and N_n being the number of elements of the mesh $\mathcal{T}_n$ at the n-th timestep. We refer to [ZCW06] for more details.

5 The time-domain acoustic scattering problem

We consider the acoustic scattering problem with the sound-hard boundary condition on the obstacle

$$\frac{\partial u}{\partial t} = -\text{div}\mathbf{p}, \qquad \frac{\partial \mathbf{p}}{\partial t} = -\nabla u \quad \text{ in } [\mathbb{R}^2\backslash\bar{D}]\times(0,T), \tag{26}$$

$$\mathbf{p}\cdot\mathbf{n}_D = g \quad \text{ on } \Gamma_D\times(0,T), \tag{27}$$

$$\sqrt{r}(u - \mathbf{p}\cdot\hat{\mathbf{x}}) \to 0, \quad \text{ as } r = |x| \to \infty, \ \text{ a.e. } t\in(0,T), \tag{28}$$

$$u|_{t=0} = u_0, \qquad \mathbf{p}|_{t=0} = \mathbf{p}_0. \tag{29}$$

Here u is the pressure and $\mathbf{p}$ is the velocity field of the wave. $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, g is determined by the incoming wave, $\hat{x} = x/|x|$, and $\mathbf{n}_D$ is the unit outer normal to Γ_D. $u_0, \mathbf{p}_0$ are assumed to be supported in some circle $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$ for some $R > 0$. (9) is the radiation condition which corresponds to the well-known Sommerfeld radiation condition in the frequency domain. We remark that the results in this paper can be easily extended to solve the scattering problems with other boundary conditions such as the sound-soft or the impedance boundary condition on Γ_D.

One of the fundamental problem in the efficient simulation of the wave propagation is the reduction of the exterior problem which is defined in the unbounded domain to the problem in the bounded domain. For any $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 0$, let $u_{\mathrm{L}} = \mathscr{L}(u)$ and $\mathbf{p}_{\mathrm{L}} = \mathscr{L}(\mathbf{p})$ be the Laplace transform of u and $\mathbf{p}$ in time

$$u_{\mathrm{L}}(x,s) = \int_0^\infty e^{-st} u(x,t)dt, \qquad \mathbf{p}_{\mathrm{L}}(x,s) = \int_0^\infty e^{-st} \mathbf{p}(x,t)dt.$$

Since u_0 and $\mathbf{p}_0$ are supported inside the circle B_R, we know that u_{L} satisfies the following Helmholtz equation outside B_R

$$-\Delta u_{\mathrm{L}} + s^2 u_{\mathrm{L}} = 0.$$

Moreover, (9) implies that u_{L} satisfies the radiation condition

$$\sqrt{r}\left(\frac{\partial u_{\mathrm{L}}}{\partial r} + s u_{\mathrm{L}}\right) \to 0, \qquad \text{as } r \to \infty.$$

Thus we have the following series representation for u_{L} outside B_R

$$u_{\mathrm{L}} = \sum_{n=-\infty}^{\infty} \frac{K_n(sr)}{K_n(sR)} u_{\mathrm{L}}^n(R,s) e^{\mathbf{i}n\theta}, \tag{30}$$

where $u_{\mathrm{L}}^n(R,s) = \frac{1}{2\pi}\int_0^{2\pi} u_{\mathrm{L}}(R,\theta,s) e^{-\mathbf{i}n\theta} d\theta$, and $K_n(z)$ is the modified Bessel function of order n. Since $\mathbf{p}_0$ is supported in B_R, we have

$$\mathbf{p}_{\mathrm{L}} \cdot \hat{\mathbf{x}} + \sum_{n=-\infty}^{\infty} \frac{K_n'(sR)}{K_n(sR)} u_{\mathrm{L}}^n(R,s) e^{\mathbf{i}n\theta} = 0 \qquad \text{on } \Gamma_R.$$

By taking the inverse Laplace transform we obtain the following Dirichlet-to-Neumann boundary condition for the solution of the scattering problem (26)-(29) on $\Gamma_R \times (0,T)$

$$\mathbf{p} \cdot \hat{\mathbf{x}} + \sum_{n=-\infty}^{\infty} \left[\mathscr{L}^{-1}\left(\frac{K_n'(sR)}{K_n(sR)}\right) * u_n(R,t)\right] e^{\mathbf{i}n\theta} = 0, \tag{31}$$

where $u_n(R,t) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(R,\theta,t) e^{-\mathbf{i}n\theta} d\theta$ is the Fourier coefficient of u on Γ_R.

