Some highlights on the domain of convergence of a sequence of functions

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Aim of this note

The aim of this note is to find the domain of convergence of the sequence of functions $f_n(x) = \sin nx$ defined on $\mathbb{R}$. We will show that this sequence of functions converges only on the set:

$$I = \pi\mathbb{Z} = \{k\pi, k \in \mathbb{Z}\} \tag{1}$$

The limit function is the function which vanishes over I.
Since $f_n(x) = \sin nx$ vanishes over I, then the serie of functions $\sum_{n\geq 0} f_n(x)$ converges only on the subset I. The limit is again the the function which vanishes over I.

1 Details

Since, f_n vanishes over I, for each n, then the sequence of functions $f_n(x) = \sin nx$ converges over I given by (1).
Assume now that there exists $x \notin I$ and prove that $f_n(x)$ does not converges as $n \to +\infty$. We will assume the there exists a limit l such that

$$\lim_{n+\infty} \sin nx = l.$$

The limit l should be in $[-1, 1]$ since $-1 \leq \sin nx \leq 1$. This implies that

$$\lim_{n+\infty} (\sin(n+1)x - \sin(n-1)x) = 0. \tag{2}$$

Which is equivalent to

$$2 \lim_{n+\infty} \cos nx \sin x = 0. \tag{3}$$

Since $\sin x \neq 0$ over I, then

$$\lim_{n+\infty} \cos nx = 0. \tag{4}$$

This implies that

$$\lim_{n+\infty} \cos(n+1)x = 0. \tag{5}$$

This gives, using (4)

$$-\lim_{n+\infty} \sin nx \sin x = 0. \tag{6}$$

Which gives, sin $\sin x \neq 0$ over I

$$\lim_{n+\infty} \sin nx = 0. \tag{7}$$

Gathering this limit with (4) gives

$$\lim_{n+\infty} \sin^2 nx + \cos^2 nx = 0. \tag{8}$$

Which is a contradiction since $\sin^2 nx + \cos^2 nx = 1$.
Consider now the serie of $\sum_{n\geq 0} f_n(x)$. In order that this serie converges, $f_n(x)$ should converge to 0. This is satisfied only when $x \in I$.

Remark 1.1 *It is proved in [1] that $f_n(x)$ does not converge to 0 when $x \notin I$. However, the proof presented here justifies that $f_n(x)$ does not converge to any limit when $x \notin I$. The techniques used in this note is similar to those of [1] in the sense that the techniques here use the known properties of the functions* sin *and* cos.

References

[1] K. Allab, Elément d'Analyse: Fonction d'une Variable Réelle. OPU, 1990