Introduction pour le memoire ”L’approximation Numérique des Equations Différentielles Fractionnaires”

Bradji, Abdallah

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1 Introduction and an overview on the work

Fractional differntial and partial differential equations have been discused by many authors as generalizations of classical differntial and partial differential equations. They appear in fluid flow, finance, biological sciences and others. Fractional differntial and partial differential equations have attracted considerable attention because of their applications in several areas.

Consequently, it is useful to look on the approximation of this large class of differential and partial differential equations.

The aim of this work is to initiate for the numerical approximation of the fractional differential and partial differential equations. Since, the basic numerical method is Finite Difference methods, we have tried to study first the finite difference approximation of fractional differential and partial differential equations.

As model, we have studied the following fractional partial differential equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^\alpha u}{\partial x^\alpha}(x,t) + f(x,t),\ (x,t) \in \Omega \times (0,T), \tag{1}$$

where $\Omega = (0,1)$, $1 < \alpha < 2$, $T > 0$ is given and the fractional differential operator $\dfrac{\partial^\alpha u}{\partial x^\alpha}$ is given by

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{x^2} \int_0^x \frac{u(\xi)}{(x-\xi)^{\alpha-1}} d\xi. \tag{2}$$

An initial condition is given by

$$u(x,0) = u^0(x),\ x \in \Omega. \tag{3}$$

Homogeneous Dirichlet boundary conditions are given by

$$u(0,t) = u(1,t) = 0,\ t \in (0,T). \tag{4}$$

Well posedness (Existence and uniqueness of u) of [1]–[4] can be justified in the framework of semi–groupe, see [BRE 99].

We aimed from this work to provide a finite volume method for [1]–[4] which has not been treated before.

For the sake of completeness, we have began by the cases $\alpha = 1$ which is hyperbolic case, and $\alpha = 2$ which is parabolic case and then we moved to the fractional case $\alpha \in (1,2)$.

For the sake of simplicity, we considered some times the explicit schemes.
We should mention finally that we would like to present the first ideas on a finite volume scheme, see [EYM 00], for [1]–[4], which is very new, but because of limited time we only tried to understand the basic ideas of the approximation of fractional differential and partial differential equations. The task of finite volume method for [1]–[4] will be the subject of a future research.

2 Conclusion and perspectives

We considered a time dependent fractional partial differential equation of order α. For the sake of completeness and in order to understand the basic ideas of the approximation, we first studied some simple cases of finite difference schemes approximating the problem under consideration when $\alpha = 1$, hyperbolic case, and $\alpha = 2$, parabolic case. We moved after to the case when $1 < \alpha < 2$. We presented a simple explicit scheme and we provided its convergence. The convergence of this implict scheme is based on the well known theorem which states that consistency and stability implies convergence of the finite difference schemes.
This work is initiation to a long work which aims to study the availables schemes (in finite differences and finite elements methods) and then to move to present finite volume schemes approximating time dependent fractional partial differential equations which is new and has not been treated before.

References

[BAE 98] Baeumer, Boris; Kovács, Mihály; Meerschaert, Mark M.: Numerical solutions for fractional reaction-diffusion equations. *Comput. Math. Appl.*, **55**, No. 10, 2212–2226 (2008).

[BRE 99] Brezis, H.: Analyse Fonctionnelle: Théorie et Applications", *D*unod, Paris, 1999.

[CHO 10] Choi, Chung; Lee: Numerical solution for space fractional dispersion equations with nonlinear source terms. *Bull. Korean Math. Soc.*, **47**, No. 4, 1256–1234 (2010).

[EYM 00] Eymard, R., Gallouët, T. and Herbin, R.: Finite Volume Methods. *Handbook of Numerical Analysis. P. G. Ciarlet and J. L. Lions (eds.)*, **VII** , 723–1020, 2000.

[MEE 04] Meerschaert, Tadjeran: Finite difference approximations for franctional advection–dispersion flow equations. *J. Compt. Appl. Math.* , **172**, No. 1, 65–77 (2004).

[MEE 06] Meerschaert, Tadjeran: Finite difference approximations for two sided space franctional partial differential equations. *Appl. Numer. Math.*, **56**, No. 1, 80–90 (2006).

[MEE 06] Meerschaert, Scheftler, Tadjeran: Finite difference methods for two dimensional franctional dispersion equations. *J. Comput. Phys.*, **211**, No. 1, 249–261 (2006).