Theorem 6. *Assume that* $u_0 \in H^2(\Omega_R)$, $\mathbf{p}_0 \in H(\mathrm{div};\Omega_R), \mathrm{div}\mathbf{p}_0 \in H^2(\Omega_R)$ *so that* $\mathrm{supp}(u_0) \subset B_R, \mathrm{supp}(\mathbf{p}_0) \subset B_R$, *and* $g \in H^2(0,T;H^{-1/2}(\Gamma_D))$. *Let the following compatibility conditions are satisfied* $g|_{t=0} = \mathbf{p}_0 \cdot \mathbf{n}_D$, $\partial_t g|_{t=0} = -\nabla u_0 \cdot \mathbf{n}_D$ *on* Γ_D. *Then the problem* (26)-(27), (31), (29) *has a unique solution* $u \in L^2(0,T;H^1(\Omega_R)) \cap H^1(0,T;L^2(\Omega_R))$, $\mathbf{p} \in L^2(0,T;H(\mathrm{div},\Omega_R)) \cap H^1(0,T;L^2(\Omega_R))$ *such that* $u|_{t=0} = u_0$, $\mathbf{p}|_{t=0} = \mathbf{p}_0$, *and for any* $v \in L^2(0,T;H^1(\Omega_R))$, $\mathbf{q} \in L^2(0,T;L^2(\Omega_R))$,

$$\int_0^T \left[\left(\frac{\partial u}{\partial t}, v\right) - (p, \nabla v) - \langle(\mathscr{L}^{-1} \circ G \circ \mathscr{L})(u), v\rangle_{\Gamma_R}\right] dt = \int_0^T \langle g, v\rangle_{\Gamma_D} dt,$$
$$\int_0^T \left[\left(\frac{\partial \mathbf{p}}{\partial t} \cdot \mathbf{q}\right) + (\nabla u \cdot \mathbf{q})\right] dt = 0.$$

Here $(\mathscr{L}^{-1} \circ G \circ \mathscr{L})(u) \in L^2(0,T;H^{-1/2}(\Gamma_R))$. *Moreover,* $(u,\mathbf{p})$ *satisfies the following stability estimate*

$$\left[\int_0^T \left(\| \partial_t u \|^2_{L^2(\Omega_R)} + \| \nabla u \|^2_{L^2(\Omega_R)} + \| \partial_t \mathbf{p} \|^2_{L^2(\Omega_R)} + \| \mathrm{div}\mathbf{p} \|^2_{L^2(\Omega_R)}\right) dt\right]^{1/2}$$
$$\leq C \max(1, T^{3/2}) \|(u_0, \mathbf{p}_0)\|_{\Omega_R} + C \max(1,T) \|g\|_{H^2(0,T;H^{-1/2}(\Gamma_D))},$$

where $\|(u_0,\mathbf{p}_0)\|_{\Omega_R} = \| u_0 \|_{H^2(\Omega_R)} + \| \mathrm{div}\mathbf{p}_0 \|_{H^2(\Omega_R)}$.

The proof of the theorem can be found in [C07], which depends on the abstract inversion theorem of the Laplace transform and sharp a priori estimate for the Helmholtz equations. To the author's best knowledge, this is the first result of that kind for the time-domain scattering problems in the literature.

The exact non-local boundary condition (31) is the starting point of various approximate absorbing boundary conditions which have been proposed and studied in the literature, see the review paper Hagstrom [H99] and the references therein. An interesting alternative to the method of absorbing boundary conditions is the method of perfectly matched layer (PML). Since the work of Berenger [B94] which proposed a PML technique for solving the time-dependent Maxwell equations in the Cartesian coordinates, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Turkel and Yefet [TY98], Teixeira and Chew [TC01] for the reviews). Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that would either slow down or attenuate all the waves that propagate from inside the computational domain.

