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A new stable method for singularly perturbed convection–diffusion equations

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Abstract

The aim of th present work is to give a review for the the nice article entitled "A new stable method for singularly perturbed convection–diffusion equations", *Computer methods in applied mechanics and engineering*, **197**, 1507–1524, 2008.

1 Introduction

Let Ω be an open polygonal subset (for the sake of simplicity, even the authors introduce their results when Ω is a bounded domain with Lipschitz boundary) of $\mathbb{R}^2$. We introduce the following model of convection–diffusion problem:

$$\mathcal{L}u = -\varepsilon\Delta u(\mathbf{x}) + \mathbf{v}\cdot\nabla u(\mathbf{x}) + ru(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x}\in\Omega, \qquad [1]$$

with Dirichlet boundary condition

$$u(\mathbf{x}) = 0,\ \mathbf{x}\in\partial\Omega. \qquad [2]$$

We assume for simplicity that $f \in L^2(\Omega)$, the velocity $\mathbf{v}$ and reaction term r are continuous. Since the case $0 < \varepsilon$ is constant not small enough is treated in the classical finite element method, the authors are interested with the case $0 < \varepsilon \ll 1$. As usual, we look for $u \in H_0^1(\Omega)$ such that

$$B(u,v) = F(v),\ \forall v \in H_0^1(\Omega), \tag{3}$$

where $B(\cdot,\cdot)$ is the bilinear form defined by

$$B(u,v) = \varepsilon\left(\nabla u, \nabla v\right) + \left(\mathbf{v}\cdot\nabla u, v\right) + \left(ru, v\right), \tag{4}$$

and $F(\cdot)$ is the linear form defined by

$$F(v) = (f, v) \tag{5}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$.

To prove the existence and uniqueness of , we apply the known Lemma of Lax–Milgram. Indeed.

- Continuity of F, thanks to the Cauchy Schwarz inequality, we have

$$\begin{aligned} |F(v)| &\le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\le \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}. \end{aligned} \tag{6}$$

- Continuity of $B(\cdot,\cdot)$

$$\begin{aligned} |B(u,v))| &\le \varepsilon\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} + \max_{\mathbf{x}\in\bar{\Omega}}|\mathbf{v}(\mathbf{x})|\|u\|_{H^1(\Omega)}\|v\|_{L^2(\Omega)} + \max_{\bar{\Omega}}|r|\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\le \left(\varepsilon + \max_{\mathbf{x}\in\bar{\Omega}}|\mathbf{v}(\mathbf{x})| + \max_{\mathbf{x}\in\bar{\Omega}}|r|\right)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \end{aligned} \tag{7}$$

- H_0^1–ellipticity, thanks to a Green's formula, we have

$$|B(u,u))| = \varepsilon\|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega \left(r(\mathbf{x}) - \frac{\nabla\mathbf{v}(\mathbf{x})}{2}\right)u(\mathbf{x})d\mathbf{x}. \tag{8}$$

To get the H_0^1–ellipticity, i.e. $|B(u,u))| \ge \alpha\|\nabla u\|^2_{L^2(\Omega)}$ (recall that, thanks to Poincaré inequality, $\|\nabla u\|_{L^2(\Omega)}$ is equivalent to the norme $\|u\|_{H^1(\Omega)}$ of $H^1(\Omega)$), it suffices to assume that

$$r(\mathbf{x}) - \frac{\nabla\mathbf{v}(\mathbf{x})}{2} \ge 0,\ a.e.\mathbf{x} \in \Omega. \tag{9}$$

Which gives with 8,

$$|B(u,u))| \geq \varepsilon \|\nabla u\|^2_{L^2(\Omega)}. \quad [10]$$

Let us a finite element discretization $\mathcal{T}$ using triangles in such that, for the sake of simplicity, $\Omega^h = \Omega$, where h is a positive parameter tends to zeror and Ω^h is the discretized domain, that is $\Omega^h = \cup\{\tau; \tau \in \mathcal{T}\}$.

To get a finite element discretization for [3], we introduce a finite dimensional subspaces $\mathcal{V}_h^p \subset H_0^1(\Omega)$, in which the finite element approximate solutions belongs to, and $\mathcal{W}_h^p \subset H_0^1(\Omega)$, in which the space of test function included in. The elements of these two spaces are polynomials of dgree up to p. The discretization of [3] is : looking for $u_h \in \mathcal{V}_h^p$ such that

$$B(u_h, v_h) = F(v_h), \ \forall v_h \in \mathcal{W}_h^p. \quad [11]$$

The space $\mathcal{V}_h^p$ is known as the trial space, and $\mathcal{W}_h^p$ is known as the space of test functions.

When $\mathcal{V}_h^p = \mathcal{W}_h$, then the method [11] called the *standard Gelerkin method*. Otherwise, the method [11] is called the *Petrov Galerkin method*.

We set $v = v_h$ in [3], substracting the reulting equation from [11], we get the so called *Galerkin orthogonality property*

$$B(u_h - u, v_h) = 0, \ \forall v_h \in \mathcal{W}_h^p. \quad [12]$$

From [12], we get the error estimate. Indeed, in case of standard finite element method, [12] implies that

$$B(u_h - \pi u, v_h) = B(u - \pi u, v_h), \ \forall v_h \in \mathcal{W}_h^p, \quad [13]$$

where Set $v_h = u_h - \pi u$ in [13], we get

$$B(u_h - \pi u, u_h - \pi u) = B(u - \pi u, u_h - \pi u). \quad [14]$$

Thanks to [10] and [7], we get

$$\varepsilon \|\nabla u_h - \pi u\|^2_{L^2(\Omega)} \leq \left(\varepsilon + \max_{\mathbf{x}\in\bar{\Omega}} |\mathbf{v}(\mathbf{x})| + \max_{\mathbf{x}\in\bar{\Omega}} |r|\right) \|u_h - \pi u\|_{H^1(\Omega)} \|u - \pi u\|_{H^1(\Omega)}. \quad [15]$$

This with Poincaré inequality and interpolation error, we get

$$\left(\|\nabla u_h - \pi u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \leq Ch^p \frac{(\varepsilon + \max_{\mathbf{x}\in\bar{\Omega}} |\mathbf{v}(\mathbf{x})| + \max_{\mathbf{x}\in\bar{\Omega}} |r(\mathbf{x})|)}{\varepsilon} |u|_{p+1,\Omega}, \quad [16]$$

where C is only depending on Ω.