On the computation of a Fourier transform

Written by Prof. Bradji, Abdallah

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Provisional home page: `http://www.cmi.univ-mrs.fr/~bradji`

Aim of this note

The aim of this note is to compute a Fourier transform of a function without make appeal to the residual method.

1 Details

Let f be the real function given by

$$f(t) = \frac{1}{1+t^2}. \tag{1}$$

The function f is even. So, the Fourier transform $\hat{f}$ of f can be computed using the following simple formula

$$\begin{aligned}\hat{f}(\alpha) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(\alpha t)dt \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos(\alpha t)}{1+t^2}dt. \end{aligned} \tag{2}$$

How to compute the integral $\int_0^\infty \frac{\cos(\alpha t)}{1+t^2}dt$ without make appeal to the residual method ?

To this end, we consider the function φ given by

$$\varphi(t) = \exp\left(-|t|\right). \tag{3}$$

Using the fact that the function φ is even, we find the following expansion for the Fourier transform $\hat{\varphi}$ of φ:

$$\begin{aligned}\hat{\varphi}(\alpha) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \varphi(t)\cos(\alpha t)dt \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{1+\alpha^2}. \end{aligned} \tag{4}$$

Since $\hat{\varphi}$ is even and φ is continuous, then

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{\varphi}(t)\cos(\alpha t)d\alpha = \varphi(t) \tag{5}$$

$$= \exp\left(-|t|\right). \tag{6}$$

Replacing the value of $\hat{\varphi}(\alpha)$ given in (4) in (5) gives

$$\int_0^\infty \frac{\cos(\alpha t)}{1+\alpha^2}d\alpha = \frac{\pi}{2}\exp\left(-|t|\right). \tag{7}$$

Consequently

$$\int_0^\infty \frac{\cos(\alpha t)}{1+t^2}dt = \frac{\pi}{2}\exp\left(-|\alpha|\right). \tag{8}$$

Therefore

$$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}}\exp\left(-|\alpha|\right). \tag{9}$$