There are two classes of time-domain PML methods for the wave scattering problems. The first class, called "split-field PML method" in the engineering literature, includes the original Berenger PML method. It is shown in Abarbanel and Gottlieb [AG97] that the Berenger PML method is only weakly well-posed and thus may suffer instability in practical applications. The second class, the so-called "unsplit-field PML formulations" in the engineering literature, is however, strongly well-posed. One such successful method is the uniaxial PML method developed in Sacks et al [SKLL95] and Gedney [G96] for the Maxwell equations in the Cartesian coordinates. The unsplit-field PML methods in the curvilinear coordinates are introduced in Petropoulos [P00] and Teixeira and Chew [TC01] for Maxwell equations.

Now we describe briefly the unsplit-field PML method for (26)-(29) to be studied in this paper. Let $\alpha(r) = \eta(r) + s^{-1}\sigma(r)$ be the artificial medium property, where $\eta = 1 + \sigma$ and $\sigma \in C(\mathbb{R})$ such that $\sigma \geq 0$ for $r \in \mathbb{R}$ and $\sigma = 0$ for $r \leq R$. Denote by $\tilde{r}$ the complex radius

$$\tilde{r} = \tilde{r}(r) = \begin{cases} r & \text{if } r \leq R, \\ \int_0^r \alpha(\tau) d\tau = r\beta(r) & \text{if } r \geq R, \end{cases}$$

where $\beta(r) = \hat{\eta}(r) + s^{-1}\hat{\sigma}(r)$, and $\hat{\eta}(r) = \dfrac{1}{r}\displaystyle\int_R^r \eta(\tau)d\tau$, $\hat{\sigma}(r) = \dfrac{1}{r}\displaystyle\int_R^r \sigma(\tau)d\tau$.

The starting point is the series representation of $u_{\rm L} = \mathscr{L}(u)$ for $r > R$ in (30). Based on the observation that $K_n(s\tilde{r}) = K_n(sr\hat{\eta} + r\hat{\sigma})$ decays exponentially for $\hat{\sigma}$ since $K_n(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}$ as $|z| \to \infty$, we define the PML extension $(\tilde{u}_{\rm L}, \tilde{\mathbf{p}}_{\rm L})$ of $(u_{\rm L}, \mathbf{p}_{\rm L})$ as

$$\tilde{u}_{\rm L}(r,\theta,s) = \sum_{n=-\infty}^{\infty} \frac{K_n(s\tilde{r})}{K_n(sR)} u_{\rm L}^n(R,s) e^{\mathbf{i}n\theta}, \qquad \forall r > R,$$

$$s\tilde{\mathbf{p}}_{\rm L} = -\tilde{\nabla}\tilde{u}_{\rm L} = -\left(\frac{\partial \tilde{u}_{\rm L}}{\partial \tilde{r}}\mathbf{e}_r + \frac{1}{\tilde{r}}\frac{\partial \tilde{u}_{\rm L}}{\partial \theta}\mathbf{e}_\theta\right), \qquad \forall r > R,$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors of the polar coordinates. Since $\tilde{u}_{\rm L}$ satisfies $-\tilde{\Delta}\tilde{u}_{\rm L} + s^2\tilde{u}_{\rm L} = 0$ outside B_R, where $\tilde{\Delta} = \dfrac{1}{\tilde{r}}\dfrac{\partial}{\partial \tilde{r}}\left(\tilde{r}\dfrac{\partial}{\partial \tilde{r}}\right) + \dfrac{1}{\tilde{r}^2}\dfrac{\partial^2}{\partial^2\theta}$ is the Laplace operator with respect to $(\tilde{r},\theta)$, we know that, where $\tilde{\mathbf{p}}_{\rm L} = \tilde{\mathbf{p}}_{{\rm L},r}\mathbf{e}_r + \tilde{\mathbf{p}}_{{\rm L},\theta}\mathbf{e}_\theta$,

$$s\tilde{u}_{\rm L} = -\tilde{\nabla}\cdot\tilde{\mathbf{p}}_{\rm L} = -\left[\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{\mathbf{p}}_{{\rm L},r}) + \frac{1}{\tilde{r}}\frac{\partial \tilde{\mathbf{p}}_{{\rm L},\theta}}{\partial \theta}\right].$$

