University of Annaba–"M.I. Mathematiques et Informatique LMD"
First year undergraduation **2011–2012**

Analyse

Sequences

Not finished yet; last update Thursday 3rd Nov. 2011

Exercice 1. (Definition of the limit) Use the definition of the limit to prove that

1.
$$\lim_{n\to+\infty} \frac{1-2n}{n+1} = -2. \tag{1}$$

2.
$$\lim_{n\to+\infty} \exp(-n) = 0. \tag{2}$$

3.
$$\lim_{n\to+\infty} \sqrt{n^2-1} = +\infty. \tag{3}$$

Exercice 2. (Some properties of the limit)

1. Prove that if $\lim_{n\to+\infty} u_n = l$ and $u_n \geq 0$ for all $n \geq k$, with a given integer k, then $l \geq 0$.
2. Prove that $\lim_{n\to+\infty} u_n = l$ if and only if $\lim_{n\to+\infty} |u_n - l| = 0$.
3. Prove that if $|u_n - l| \leq v_n$ for large n and $\lim_{n\to+\infty} v_n = 0$, then $\lim_{n\to+\infty} u_n = l$.

Exercice 3. (Computation of the limit and prove it) Compute the limit of the following sequences and prove the limit using the definition:

1.
$$2 + \frac{1}{n+1}, \quad \frac{n+2}{n+1}, \quad \frac{1}{n}\sin\frac{n\pi}{4}. \tag{4}$$

2.
$$\frac{n}{2n+\sqrt{n+1}}, \quad \frac{\sin n}{\sqrt{n}}. \tag{5}$$

Exercice 4. (N does not depend on ε!) Recall that $\lim_{n\to+\infty} u_n = l$ if and only if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|u_n - l| < \varepsilon$ for all $n \geq N$.
State necessary and sufficient conditions on a convergent sequence u_n such that the integer N in the previous definition does not depend ε.

Exercice 5. (Monotonocity) Study the monotonocity (decreased or increased) of the following sequences:

1.
$$u_n = \exp(n); \quad u_n = \frac{1}{\sqrt{n+1}}; \quad u_n = \frac{1}{(n-3)!} \tag{6}$$

2.
$$u_n = (-1)^n; \quad u_n = \frac{n^2}{n!}; \quad u_{n+1} = 2 - \frac{1}{u_n}, \ u_0 = 2. \tag{7}$$

Exercice 6. (Monotonocity and convergence) Study the monotonocity and then the convergence of the following sequences:

$$u_n = \frac{1.3.5\dots(2n+1)}{2.4.6\dots(2n+2)};\ \ u_n = \frac{\sqrt{(n-1)!}}{(1+\sqrt{1})(1+\sqrt{2})\dots(1+\sqrt{n})}. \tag{8}$$

Exercice 7. (Convergence) Study the convergence of the following sequences:

1. $$u_n = \frac{7^{n+1}+6^{n+1}}{7^n+6^n};\ \ u_n = \sum_{k=0}^{n} \ln\left(1+\frac{1}{n+k}\right);\ \ u_n = (n)^{\frac{1}{n}}. \tag{9}$$

2. $$u_n = \frac{1!+2!+\dots+(n+1)!}{(n+1)!};\ \ u_n = \frac{1!+2!+\dots+(n+1)!}{(n+2)!};\ \ u_n = \frac{1!+2!+\dots+(n+1)!}{n!}. \tag{10}$$

Exercice 8. (Convergence) Let u_n and v_n be two sequences in $[0,1]$ such that $\lim\limits_{n\to+\infty} u_n v_n = 1$. Study the convergence of u_n and v_n.

Exercice 9. (Series, increased and bounded above) Let u_n and v_n be two sequences such that $0 \leq u_n \leq v_n$ and $S_n = \sum\limits_{k=0}^{n} v_n$. Show that the sequence $T_n = \sum\limits_{k=0}^{n} u_n$ converges.

Exercice 10. (increased and bounded above, decreased and bounded below) Let $u_n = \dfrac{-1}{3+u_n}$, with $u_0 = 1$.

1. Compute the first four terms of u_n.
2. Show that u_n is a nonincreasing sequence and it is bounded below by -1.
3. Deduce that u_n converges and compute its limit.

Exercice 11. (Three sequences) Let u_n, v_n and w_n be three sequences such that

$$\lim_{n\to+\infty}(u_n+v_n+w_n) = 3a, \tag{11}$$

and

$$\lim_{n\to+\infty}(u_n^2+v_n^2+w_n^2) = 3a^2, \tag{12}$$

where a is a given real.
Show that

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = \lim_{n\to+\infty} w_n = a. \tag{13}$$

Exercice 12. (Arithmetic sequence) Compute the following sums:

1. $$3+7+11+\dots+2011. \tag{14}$$

2. $$1+11+11+\dots+11\dots1, \tag{15}$$

   where the number 1 is reapeted 100 times in the last term of the previous sum.

Exercice 13. (Adjacent sequences) Let $v_0 \leq u_0 < 0$ and $0 < q < p$. We set

$$u_{n+1} = \frac{pu_n+qv_n}{p+q}, \tag{16}$$

and

$$v_{n+1} = \frac{pv_n + qu_n}{p + q}. \tag{17}$$

Prove that u_n and v_n are two adjacent sequences.