Since $\tilde{r} = r\beta$ and $\dfrac{d\tilde{r}}{dr} = \alpha$, for $r \geq R$, by using the chain rule, we obtain

$$s\tilde{u}_{\rm L} = -\left(\frac{1}{\alpha\beta r}\frac{\partial}{\partial r}(\beta r\tilde{\mathbf{p}}_{{\rm L},r}) + \frac{1}{\beta r}\frac{\partial \tilde{\mathbf{p}}_{{\rm L},\theta}}{\partial \theta}\right), \quad s\tilde{\mathbf{p}}_{{\rm L},r} = -\frac{1}{\alpha}\frac{\partial \tilde{u}_{\rm L}}{\partial r}, \quad s\tilde{\mathbf{p}}_{{\rm L},\theta} = -\frac{1}{\beta r}\frac{\partial \tilde{u}_{\rm L}}{\partial \theta}.$$

Heuristically $(\tilde{u}_{\rm L}, \tilde{\mathbf{p}}_{\rm L})$ decays exponentially for $r > R$ and its inverse Laplace transform $(\tilde{u}, \tilde{\mathbf{p}})$ will also decay exponentially in the time domain. The desired time-domain PML system will be obtained by taking the inverse Laplace transform of above equations.

$$\eta\hat{\eta}\frac{\partial \hat{u}}{\partial t} + \mathrm{div}\hat{\mathbf{p}} + (\sigma\hat{\eta} + \hat{\sigma}\eta)\hat{u} + \sigma\hat{u}_\Delta = 0 \quad \text{in } \Omega_\rho \times (0,T),$$

$$M\frac{\partial \hat{\mathbf{p}}}{\partial t} + \nabla\hat{u} + \Lambda_\Delta(\hat{\mathbf{p}} - \hat{\mathbf{p}}_\Delta) = 0 \quad \text{in } \Omega_\rho \times (0,T),$$

$$\frac{\partial \hat{u}_\Delta}{\partial t} - \hat{\sigma}\hat{u} = 0, \quad \frac{\partial \hat{\mathbf{p}}_\Delta}{\partial t} + \Lambda(\hat{\mathbf{p}}_\Delta - \hat{\mathbf{p}}) = 0 \quad \text{in } \Omega_\rho \times (0,T),$$

$$\hat{\mathbf{p}}\cdot\mathbf{n}_D = g \quad \text{on } \Gamma_D \times (0,T), \quad \hat{u} = 0 \quad \text{on } \Gamma_\rho \times (0,T),$$

$$\hat{u}|_{t=0} = u_0, \quad \hat{\mathbf{p}}|_{t=0} = \mathbf{p}_0, \quad \hat{u}_\Delta|_{t=0} = 0, \quad \hat{\mathbf{p}}_\Delta|_{t=0} = 0 \quad \text{in } \Omega_\rho.$$

By the construction of the PML problem, $(\hat{u}, \hat{\mathbf{p}})$ is designed to approximate the solution of the original scattering problem $(u, \mathbf{p})$ in the domain $\Omega_R \times (0,T)$.

Although the tremendous attention and success in the application of PML methods in the engineering literature, there are few mathematical results on the convergence of the PML methods. For the Helmholtz equation in the frequency domain, it is proved in Lassas and Somersalo [LS98], Hohage *et al* [HSZ03] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinity. In Chen and Wu [CW03], Chen and Liu [CL05], an adaptive PML technique is proposed and studied in which a posteriori error estimate is used to determine the PML parameters. In particular, it is shown that exponential convergence can be achieved for fixed thickness of the PML layer by enlarging PML medium properties. For the time-domain PML method, not much mathematical convergence analysis is known except the work in Hagstrom [H99] in which the planar PML method in one space direction is considered for the wave equation. In de Hoop *et al* [DBR02] and Diaz and Joly [DJ06] the PML system with point source is analyzed based on the Cagniard - de Hoop method.

Our convergence analysis makes use of the following uniform exponential decay property of the modified Bessel function $K_n(z)$.

Lemma 4. *For any* $\nu \in \mathbb{R}$, $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 0$, $\rho > R > 0$, *and* $\tau > 0$, *we have*

$$\frac{|K_\nu(s\rho + \tau)|}{|K_\nu(sR)|} \le e^{-\tau\left(1 - \frac{R^2}{\rho^2}\right)}.$$

The proof which can be found in [C07] depends on the Macdonald formula for the integral representation of the product of modified Bessel functions and extends our earlier uniform estimate in [CL05] for the first Hankel function $H^1_\nu(z)$, $\nu \in \mathbb{R}$.

Now for $r > R$, let $\tilde{u} = \mathscr{L}^{-1}(\tilde{u}_{\mathrm{L}})$, where $\tilde{u}_{\mathrm{L}}$ is the PML extension

$$\tilde{u}(r,\theta,t) = \sum_{n=-\infty}^{\infty} \left[\mathscr{L}^{-1}\left(\frac{K_n(s\tilde{r})}{K_n(sR)}\right) * u_n(R,t)\right] e^{\mathbf{i}n\theta},$$

where $u_n(R,t) = \mathscr{L}^{-1}(u^n_{\mathrm{L}}(R,s)) = \frac{1}{2\pi}\int_0^{2\pi} u_{\mathrm{L}}(R,\theta,t)e^{-\mathbf{i}n\theta}d\theta$. Since $s\tilde{\rho} = s\rho\hat{\eta}(\rho) + \rho\hat{\sigma}(\rho)$, by using the convolution estimate,

$$
\begin{aligned}
&\| \tilde{u} \|^2_{L^2(0,T;H^{1/2}(\Gamma_\rho))} \\
&= \rho \sum_{n=-\infty}^{\infty} (1+n^2)^{1/2} \left\| \mathscr{L}^{-1} \left(\frac{K_n(s\tilde{\rho})}{K_n(sR)} \right) * u_n(R,t) \right\|^2_{L^2(0,T)} \\
&\leq \rho e^{2s_1 T} \sum_{n=-\infty}^{\infty} (1+n^2)^{1/2} \max_{-\infty<s_2<\infty} \left| \frac{K_n(s\tilde{\rho})}{K_n(sR)} \right|^2 \| u_n(R,t) \|^2_{L^2(0,T)} \\
&\leq \frac{\rho}{R} e^{2s_1 T} \max_{-\infty<n<\infty} \max_{-\infty<s_2<\infty} \left| \frac{K_n(s\tilde{\rho})}{K_n(sR)} \right|^2 \| u \|^2_{L^2(0,T;H^{1/2}(\Gamma_R))} \\
&\leq \frac{\rho}{R} e^{2s_1 T} e^{-2\rho\hat{\sigma}(\rho)\left(1-\frac{R^2}{\rho^2\hat{\eta}(\rho)^2}\right)} \| u \|^2_{L^2(0,T;H^{1/2}(\Gamma_R))}.
\end{aligned}
$$

This implies

$$
\| \tilde{u} \|_{L^2(0,T;H^{1/2}(\Gamma_\rho))} \leq \left(\frac{\rho}{R}\right)^{1/2} e^{s_1 T - \rho\hat{\sigma}(\rho)\left(1-\frac{R^2}{\rho^2\hat{\eta}(\rho)^2}\right)} \| u \|_{L^2(0,T;H^{1/2}(\Gamma_R))}.
$$

Since the above estimate is valid for any $s_1 > 0$, we conclude by letting $s_1 \to 0$ that

$$
\| \tilde{u} \|_{L^2(0,T;H^{1/2}(\Gamma_\rho))} \leq \left(\frac{\rho}{R}\right)^{1/2} e^{- \rho\hat{\sigma}(\rho)\left(1-\frac{R^2}{\rho^2\hat{\eta}(\rho)^2}\right)} \| u \|_{L^2(0,T;H^{1/2}(\Gamma_R))}.
$$

We have the following theorem on the convergence of the PML system.

Theorem 7. *Let* $(u, \mathbf{p})$ *be the solution of the original scattering problem and* $(\hat{u}, \hat{\mathbf{p}},\ \hat{u}_\Delta, \hat{\mathbf{p}}_\Delta)$ *be the solution of the PML problem. Then there exists a constant* $C > 0$ *depending only on* ρ/R *but independent of* σ, η, R, ρ*, and* T *such that*

$$
\begin{aligned}
&\| u - \hat{u} \|_{L^2(0,T;L^2(\Omega_R))} + \| \mathbf{p} - \hat{\mathbf{p}} \|_{L^2(0,T;L^2(\Omega_R))} \\
&\leq C(\eta_{\mathrm{m}} T) e^{-\rho\hat{\sigma}(\rho)\left(1-\frac{R^2}{\rho^2\hat{\eta}(\rho)^2}\right)+T} \| u \|_{H^1(0,T;H^{1/2}(\Gamma_R))}.
\end{aligned}
$$

where $\eta_{\mathrm{m}} = \max_{R\leq r\leq\rho} \eta(r)$*.*

Long time stability of the PML methods is also a much studied topic in the literature (see e.g. Bécache and Joly [BJ02], Bécache *et al* [BPG04], Appelö *et al* [AHK06]). For a PML method to be practically useful, it must be stable in time, that is, the solution should not grow exponentially in time. We remark that the well-posedness of the PML system which follows from the theory of symmetric hyperbolic systems allows the exponential growth of the solutions. In [BJ02, BPG04, AHK06] the stability of the PML systems is considered under the assumption of constant PML medium property which, however, violates the property of perfect matchness of the associated PML system. Thus those studies do not fully explain the success of practical applications of the PML methods. The strategy to prove the stability of the PML method is based on the combination of the stability of the original scattering problem in

Theorem 6 and the convergence of the PML method in Theorem 7. We refer to [C07] for more details.

Acknowledgement. The author is grateful to Mr. Junqing Chen, Dr. Long Wang and Dr. Weiying Zheng for the joint works on which some part of this paper is based. The author is also grateful to the support of China NSF under the grant 10428105 and of the National Basic Research Project under the grant 2005CB321701.

References

[AHK06] Appelö, D., Hagstrom, T., Kreiss, G.: Perfectly matched layers for hyperbolic systems: General formulation, well-posedness and stability. Preprint, University of New Mexico (2006)

[AG97] Abarbanel S., Gottlieb, D.: A Mathematical Analysis of the PML Method. J. Comput. Phys. **134**, 357-363 (1997)

[ABDG98] Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three-dimensional non-smooth domains. Math. Meth. Appl. Sci. **21** 823-864 (1998)

[ACS98] Assous, F., Ciarlet, Jr. P., Sonnendrücher E.: Resolution of the Maxwell equations in a domain with reentrant corners. Math. Model. Numer. Anal. **32**, 359-389 (1998)

[BA73] Babuška, I., Aziz, A.: Survey Lectures on Mathematical Foundations of the Finite Element Method. In: Aziz, A. (ed) The Mathematical Foundations of the Finite Element Method with Application to Partial Differential Equations. Academic Press, New York (1973)

[BR78] Babuška, I., Rheinboldt, C.: Error estimates for adaptive finite element computations. SIAM J. Numer. Anal. **15**, 736-754 (1978)

[BW05] Bao G., Wu, H.J.: On the convergence of the solutions of PML equations for Maxwell's equations. SIAM J. Numer. Anal. **43** 2121-2143, (2005)

[BHHW00] Beck, R., Hiptmair, R., Hoppe R.H.W., Wohlmuth, B.: Residual based a posteriori estimators for eddy current computation. M2AN Math. Modeling and Numer. Anal. **34**, 159-182 (2000)

[BJ02] Bécache E., Joly P.: On the analysis of Berenger's perfectly matched layers for Maxwell equations. Math. Modelling Numer. Anal. **36**, 87-119 (2002)

[BPG04] Bécache, E., Petropoulos, P.G., Gedney, D.: On the long-time behavior of unsplit perfectly matched layers. IEEE Trans. Antennas and Propagation **52**, 1335-1342 (2004)

[B94] Bérénger, J.-P.: A perfectly matched layer for the absorption of electromagnetic waves. J. Comput. Phys. **114** 185-200, (1994)

[BS87] Birman, M.Sh., Solomyak, M.Z.: L^2-Theory of the Maxwell operator in arbitary domains. Uspekhi Mat. Nauk **42**, 61-76 (1987) (in Russian); Russian Math. Surveys **43**, 75-96 (1987) (in English)

[BS94] Birman, M.Sh., Solomyak, M.Z.: On the main singularities of the electric component of the electro-magnetic field in regions with screens. St. Petersburg Math. J. **5**, 125-139 (1994)

[BP06] Bramble, J.H., Pasciak, J.E.: Analysis of a finite PML approximation for the three dimensional time-harmonic Maxwell and acoustic scattering problems. Math. Comp. (to appear)

[CC06] Chen, J.Q., Chen, Z.: An adaptive perfectly matched layer technique for 3D time-harmonic electromagnetic scattering problem. (to appear)

[C07] Chen, Z.: Stability and convergence of the time-domain perfectly matched layer method for acoustic scattering problems. (to appear)

[CL05] Chen, Z., Liu, X.Z.: An adaptive perfectly matched layer technique for time-harmonic scattering problems. SIAM J. Numer. Anal. **43**, 645-671, (2005)

[CWZ07] Chen, Z., Wang, L., Zheng, W.: An adaptive multilevel method for time-harmonic Maxwell equations with singularities, SIAM J. Sci. Comput. **29**, 118-138, (2007)

[CW03] Chen, Z., Wu, H.J.: An adaptive finite element method with perfectly matched absorbing layers for the wave scattering by periodic structures. SIAM J. Numer. Anal. **41**, 799-826, (2003)

[CD00] Costabel, M., Dauge, M.: Singularities of electromagnetic fields in polyhedral domains. Arch. Ration. Mech. Anal. **151**, 221-276 (2000)

[DBR02] de Hoop, A.T., van den Berg, P.M., Remis R.F.: Absorbing boundary conditions and perfectly matched layers – An analytic time-domain performance analysis. IEEE Trans. Mag. **38** 657-660 (2002)

[DJ06] Diaz J., Joly P.: A time domain analysis of PML models in acoustics, Comput. Methods Appl. Mech. Engrg. **195** 3820-2853 (2006)

[DHL99] Dhia, A.B., Hazard, C., Lohrengel, S.: A singular field method for the solution of Maxwell's equations in polyhedral domains. SIAM J. Appl. Math. **59** 2028-2044, (1999)

[G96] Gedney, S.D.: An anisotropic perfectly matched layer absorbing media for the truncation of FDTD latices. IEEE Trans. Antennas Propagat. AP-**44** 1630-1639 (1995)

[H99] Hagstrom, T.: Radiation boundary conditions for the numerical simulation of waves. Acta Numerica 47-106 (1999)

[HSZ03] Hohage, T., Schmidt, F., Zschiedrich, L.: Solving time-harmonic scattering problems based on the pole condition. II: Convergence of the PML method. SIAM J. Math. Anal. **35** 547–560, (2003)

[LS98] Lassas M., Somersalo, E.: On the existence and convergence of the solution of PML equations. Computing **60** 229-241, (1998)

[M98] Monk, P.: A posteriori error indicators for Maxwell's equations. J. Comp. Appl. Math. **100**, 173-190 (1998)

[M03] Monk, P.: Finite Element Methods for Maxwell's Equations. Clarendon Press, Oxford (2003)

[N80] Nédélec, J.: Mixed finite elements in R^3. Numer. Math. **35**, 315-341 (1980)

[P00] Petropoulos. P.G.: Reflectionless sponge layers as absorbing boundary conditions for the numerical solution of Maxwell equations in Rectangular, cylindrical, and spherical coordinates. SIAM J. Appl. Math. **60**, 1037-1058 (2000)

[SKLL95] Sacks, Z.S., Kingsland, D.M., Lee, D.M., Lee. J.F.: A perfectly matched anisotropic absorber for use as an absorbing boundary condition. IEEE Trans. Antennas Propagat. AP-**43** 1460-1463 (1995)

[SS00] Schmidt, A., Siebert K.G.: Design of adaptive finite element software: The finite element Toolbox ALBERTA. Lecture Notes in Computational Science and Engineering 2, Springer Verlag, Berlin (2005)

[SC94] Scott, L.R., Zhang, Z.: Finite element interpolation of nonsmooth functions satisfying boundary conditions. Math. Comp. **54**, 483-493 (1994)

[TC01] Teixeira, F.L., Chew, W.C.: Advances in the theory of perfectly matched layers. In: W.C. Chew et al, (ed), Fast and Efficient Algorithms in Computational Electromagnetics. Artech House, Boston, (2001)

[TY98] Turkel, E., Yefet, A.: Absorbing PML boundary layers for wave-like equations. Appl. Numer. Math. **27** 533-557 (1998)

[ZCW06] Zheng, W.Y., Chen, Z., Wang, L.: An adaptive finite element method for the H-ψ formulation of time-dependent eddy current problems. Numer. Math. **103**, 667-689, (2006)